International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:9, No:9, 2015

Sixth-Order Two-Point Efficient Family of Super-Halley Type Methods

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***Abstract*—**The main focus of this manuscript is to provide a highly efficient two-point sixth-order family of super-Halley type methods that do not require any second-order derivative evaluation for obtaining simple roots of nonlinear equations, numerically. Each member of the proposed family requires two evaluations of the given function and two evaluations of the first-order derivative per iteration. By using Mathematica-9 with its high precision compatibility, a variety of concrete numerical experiments and relevant results are extensively treated to confirm t he t heoretical d evelopment. From their basins of attraction, it has been observed that the proposed methods have better stability and robustness as compared to the other sixth-order methods available in the literature.

***Keywords*—**Basins of attraction, nonlinear equations, simple roots, Super-Halley.

I. Introduction

EFFICIENT solution techniques are required for finding simple roots of nonlinear equation $f(x) = 0$, which partake of scientific, engineering and various other models. One of the best known one-point optimal method is classical Newton's method [1], [2]. With the advancements of computer algebra, researchers [3]-[8], from the worldwide proposed three-point sixth-order methods that are known as the extensions of Newton's method at the expense of additional evaluations of functions, derivatives and changes in the points of iterations.

But, the body structures of these three-point sixth-order methods are not simple as compared to two-point methods [9], [10]. Further, it is not easy to find two-point methods whose order of convergence greater than four [11].

Therefore, our primary aim is to develop a new highly efficient two-point sixth-order family of super-Halley type methods, that do not require any second-order derivative. It is also observed that the body structures of our proposed methods are simpler than the existing three-point sixth-order methods. Further, our proposed methods are more effective in all the tested examples to the existing methods available in the literature. The dynamic study of our proposed methods which is given in Section V, to cross verify the theoretical aspects.

II. Development of Two-Point Sixth-Order Methods

In this section, we intend to develop several new families of sixth-order super-Halley type methods. For this purpose, we consider the following well known third-order super-Halley method [1], [2]

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}\left(\frac{2\{f'(x_n)\}^2 - f(x_n)f''(x_n)}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}\right). \quad (1)$$

Further, we consider $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, a Newton's iterate. With the help of Taylor series, we expand the function $f(y_n)$ about a point $x = x_n$ as follows:

$$f''(x_n) \approx \frac{2\{f'(x_n)\}^2 f(y_n)}{\{f(x_n)\}^2}. \quad (2)$$

Similarly, expanding the function $f'(y_n) = f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$ about a point $x = x_n$ by Taylor series, we have $f'(y_n) \approx f'(x_n) + f''(x_n)(y_n - x_n)$, which further yields

$$f''(x_n) \approx \frac{f'(x_n)\big(f'(x_n) - f'(y_n)\big)}{f(x_n)}. \quad (3)$$

From (2) and (3), we have

$$f''(x_n) \approx \frac{\frac{2\{f'(x_n)\}^2 f(y_n)}{\{f(x_n)\}^2} + \frac{f'(x_n)\big(f'(x_n) - f'(y_n)\big)}{f(x_n)}}{2}. \quad (4)$$

Using this approximate value of $f''(x_n)$ in formula (1) and using the weight function on the second step, we get a modified family of methods free from second-order derivative as follows:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{2f'(x_n)}\left[\dfrac{3f(x_n)f'(x_n) + f(x_n)f'(y_n) - 2f'(x_n)f(y_n)}{f'(y_n)f(x_n) + f(x_n)f'(x_n) - 2f'(x_n)f(y_n)}\right] \\ \qquad \times L_f(u,\ v) \end{cases} \quad (5)$$

where the weight function L_f is sufficient differential function in a neighborhood of $(1,\ 0)$ with $u = \frac{f'(x_n)}{f'(y_n)} = 1 + O(e_n)$ and $v = \frac{f(y_n)}{f(x_n)} = O(e_n)$. Theorem III indicates that under what choices on the weight function which is proposed in (5), the order of convergence will reach at six without using any more functional evaluations.

III. Order of Convergence

Theorem 1: Let a sufficiently smooth function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ has a simple zero ξ in the open interval D. Then, the iterative scheme defined by (5) has sixth-order convergence when it satisfies the following conditions

$$\begin{aligned} &L_{00} = 1,\ Ł_{01} = \frac{1}{2},\ L_{10} = -\frac{1}{4},\ L_{02} = -\frac{3}{2},\ L_{11} = \frac{1}{2},\ L_{20} = \frac{3}{8}, \\ &L_{12} = -\frac{1}{2} - 4L_{21} - 4L_{30},\quad L_{03} = -9 + 12L_{21} + 16L_{30}, \\ &L_{04} = -8(9 + L_{13} + 3L_{22} + 4L_{31} + 2L_{40}), \end{aligned} \quad (6)$$

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where $L_{ij} = \frac{\partial^{i+j}}{\partial u^i \partial v^j} L_f(u,\ v)|_{(u=1,\ v=0)}$. It satisfies the following error equation

$$e_{n+1} = -\frac{c_2}{12}\Big[4(9 + 2L_{13} + 24L_{21} + 12L_{22} + 48L_{30} + 24L_{31} + 16L_{40})c_2^4 - 2(L_{13} + 24L_{21} + 6L_{22} + 48L_{30} + 12L_{31} + 8L_{40} - 12)c_2^2c_3 + 3(3 + 2L_{21} + 4L_{30})c_3^2 - 12c_2c_4\Big]e_n^6 + O(e_n^7). \tag{7}$$

Proof: Let ξ be a simple zero of $f(x)$. With the help of Taylor's series, we get the following expansion of $f(x_n)$ and $f'(x_n)$ around $x = \xi$

$$f(x_n) = f'(\xi)\big(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + e_n^5c_5 + e_n^6c_6 + O(e_n^7)\big), \tag{8}$$

and

$$f'(x_n) = f'(\xi)\big(1 + 2e_nc_2 + 3e_n^2c_3 + 4e_n^3c_4 + 5e_n^4c_5 + 6e_n^5c_6 + 7e_n^6c_7 + O(e_n^7)\big), \tag{9}$$

respectively. From (8) and (9), we obtain

$$y_n = c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (20c_2^2c_3 - 8c_2^4 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 + O(e_n^6). \tag{10}$$

By using (10) and with the help of Taylor series, we get the following expansions of $f(y_n)$ and $f'(y_n)$ about $x = \xi$

$$f(y_n) = f'(\xi)\Big(c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 - 2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 + O(e_n^6)\Big) \tag{11}$$

and

$$f'(y_n) = f'(\xi)\Big(1 + 2c_2^2e_n^2 - 4(c_2^3 - c_2c_3)e_n^3 + c_2(8c_2^3 - 11c_2c_3 + 6c_4)e_n^4 - 4c_2(4c_2^4 - 7c_2^2c_3 + 5c_2c_4 - 2c_5)e_n^5 + O(e_n^6)\Big) \tag{12}$$

By using (8)-(12), we obtain

$$h = \frac{f'(y_n)}{f'(x_n)} = 1 + 2c_2e_n + (-2c_2^2 + 3c_3)e_n^2 + (-4c_2c_3 + 4c_4)e_n^3 + (4c_2^4 - 3c_2^2c_3 - 6c_2c_4 + 5c_5)e_n^4 + \big(22c_2^3c_3 - 8c_2^5 - 4c_2(3c_3^2 + 2c_5) + 6c_6\big)e_n^5 + O(e_n^6). \tag{13}$$

and

$$k = \frac{f(y_n)}{f(x_n)} = c_2e_n + (-3c_2^2 + 2c_3)e_n^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e_n^3 + (-20c_2^4 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4 + 4c_5)e_n^4 + \big(48c_2^5 - 118c_2^3c_3 + 51c_2^2c_4 - 22c_3c_4 + c_2(55c_3^2 - 18c_5) + 5c_6\big)e_n^5 + O(e_n^6). \tag{14}$$

Since it is noteworthy from the above mention equations namely, (13) and (14), $u = 1 + p$ and $v = O(e_n)$. Then, from these equations, we get the remainder $p = u - 1$ and v are infinitesimal with the same order of e_n. Therefore, we can expand weight function $L_f(u,\ v)$ in the neighborhood of $(1,\ 0)$ by Taylor series expansion up to fourth-order terms as follow

$$L_f(u,\ v) = L_{00} + L_{10}p + L_{01}v + \frac{1}{2!}\big(L_{20}p^2 + 2L_{11}pv + L_{02}v^2\big) + \frac{1}{3!}\big(L_{30}p^3 + 3L_{21}p^2v + 3L_{12}pv^2 + L_{03}v^3\big) + \frac{1}{4!}\big(L_{40}p^4 + 4L_{31}p^3v + 6L_{22}p^2v^2 + 4L_{13}pv^3 + L_{04}v^4\big) + O(e_n^5). \tag{15}$$

Using (8)-(15), in scheme (5), we get

$$e_{n+1} = (1 - L_{00})e_n - c_2(L_{01} + 2L_{10})e_n^2 + \sum_{l=3}^{6} M_le_n^l, \tag{16}$$

where $M_l = M_l(c_2, c_3, \ldots, c_6)L_{ij}$, for $0 \le i,\ j \le 4$.

We will get at least third-order convergence if we insert the following values of L_{00} and L_{01} in (16),

$$L_{00} = 1, \quad L_{01} = -2L_{10}. \tag{17}$$

Further, using (17) into $M_3 = 0$, we find two independent relation as follows:

$$(L_{02} + 4(2L_{10} + L_{11} + L_{20})) = 0, \quad (1 + 4L_{10}) = 0 \tag{18}$$

Solving the equations defined in (18) for L_{11} and L_{10}, we have

$$L_{11} = -\frac{1}{4}(-2 + L_{02} + 4L_{20}), \quad L_{10} = -\frac{1}{4} \tag{19}$$

By inserting (17) and (19) into $M_4 = 0$, we obtain

$$(3 + L_{02} - 4L_{20}) = 0, \quad \left(1 + L_{02} - \frac{L_{03}}{6} - L_{12} - 4L_{20} - 2L_{21} - \frac{4L_{30}}{3}\right) = 0. \tag{20}$$

Further, solve the above equation namely, (20) for L_{02} and L_{03}, we get

$$L_{02} = 4L_{20} - 3, \quad L_{03} = -(12 + 6L_{12} + 12L_{21} + 8L_{30}). \tag{21}$$

By substituting (17), (19) and (21) into $M_5 = 0$, we obtain

$$\begin{cases} (3 - 8L_{20}) = 0, \\ (-5 + 2L_{12} + 16L_{20} + 8L_{21} + 8L_{30}) = 0, \\ \big[L_{04} + 8(3 + 6L_{12} + L_{13} + 24L_{20} + 24L_{21} + 3L_{22} \\ \quad + 24L_{30} + 4L_{31} + 2L_{40})\big] = 0. \end{cases} \tag{22}$$

Solving the above equation for $L_{20},\ L_{12}$ and L_{04}, we further yield

$$\begin{cases} L_{20} = \frac{3}{8}, \\ L_{12} = -\frac{1}{2}(1 + 8L_{21} + 8L_{30}), \\ L_{04} = -8(9 + L_{13} + 3L_{22} + 4L_{31} + 2L_{40}). \end{cases} \tag{23}$$

We can easily obtain the following error equation, by using (17), (19), (21) and (23) into (16)

$$e_{n+1} = -\frac{c_2}{12}\Big[4(9 + 2L_{13} + 24L_{21} + 12L_{22} + 48L_{30} + 24L_{31} + 16L_{40})c_2^4 - 2(L_{13} + 24L_{21} + 6L_{22} + 48L_{30} + 12L_{31} + 8L_{40} - 12)c_2^2c_3 + 3(3 + 2L_{21} + 4L_{30})c_3^2 - 12c_2c_4\Big]e_n^6 + O(e_n^7). \tag{24}$$

This reveals that our proposed scheme (5) has sixth-order of convergence while using only four functional evaluations (viz $f(x_n)$ $f'(x_n)$ $f(y_n)$ and $f'(y_n)$) per full iteration. Hence, this completes the proof of above Theorem III. ■

IV. Special Cases

In this section, we discuss some interesting special cases of weight function $L_f(u,\ v)$, which are defined as follows:
(1) For $L_{21}=0,\ L_{30}=0,\ L_{13}=0,\ \ L_{22}=0$ and $L_{31}=0$ in (15), we get the following weight function

$$\begin{aligned}L_f(u,\ v) =&1-\frac{p}{4}+\frac{3p^2}{16}+\frac{L_{40}}{24}p^4+\frac{4+4p+3p^2}{8}v\\&-\frac{3+7p+3p^2}{4}v^2-\frac{2L_{40}}{3}v^4,\end{aligned}\tag{25}$$

where L_{40} is a free variable and for the sake of simplicity $p=u-1$. With the help of this disposable parameter, we can easily obtain various different types of weight functions as well as two-point sixth-order methods.
(2) For $L_{21}=0,\ L_{40}=0,\ L_{13}=0,\ \ L_{22}=0$ and $L_{31}=0$ in (15), we obtain

$$\begin{aligned}L_f(u,\ v) =&1-\frac{p}{4}+\frac{3p^2}{16}+\frac{L_{30}}{6}p^3+\frac{p+1}{2}v\\&-\frac{(1+8L_{30})p+3}{4}v^2+\frac{16L_{30}-9(1+p)}{6}v^3,\end{aligned}\tag{26}$$

where L_{30} is a free variable.
(3) We consider the following weight function, which satisfies all the conditions defined in theorem III

$$\begin{aligned}L_f(u,\ v) =&\frac{1}{16u^2}\bigg(22u^3v-3u^4v+u(6+8v)-u^2(19v\\&+12v^2+36v^3-11)-1\bigg).\end{aligned}\tag{27}$$

V. Numerical Experiments

In this section, we will check the validity and efficiency of theoretical results. Therefore, we apply our methods for ($L_{40}=0$ & $L_{40}=\frac{9}{16}$) in scheme (25) and for ($L_{30}=0$ & $L_{30}=\frac{3}{16}$) in scheme (26) are denoted by $OM1$, $OM2$, $OM3$, and $OM4$, respectively, to solve some nonlinear equations given in Table I. We compare them with a three-point sixth-order method proposed by Sharma and Ghua [3], method (3) for $(a=2)$ denoted by (SG). In addition, we also compare our schemes with a method namely, method (5) for (c=0, d=1, r=0) called (WM) which is given by Wang and Liu in [5]. Finally, we will also compare them with a two-point family of sixth-order methods that is very recently proposed by Guem et al. [11], between them we will choose their best expression (3.8 and 3.12) denoted by ($KM1$ and $KM2$), respectively. For better comparisons of our proposed methods, we have given three comparison tables in each example: one is corresponding to absolute error in Table II, the second one is with respect to number of iterations in Table III and third one is regarding their computational order of convergence in Table IV respectively. All computations have been performed using the programming package $Mathematica$ 9 with multiple precision arithmetic. The meaning of $a(-b)$ is $a\times10^{-b}$ in Table II. We use $\epsilon=10^{-34}$ as a tolerance error. The following stopping criteria are used for computer programs:
$(i)|x_{n+1}-x_n|<\epsilon$ and $(ii)|f(x_{n+1})|<\epsilon$.

VI. Attractor Basins in the Complex Plane

Here, we further investigate the comparison of the attained simple root finders in the complex plane using basins of attraction. It is known that the corresponding fractal of an iterative root-finding method is a boundary set in the complex plane, which is characterized by the iterative method applied to a fixed polynomial $p(z)\in\mathbb{C}$, see e.g. [12], [13]. The aim herein is to use basin of attraction as another way for comparing the iteration algorithms.

From the dynamical point of view, we consider a rectangle $D=[-3,\ 3]\times[-3,\ 3]\in\mathbb{C}$ with a 400×400 grid, and we assign a color to each point $z_0\in D$ according to the simple root at which the corresponding iterative method starting from z_0 converges, and we mark the point as black if the method does not converge. In this section, we consider the stopping criterion for convergence to be less than 10^{-4} wherein the maximum number of full cycles for each method is considered to be 200. In this way, we distinguish the attraction basins by their colors for different methods.

Test problem 1. Let $p_1(z)=(z^4+1)$, having simple zeros $\{-0.707107-0.707107i,\ \ -0.707107+0.707107i,\ \ 0.707107-0.707107i, 0.707107+0.707107i\}$. It is straight forward to see from Figs. 1 and 2 that our methods namely, $OM1$ and $OM3$ contain lesser number of non convergent points in comparison to the methods, namely SG, WM $KM1$ and $KM2$. Further our methods they have larger and brighter basin of attraction in comparison to the mentioned methods.

Test Problem 2. Let $p_2(z)=(z^3+z+1)$, having simple zeros $\{-0.682328, 0.341164-1.16154i, 0.341164+1.16154i\}$. We can easily note from Figs. 3 and 4 that our proposed methods namely, $OM1$ and $OM3$ have less number of non convergent points and have larger and brighter basin of attraction in comparison to methods WM, $KM1$.

VII. Conclusions

In this paper, we contributed further to the development of the theory of iteration processes and proposed several second-derivative free two-point sixth-order family of super-Halley type methods based on weight function and arithmetic means of second-order derivative. By assigning particular values to the disposable parameters namely, (L_{40}) and (L_{30}) in schemes (25) and (26), respectively, we can obtain several new weight functions as well as new two-point sixth-order methods. By considering different type of weight functions, we further yield so many new two-point sixth-order methods. Finally, on accounts of the results obtained, it can be concluded that our proposed methods are highly efficient as compared to the existing methods. From their basins of attraction, it has been observed that the proposed methods have better stability and robustness as compared to the other sixth-order methods available in the literature.

TABLE I
TEST PROBLEMS

$f(x)$	$Root(r)$
$f_1(x) = xe^{x^2} - \sin x^2 + 3\cos x + 5$	$-1.2015761120922993992523943629089366$
$f_2(x) = e^{-x} + \sin x$	$3.1830630119333635919391869956363946$
$f_3(x) = (x-2)^2 - \log x - 33x$	$36.989473582944669865344734734912736$
$f_4(x) = \cos x - x$	$0.7390851332151606416553120876738734$0
$f_5(x) = \tan^{-1}(x^2 - x)$	$1.0000000000000000000000000000000000$
$f_6(x) = e^{-x^2+x+2} - 1$	$2.0000000000000000000000000000000000$

TABLE II
COMPARISON OF DIFFERENT SIXTH-ORDER METHODS WITH THE SAME TOTAL NUMBER OF FUNCTIONAL EVALUATIONS (TNFE=12)

$f(x)$	I.G.	SG	WM	$KM1$	$KM2$	$OM1$	$OM2$	$OM3$	$OM4$
1.	−2	9.2−18	1.7−9	2.6−9	4.8−9	4.0−17	9.0−24	4.1−22	6.4−21
	−1.6	1.0−65	1.2−43	8.9−43	7.1−42	1.7−63	2.3−80	1.2−75	1.2−97
2.	2.5	1.0−131	2.0−130	8.9−146	8.9−146	1.3−138	1.3−136	4.9−157	6.9−144
	4.0	3.8−79	2.5−61	8.7−82	1.3−86	2.7−109	5.4−91	1.9−101	4.2−108
3.	34	3.3−188	8.3−172	1.5−169	1.5−169	4.9−203	6.8−249	4.9−203	4.8−247
	39	1.8−207	2.8−232	6.6−230	6.6−230	3.7−256	2.6−315	3.8−256	3.6−312
4.	1.5	1.7−150	3.0−151	3.6−164	1.3−165	3.7−167	9.2−165	6.7−175	2.5−171
	1.6	1.3−144	6.5−146	1.4−162	4.7−174	1.4−156	6.2−155	9.0−162	9.2−161
5	0.85	2.0−115	1.7−93	3.9−95	1.1−95	5.1−137	8.8−140	7.8−135	5.0−143
	1.6	2.6−74	3.2−72	1.7−107	2.5−63	7.6−77	2.9−81	9.2−112	1.7−117
6	1.0	2.7−49	5.1−57	3.8−63	5.7−85	2.8−43	3.1−41	7.2−49	3.0−42
	1.2	3.4−49	1.2−47	1.1−50	9.5−53	1.3−83	1.4−94	1.6−73	1.2−105
	2.25	5.8−66	7.5−24	1.2−30	8.9−29	1.4−79	1.1−66	1.0−91	1.4−67

TABLE III
COMPARISON OF DIFFERENT SIXTH-ORDER METHODS WITH RESPECT TO NUMBER OF ITERATIONS

$f(x)$	I.G.	SG	WM	$KM1$	$KM2$	$OM1$	$OM2$	$OM3$	$OM4$
1.	−2	5	5	5	5	5	5	5	5
	−1.6	4	4	4	4	4	4	4	4
2.	2.5	4	4	4	4	4	4	4	4
	4.0	4	4	4	4	4	4	4	4
3.	34	4	4	4	4	4	4	4	4
	39	4	4	4	4	4	4	4	4
4.	1.5	4	4	4	4	4	4	4	4
	1.6	4	4	4	4	4	4	4	4
5	0.85	4	4	4	4	4	4	4	4
	1.6	4	4	4	4	4	4	4	4
6	1.0	4	4	4	4	4	4	4	4
	1.2	4	4	4	4	4	4	4	4
	2.25	4	4	5	5	4	4	4	4

TABLE IV
COMPUTATIONAL ORDER OF CONVERGENCE OF DIFFERENT SIXTH-ORDER METHODS

$f(x)$	I.G.	SG	WM	$KM1$	$KM2$	$OM1$	$OM2$	$OM3$	$OM4$
1.	−2	6.000	5.997	5.997	5.996	6.000	6.000	6.000	6.000
	−1.6	5.999	6.000	5.989	5.988	5.999	6.000	5.996	6.000
2.	2.5	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000
	4.0	5.990	5.931	5.958	5.995	6.001	6.003	5.999	6.001
3.	34	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000
	39	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000
4.	1.5	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000
	1.6	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000
5	0.85	6.000	5.999	5.999	5.999	6.000	6.000	6.000	6.000
	1.6	6.000	6.003	6.000	5.993	6.000	6.000	6.000	6.000
6	1.0	5.989	5.993	5.996	6.001	6.019	6.061	6.010	6.052
	1.2	5.988	5.984	5.988	5.990	5.999	5.999	6.001	6.000
	2.25	5.997	6.000	6.000	6.000	5.999	6.008	5.999	6.007

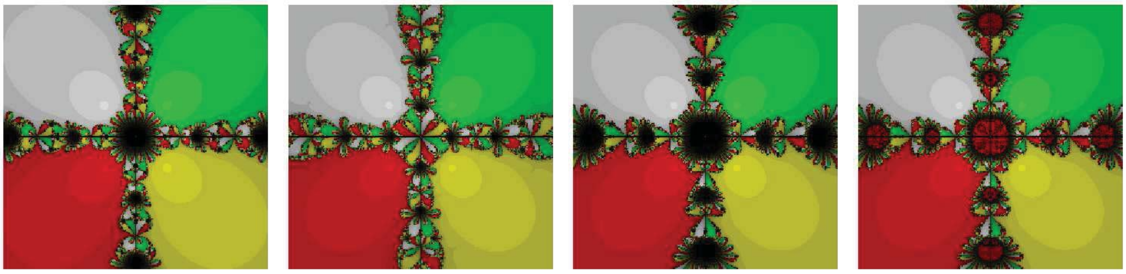

Fig. 1 The methods SG, WM, $KM1$ and $KM2$, respectively for test problem 1

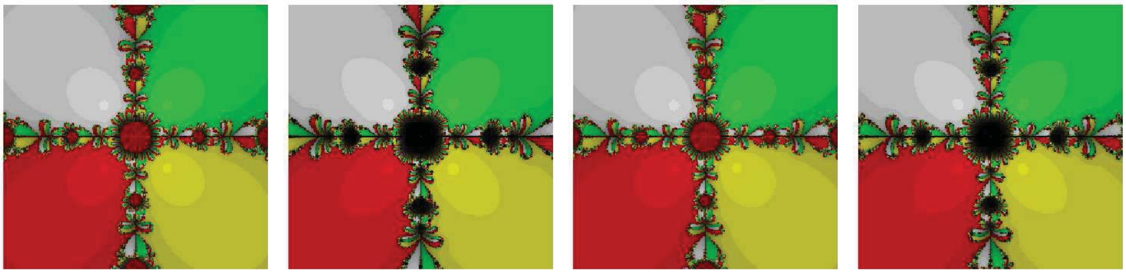

Fig. 2 Our methods $OM1$, $OM2$, $OM3$ and $OM4$, respectively for test problem 1

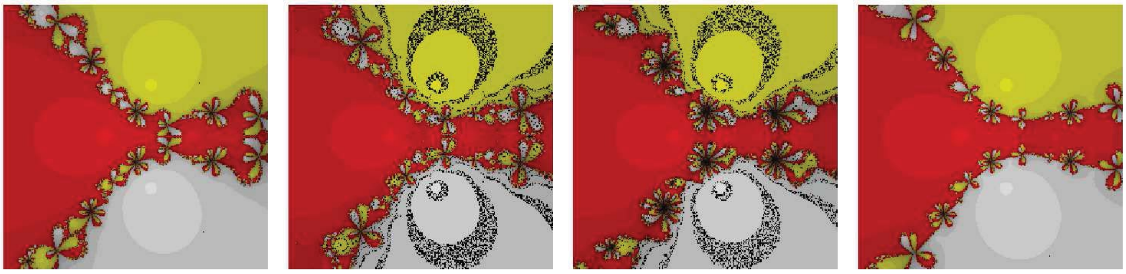

Fig. 3 The methods SG, WM, $KM1$ and $KM2$, respectively for test problem 2

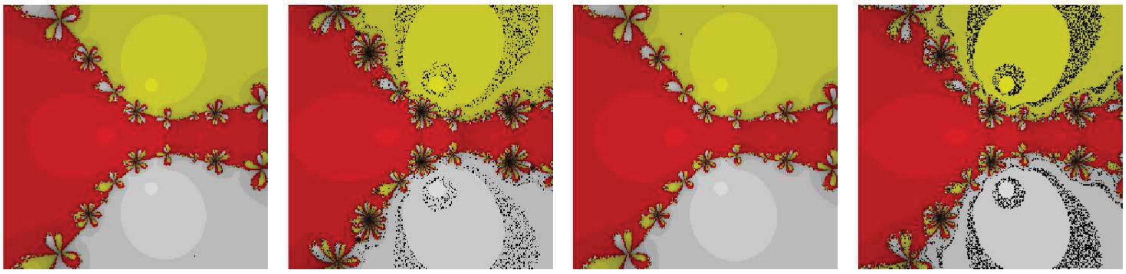

Fig. 4 Our methods $OM1$, $OM2$, $OM3$ and $OM4$, respectively for test problem 2

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