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Single-Crystal Kerfless 2D Array Transducer for Volumetric Medical Imaging: Theoretical Study

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***Abstract*—**The aim of this work is to present a theoretical analysis of a 2D ultrasound transducer comprised of crossed arrays of metal strips placed on both sides of thin piezoelectric layer (a). Such a structure is capable of electronic beam-steering of generated wavebeam both in elevation and azimuth. In this paper a semi-analytical model of the considered transducer is developed. It is based on generalization of the well-known BIS-expansion method. Specifically, applying the electrostatic approximation, the electric field components on the surface of the layer are expanded into fast converging series of double periodic spatial harmonics with corresponding amplitudes represented by the properly chosen Legendre polynomials. The problem is reduced to numerical solving of certain system of linear equations for unknown expansion coefficients.

***Keywords*—**Beamforming, transducer array, BIS-expansion.

I. Introduction

RECENTLY 3D ultrasound has become one of the most promising imaging modality for clinical diagnosis providing orientations and slices not available with traditional 2D ultrasound. It leads to more efficient and faster examination, diagnostic and monitoring of therapeutic procedures free of potential inaccuracies related to subjective operator dependent treatment in contrast to 2D case, where the sequence of 2D images is transformed by the operator in his mind to obtain the impression of 3-D viewing. Therefore, there has been a high demand for 2D transducer arrays for medical ultrasonography. The systems using fully sampled 2D transducer arrays capable of electronic scanning can provide 3D images of a volume of interest in real-time. But such 2D probes are not easily accessible because their fabrication is still a challenging task. Specifically, to handle typical 2D array consisting of 256×256 elemental transducers, as many as 65536 signal channels are required which introduces considerable technological difficulties, especially at high frequencies (above 3-5 MHz). To overcome these problems recently in the literature conceptually different 2-D transducer array architecture has been considered. Specifically, a 2D structure of an edge-connected, crossed-electrode array was considered in [1], [2]. The electrode patterns arranged on both sides of piezoelectric layer is illustrated in Fig. 1. The proposed single-crystal transducer is capable of control $N \times N$ elements with $2N$ signal channels, which in case of 256×256 matrix transducers allows the number of connecting wiring to be reduced from above mentioned 65536 to only 512. This simplifies design and fabrication of such 2D transducer probes and external electronics considerably.

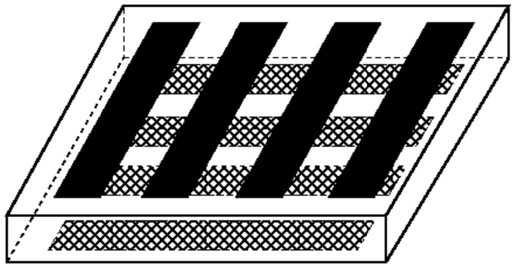

Fig. 1 Single-crystal kerfless 2D array transducer

However no profound theoretical analysis of the considered crossed-electrode array has been carried out so far. In [1] the problem was superficially approached in the signal processing framework without thorough research. The 2D array, shown in Fig. 1 is capable of electronic beam steering of generated wave both in elevation and azimuth and can, potentially, be well suited to be used in medical 3D ultrasound imaging systems. The wave beam control is achieved by addressable driving of the 2D matrix transducer through proper voltage supply of electrodes on the opposite faces of the piezoelectric layer. Assuming time-harmonic signals , with frequencies differing by Ω, applied to i^{th} upper and j^{th} bottom strips located on the opposite surfaces of piezoelectric layer the electric field and the resulting inducted normal stress will be localized near the $(i,j)^{th}$ cell (especially for such piezoelectric materials like the PZT-5H [2] PVDF [3] or PMN-PT [4]). This yields the tool for selective (addressable) excitation of given cells: only this cell will vibrate which resides between strips driven by the signals and with frequencies differing by Ω. Thus, applying different amplitudes and phase-shifts to , or frequencies difference Ω, one obtains quite flexible tool for controlling vibrations of cells and the induced stress distribution over entire matrix transducer. The shape of vibrations require detailed analysis of the electric field distribution in the layer. The main objective of this work is to present a theoretical analysis of a 2D ultrasound transducer comprised of crossed arrays of metal electrodes deposited on both sides of thin piezoelectric layer as illustrated in Fig. 1. Specifically, a semi-analytical method of analysis of the considered transducer is proposed. It is based on generalization of the so-called 'BIS-expansion' method [5]

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which was earlier exploited with great success in the theory of interdigital transducers of surface acoustic waves [6], theory of elastic wave scattering by cracks and certain advanced electrostatic problems [7].

II. Governing Equations

Consider a piezoelectric plate with its surfaces defined with $x_3 = \pm h$ in Cartesian coordinate system x_i, i=1,2,3. Without loss of generality, the further analysis concerns the PZT-5H piezoelectric layer. On the upper surface of the plate an infinite x_1-periodic system of conducting strips infinitely long in the x_2 -direction is deposited. Similarly, on the bottom surface of the layer an infinite x_2-periodic system of infinitely long strips in the x_1-direction is deposited as illustrated in Fig. 1. Without loss of generality the same strip period is assumed on both surfaces denoted here as Λ and the strip width is w.

The linear response of a dynamically excited piezoelectric plate is governed by the following set of differential equations: the stress equation of motion:

$$T_{ij,i} = \rho \ddot{u}_j, \tag{1}$$

and the electrostatic charge conservation equation:

$$D_{i,i} = 0, \tag{2}$$

where the summation conversion for repeated indices is employed and the index preceded by a comma means differentiation with respect to spatial variable, whereas the dot above the variable means time differentiation. In (1) and (2) T, u, D denote the stress tensor, mechanical displacement and electric displacement vectors, respectively, ρ – mass density of the media.

The corresponding constitutive relations for piezoelectric media are as follows:

$$\begin{aligned} T_{ij} &= c^E_{ijkl} S_{kl} - e_{kij} E_k \\ D_i &= e_{ijk} S_{jk} + \epsilon^S_{ij} E_j \end{aligned}, \tag{3}$$

where c^E_{ijkl} – the elastic constants measured at constant electric field E; e_{ijk}- the piezoelectric constants; ϵ^S_{ij} – the dielectric constants measured under constant strain S.

The strain-displacement and electric-field-potential relations are as follows:

$$\begin{aligned} S_{ij} &= \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \\ E_i &= -\varphi_{,i} \end{aligned}, \tag{4}$$

where φ is the electrostatic potential. Substituting (4) into (3) and the resulting equations into (1) and (2) one obtains:

$$\begin{aligned} c^E_{ijkl} u_{k,il} + e_{kij} \varphi_{,ik} &= \rho \ddot{u}_j \\ e_{kij} u_{j,ik} - \epsilon^S_{ij} \varphi_{,ij} &= 0 \end{aligned}. \tag{5}$$

The time-harmonic wave field being a function of $e^{-j\omega t}$ where ω is angular frequency is assumed. The time derivative in (5) is therefore: $\rho \ddot{u}_j = -\rho \omega^2 u_j$. The solution for filed components in the piezoelectric layer under crossed periodic arrays of conducting strips is sought in the form of the following Bloch series [8]:

$$\begin{gathered} u_i = \sum_{n,m} U^{(i)}_{nm} \Psi_{nm} e^{-jk_{nm} x_3}, \phi = \sum_{n,m} \varphi_{nm} \Psi_{nm} e^{-jk_{nm} x_3} \\ \Psi_{nm} \equiv e^{-j(r_n x_1 + s_m x_2)} \end{gathered}, \tag{6}$$

where Ψ_{nm} are the planar spatial harmonics defined in the plane $x_3 = 0$ parallel to strip systems. In (6):

$$r_n = r + nK, \quad s_m = s + mK, \quad \xi_{nm} = \sqrt{r_n^2 + s_m^2}, \tag{7}$$

and $K = 2\pi/\Lambda$ is a wavenumber of the strip arrays; $r \in (0, K)$ and $s \in (0, K)$ are arbitrary spatial spectrum variables reduced to one Brillouin zone for the uniqueness of representation. Similarly, ξ_{nm} is a wavenumber defined in the plane $x_3 = 0$ along the axis rotated by angle $\vartheta_{nm} = \tan^{-1}(s_m/r_n)$ with to x_1, x_2 axes. In (6) $u^{(i)}_{nm}, \phi_{nm}$ the mode amplitudes for mechanical displacement components and electrostatic potential.

Since the spatial harmonics are orthogonal, substituting (6) into (5) and taking into account that:

$$\begin{gathered} \frac{\partial}{\partial x_1} = -jr_n, \quad \frac{\partial}{\partial x_2} = -js_m, \quad \frac{\partial}{\partial x_3} = -jk_{nm} \\ \frac{\partial^2}{\partial x_1^2} = -r_n^2, \quad \frac{\partial^2}{\partial x_2^2} = -s_m^2, \quad \frac{\partial^2}{\partial x_3^2} = -k_{nm}^2 \\ \frac{\partial^2}{\partial x_1 \partial x_2} = -r_n s_m, \frac{\partial^2}{\partial x_1 \partial x_3} = -r_n k_{nm}, \frac{\partial^2}{\partial x_2 \partial x_3} = -s_m k_{nm} \end{gathered}$$

the corresponding Christoffel equations for each spatial harmonic with (n,m) indices can be deduced:

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix} \begin{pmatrix} u^1_{nm} \\ u^2_{nm} \\ u^3_{nm} \\ \varphi_{nm} \end{pmatrix} = 0, \tag{8}$$

where the coefficients of the matrix are given below for particular case of PZT-5H material:

$$
\begin{gathered}
R_{11}=c_{11}^{E}r_n^2+c_{66}^{E}s_m^2+c_{44}^{E}k_{nm}^2-\rho\omega^2\\
R_{12}=(c_{12}^{E}+c_{66}^{E})r_ns_m\\
R_{13}=(c_{13}^{E}+c_{44}^{E})r_nk_{nm}\\
R_{14}=(e_{15}+e_{31})r_nk_{nm}\\
R_{22}=c_{66}^{E}r_n^2+c_{22}^{E}s_m^2+c_{44}^{E}k_{nm}^2-\rho\omega^2\\
R_{23}=(c_{13}^{E}+c_{44}^{E})s_mk_{nm}\\
R_{24}=(e_{15}+e_{31})sk_{nm}\\
R_{33}=c_{44}^{E}(r_n^2+s_m^2)+c_{33}^{E}k_{nm}^2-\rho\omega^2\\
R_{34}=e_{15}(r_n^2+s_m^2)+e_{33}k_{nm}^2\\
R_{44}=-\epsilon_{11}^{S}(r_n^2+s_m^2)-\epsilon_{33}^{S}k_{nm}^2\\
R_{ij}=R_{ij},\ \ i,j=1\ldots4
\end{gathered}
\tag{9}
$$

where the material constants are represented in contracted form to shorten notation. It should be noted, that PZT-5H piezoelectric material is characterized by the following nonzero material constants:

-elastic stiffness $c_{11}^{E},c_{12}^{E},c_{13}^{E},c_{22}^{E},c_{33}^{E},c_{44}^{E},c_{66}^{E},\ c_{23}^{E}=c_{13}^{E}$ and $c_{55}^{E}=c_{44}^{E};(c_{IJ}^{E}=c_{JI}^{E})$

- piezoelectric constants e_{15},e_{31},e_{33} and $e_{32}=e_{31},e_{24}=e_{15}$

- dielectric constants $\epsilon_{11}^{S},\epsilon_{33}^{S}$ and $\epsilon_{22}^{S}=\epsilon_{11}^{S}$.

The system of equations (8) has nontrivial solution only if its determinant is equal to zero:

$$\left|R_{ij}(n,m)\right|=0. \tag{10}$$

In general case determinant of the matrix R_{ij} can be expanded yielding an 8th-degree polynomial with respect to k_{nm}:

$$\left|R_{ij}(n,m)\right|=a_8k_{nm}^8+\cdots+a_1k_{nm}+a_0=0,$$

and the polynomial coefficients are the functions of the wave-number components and material constants. Solving the above equations yields eight roots in general case $k_{nm}=k_{nmr},r=1\ldots8$. In the considered case of lossless material and the orientation of the layer with respect to the x_3-axis (poling direction along the x_3-axis) the resulting polynomial is even function of k_{nm} ($a_j=0,j=1,3,5,7$) with real coefficients. Therefore there are four pairs of roots which are either real or complex conjugate (in general case) representing propagating or evanescent modes in opposite directions in the layer, that is the roots satisfy the relation:

$$k_{nm(2r)}=-k_{nm(2r-1)},r=1\ldots4$$

Denoting the roots $k_{nm}=k_{nmr},r=1\ldots8,$ and inserting them in (8) the modes amplitudes $U_{nmr}^{(i)}$ and φ_{nmr} can be obtained which correspond to the partial waves. The general solution for $(n,m)^{th}$ spatial harmonic that satisfies the wave equations is a superposition of 8 partial waves:

$$
\begin{gathered}
u^{(i)}(n,m)=\sum_{r=1}^{8}(C_{nmr}U_{nmr}^{(i)})\Psi_{nm}e^{-jk_{nmr}x_3}\\
\varphi(n,m)=\sum_{r=1}^{8}(C_{nmr}\varphi_{nmr})\Psi_{nm}e^{-jk_{nmr}x_3}
\end{gathered}
\tag{11}
$$

Accounting for that in the considered case the roots come in pairs, (11) can be redefined as follows:

$$
\begin{gathered}
u^{(i)}(n,m)=\sum_{r=1}^{4}(C_{nmr}^{\pm}U_{nmr}^{(i)\pm})\Psi_{nm}e^{\pm jk_{nmr}x_3}\\
\varphi(n,m)=\sum_{r=1}^{4}(C_{nmr}^{\pm}\varphi_{nmr}^{\pm})\Psi_{nm}e^{\pm jk_{nmr}x_3}
\end{gathered}
\tag{12}
$$

In (12) summation is over r and the partial waves with ± signs are summed up for each value of r=1…4. It should be noted that in (12) there are 8 unknown constants $C_{nmr}^{\pm}$ for each $(n,m)^{th}$ spatial harmonic which have to be determined from mechanical and electric boundary conditions considered below.

III. Mechanical Boundary Conditions

In the case of thin strips deposited on the surface of piezoelectric layer, mechanical boundary conditions may be assumed uniform. Specifically, the traction-free condition on the surfaces of the layer have to be satisfied:

$$T_{i3}=0,\ \ x_3=\pm h, \tag{13}$$

where $2h$ is the layer thickness and T_{i3} is the normal stress components defined by the constitutive equations (3). Substituting (12) into the first equation in (3) and using Eq. (13) and orthogonally of spatial harmonics the system of linear equations for unknown coefficients $C_{nmr}^{\pm}$ can be deduced:

$$\hat{T}_{kl}\hat{C}_l=0,\qquad k=1\ldots6,l=1\ldots8, \tag{14}$$

where the vector of unknown coefficients $\hat{C}_l$ is defined for $(n,m)^{th}$ spatial harmonic as follows:

$$(\hat{C})\equiv(C_{nmr}^{+}\ \ C_{nmr}^{-})^T,\qquad r=1\ldots4, \tag{15}$$

and the elements of matrix $\hat{T}_{kl}$ are given by the following expressions:

$$\hat{T}_{1r} = \left(c_{44}^{E}\left(r_n U_{nmr}^{(3)+} - k_{nmr} U_{nmr}^{(1)+}\right) + e_{15} r_n \varphi_{nmr}^{+}\right) e^{jk_{nmr}h}$$
$$\hat{T}_{1,r+4} = \left(c_{44}^{E}\left(r_n U_{nmr}^{(3)-} + k_{nmr} U_{nmr}^{(1)-}\right) + e_{15} r_n \varphi_{nmr}^{-}\right) e^{-jk_{nmr}h}$$
$$\hat{T}_{2r} = \left(c_{44}^{E}\left(s_m U_{nmr}^{(3)+} - k_{nmr} U_{nmr}^{(2)+}\right) + e_{15} s_m \varphi_{nmr}^{+}\right) e^{jk_{nmr}h}$$
$$\hat{T}_{2,r+4} = \left(c_{44}^{E}\left(s_m U_{nmr}^{(3)-} + k_{nmr} U_{nmr}^{(2)-}\right) + e_{15} s_m \varphi_{nmr}^{-}\right) e^{-jk_{nmr}h}$$
$$\hat{T}_{3r} = \left(c_{13}^{E}\left(r_n U_{nmr}^{(1)+} + s_m U_{nmr}^{(2)+}\right) - k_{nmr}\left(c_{33}^{E} U_{nmr}^{(3)+} + e_{33} \varphi_{nmr}^{+}\right)\right) e^{jk_{nmr}h}$$
$$\hat{T}_{3,r+4} = \left(c_{13}^{E}\left(r_n U_{nmr}^{(1)-} + s_m U_{nmr}^{(2)-}\right) + k_{nmr}\left(c_{33}^{E} U_{nmr}^{(3)-} + e_{33} \varphi_{nmr}^{-}\right)\right) e^{-jk_{nmr}h}$$
$$\hat{T}_{4r} = \left(c_{44}^{E}\left(r_n U_{nmr}^{(3)+} - k_{nmr} U_{nmr}^{(1)+}\right) + e_{15} r_n \varphi_{nmr}^{+}\right) e^{-jk_{nmr}h}$$
$$\hat{T}_{4,r+4} = \left(c_{44}^{E}\left(r_n U_{nmr}^{(3)-} + k_{nmr} U_{nmr}^{(1)-}\right) + e_{15} r_n \varphi_{nmr}^{-}\right) e^{jk_{nmr}h}$$
$$\hat{T}_{5r} = \left(c_{44}^{E}\left(s_m U_{nmr}^{(3)+} - k_{nmr} U_{nmr}^{(2)+}\right) + e_{15} s_m \varphi_{nmr}^{+}\right) e^{-jk_{nmr}h}$$
$$\hat{T}_{5,r+4} = \left(c_{44}^{E}\left(s_m U_{nmr}^{(3)-} + k_{nmr} U_{nmr}^{(2)-}\right) + e_{15} s_m \varphi_{nmr}^{-}\right) e^{jk_{nmr}h}$$
$$\hat{T}_{6r} = \left(c_{13}^{E}\left(r_n U_{nmr}^{(1)+} + s_m U_{nmr}^{(2)+}\right) - k_{nmr}\left(c_{33}^{E} U_{nmr}^{(3)+} + e_{33} \varphi_{nmr}^{+}\right)\right) e^{-jk_{nmr}h}$$
$$\hat{T}_{6,r+4} = \left(c_{13}^{E}\left(r_n U_{nmr}^{(1)-} + s_m U_{nmr}^{(2)-}\right) + k_{nmr}\left(c_{33}^{E} U_{nmr}^{(3)-} + e_{33} \varphi_{nmr}^{-}\right)\right) e^{+jk_{nmr}h}$$

The mechanical boundary conditions (13) yields 6 equations for 8 unknown constants. The lacking equations can be obtained from the electric boundary conditions considered below.

IV. Electric Boundary Conditions

For the considered 2D array transducer electric boundary conditions are determined by the periodic strips deposited on the opposite surfaces of the layer. To find the solution of the problem the method of analysis based on the BIS-expansion known from the theory of surface acoustic waves interdigital transducers [5] or electrostatics of crossed periodic systems of conducting strips [8] can be adopted with great success.

The boundary conditions on the upper (superscript +) and bottom (superscript –) surfaces of the layer imposed on the electric field components are:

$$\begin{aligned} &E_1^{\pm} = 0,\ \ E_2^{\pm} = 0, \ \ \text{on strips}, \\ &\Delta D_3^{\pm} = 0, \quad \text{between strips}. \end{aligned} \tag{16}$$

Stating that tangential electric field vanishes on strips and between strips jump of normal electric induction equals to zero. The electrostatic potential on the surfaces of the layer can be expanded into the series of surface spatial harmonics Ψ_{nm} as follows:

$$\varphi^{\pm}(x_1, x_2) = \sum_{n,m} \hat{\varphi}_{nm}^{\pm} \Psi_{nm}, \tag{17}$$

where the surface mode amplitudes result directly from (12):

$$\begin{aligned} \hat{\varphi}_{nm}^{+} &= \sum_{r} \left(C_{nmr}^{+} \varphi_{nmr}^{+} e^{+jk_{nmr}h} + C_{nmr}^{-} \varphi_{nmr}^{-} e^{-jk_{nmr}h}\right) \\ \hat{\varphi}_{nm}^{-} &= \sum_{r} \left(C_{nmr}^{+} \varphi_{nmr}^{+} e^{-jk_{nmr}h} + C_{nmr}^{-} \varphi_{nmr}^{-} e^{+jk_{nmr}h}\right) \end{aligned} \tag{18}$$

Since electrostatic potential in the media outside the piezoelectric layer should obey the Laplace equation and should be continuous across the boundaries $x_3 = \pm h$, it can be expressed in the following form satisfying Floquet's theorem:

$$\begin{aligned} \varphi^{a} &= \sum_{n,m} \hat{\varphi}_{nm}^{+} \Psi_{nm} e^{-|\xi_{nm}|(x_3 - h)}, x_3 > h \\ \varphi^{a} &= \sum_{n,m} \hat{\varphi}_{nm}^{-} \Psi_{nm} e^{|\xi_{nm}|(x_3 + h)}, x_3 < -h \end{aligned}, \tag{19}$$

where the wavenumber ξ_{nm} is defined in (7).

The jump discontinuity induction of normal electric induction $\Delta D_3^{\pm}$ is defined as follows:

$$\begin{aligned} \Delta D_3^{+} &= D_3(x_3 = h + 0) - D_3(x_3 = h - 0) = D_3^{a} - D_3^{+} \\ \Delta D_3^{-} &= D_3(x_3 = -h - 0) - D_3(x_3 = -h + 0) = D_3^{b} - D_3^{-} \end{aligned}, \tag{20}$$

where the normal electric induction in the layer $D_3^{\pm}$ can be obtain from the constitutive equations (3). Specifically, for the $(n,m)^{\text{th}}$ spatial harmonics amplitudes one obtains:

$$\begin{aligned} D_{3nm}^{+} &= -j \sum_{r} C_{nmr}^{\pm}\left(e_{31}\left(r_n U_{nmr}^{(1)\pm} + s_m U_{nmr}^{(2)\pm}\right) \mp k_{nmr}\left(e_{33} U_{nmr}^{(3)\pm} - \epsilon_{33}^{S} \varphi_{nmr}^{\pm}\right)\right) e^{\pm jk_{nmr}h}, \\ D_{3nm}^{-} &= -j \sum_{r} C_{nmr}^{\pm}\left(e_{31}\left(r_n U_{nmr}^{(1)\pm} + s_m U_{nmr}^{(2)\pm}\right) \mp k_{nmr}\left(e_{33} U_{nmr}^{(3)\pm} - \epsilon_{33}^{S} \varphi_{nmr}^{\pm}\right)\right) e^{\mp jk_{nmr}h}. \end{aligned} \tag{21}$$

In the media outside the layer the amplitudes of the $(n,m)^{\text{th}}$ spatial harmonics of the normal electric induction are:

$$D_{3nm}^{a,b} = -\epsilon_M \left(\varphi_{,3}^{a,b}\right)_{nm} = \pm \epsilon_M |\xi_{nm}| \hat{\varphi}_{nm}^{\pm}, \tag{22}$$

where the definition of electrostatic potential outside the layer (19) was used. Substituting (21) and (22) into (20), the jump of the normal electric induction (amplitudes of the $(n,m)^{\text{th}}$ harmonics) across the boundaries can be obtained immediately:

$$\begin{aligned} \Delta D_{3nm}^{+} &= -j \sum_{r} C_{nmr}^{\pm}\left(e_{31}\left(r_n U_{nmr}^{(1)\pm} + s_m U_{nmr}^{(2)\pm}\right) \mp k_{nmr} e_{33} U_{nmr}^{(3)\pm} \pm \left(\epsilon_{33}^{S} k_{nmr} \mp j\epsilon_M |\xi_{nm}|\right) \varphi_{nmr}^{\pm}\right) e^{\pm jk_{nmr}h}, \\ \Delta D_{3nm}^{-} &= -j \sum_{r} C_{nmr}^{\pm}\left(e_{31}\left(r_n U_{nmr}^{(1)\pm} + s_m U_{nmr}^{(2)\pm}\right) \mp k_{nmr} e_{33} U_{nmr}^{(3)\pm} \pm \left(\epsilon_{33}^{S} k_{nmr} \pm j\epsilon_M |\xi_{nm}|\right) \varphi_{nmr}^{\pm}\right) e^{\mp jk_{nmr}h}. \end{aligned} \tag{24}$$

The components of tangential electric field in the plane of strips $E_1^{\pm}, E_2^{\pm}$ results directly from the electrostatic potential definition, (17). Specifically, for amplitudes of the $(n,m)^{\text{th}}$ harmonics one obtains:

$$E_{1nm}^{\pm} = -jr_n\hat{\varphi}_{nm}^{\pm}, \qquad E_{2nm}^{\pm} = -js_m\hat{\varphi}_{nm}^{\pm} \tag{25}$$

For further analysis it is convenient to consider the tangential field component in the planes of strips $x_3 = \pm h$:

$$E_{\xi}^{\pm}(n,m) = -j\xi_{nm}\hat{\varphi}_{nm}^{\pm}e^{-j\xi_{nm}\tau}, \tag{26}$$

where

$$x_1 = \tau\cos\vartheta_{nm},\ x_2 = \tau\sin\vartheta_{nm},\ \xi_{nm}^2 = r_n^2 + s_m^2,$$

and the angle $\vartheta_{nm} = \tan^{-1}(s_m/r_n)$. Taking into account the above definitions of τ, ξ_{nm} and ϑ_{nm} one can easily check by inspection that:

$$e^{-j\xi_{nm}\tau} = \Psi_{nm}.$$

Furthermore, to apply the BIS-expansion approximation the relation between the tangential electric field and the jump of normal electric induction in the planes of strips should be deduced first. For this purpose it is convenient to express the unknown coefficient $C_{nmr}^{\pm}$ in terms of 2 unknown constants to be determined from the electric boundary conditions. Specifically, by rearrangement of terms the system of equations (14) can be rewritten in the form:

$$\tilde{T}_{km}\tilde{C}_m = B_k, \quad k,m = 1\ldots6, \tag{27}$$

where $\tilde{T}_{km} = \hat{T}_{k,m+2}$ and $\tilde{C}_m = \hat{C}_{m+2}$, that is:

$$(\tilde{C}) \equiv \left(C_{nm,r+1}^{+}\ \ C_{nm,r+1}^{-}\right)^T, \qquad r = 1\ldots3. \tag{28}$$

The elements of vector B are defined as follows:

$$B_k = \bar{T}_{ki}A_i,\ \ k = 1\ldots6, i = 1{,}2\,, \tag{29}$$

where $\bar{T}_{ki} = \tilde{T}_{ki}$. In (29) the new vector comprised of 2 unknown constants is defined for each spatial harmonic (n,m):

$$(A) \equiv (C_{nm1}^{+}\ \ C_{nm1}^{-})^T. \tag{30}$$

Hence, the unknown coefficients $\tilde{C}$ can be expressed in terms of only 2 unknown constants A_i defined in (30):

$$(\tilde{C}) = (\tilde{T}^{-1}\bar{T})(A) = (\tilde{a})(A). \tag{31}$$

In what follows, it is convenient to rewrite (31) in slightly different form:

$$(\hat{C}) = (\hat{a})(A), \tag{32}$$

which allows all the unknown constants $C_{nmr}^{\pm}$ to be found in terms of $C_{nm1}^{\pm}$. It should be noted that:

$$\begin{gathered}\hat{a}_{11} = 1, \hat{a}_{12} = 0;\ \hat{a}_{21} = 0, \hat{a}_{22} = 1,\\ \hat{a}_{k+2,i} = \tilde{a}_{ki}, \qquad k = 1\ldots6, \qquad i = 1{,}2\end{gathered} \tag{33}$$

and

$$(\tilde{a}) = (\tilde{T}^{-1}\bar{T}). \tag{34}$$

Therefore only 2 coefficients ($C_{nm1}^{\pm}$) defined in (30) remain unknown which have to be determined from the electric boundary conditions. Using the unknown variables defined in (30) (the subscripts *n,m* show that the variables are defined for each spatial harmonic), the mode amplitudes of the electric potential in the planes $x_3 = \pm h$ defined in (18) can be rewritten as follows:

$$\begin{pmatrix}\hat{\varphi}_{nm}^{+}\\ \hat{\varphi}_{nm}^{-}\end{pmatrix} = \begin{pmatrix}L_{nm}^{1+} & L_{nm}^{2+}\\ L_{nm}^{1-} & L_{nm}^{2-}\end{pmatrix}\begin{pmatrix}A_{nm}^{1}\\ A_{nm}^{2}\end{pmatrix}. \tag{35}$$

In the above equation the matrix form is used to shorten notation, where the elements of matrix *L* are:

$$L_{nm}^{i\pm} = \sum_r\left(\hat{a}_{nmr}^{i+}\varphi_{nmr}^{+}e^{\pm jk_{nmr}h} + \hat{a}_{nmr}^{i-}\varphi_{nmr}^{-}e^{\mp jk_{nmr}h}\right), \tag{36}$$

where the elements of matrix $\hat{a}$ are defined in (33) for each spatial harmonic, which is denoted by the *n,m* subscripts. Similarly, the mode amplitudes of the $(n,m)^{\text{th}}$ spatial harmonic of the tangential electric field $E_{\xi nm}^{\pm}$ can be written as follows:

$$\begin{pmatrix}E_{\xi nm}^{+}\\ E_{\xi nm}^{-}\end{pmatrix} = -j\xi_{nm}\begin{pmatrix}L_{nm}^{1+} & L_{nm}^{2+}\\ L_{nm}^{1-} & L_{nm}^{2-}\end{pmatrix}\begin{pmatrix}A_{nm}^{1}\\ A_{nm}^{2}\end{pmatrix}. \tag{37}$$

Finally, for the jump of normal electric induction defined in (24) can be rewritten in the following form:

$$\begin{pmatrix}\Delta D_{3nm}^{+}\\ \Delta D_{3nm}^{-}\end{pmatrix} = -j\begin{pmatrix}M_{nm}^{1+} & M_{nm}^{2+}\\ M_{nm}^{1-} & M_{nm}^{2-}\end{pmatrix}\begin{pmatrix}A_{nm}^{1}\\ A_{nm}^{2}\end{pmatrix}, \tag{38}$$

where the matrix *M* is defined below:

$$\begin{gathered}M_{nm}^{i\pm} = \sum_r \hat{a}_{nmr}^{i+}\left(e_{31}\left(r_nU_{nmr}^{(1)+} + s_mU_{nmr}^{(2)+}\right) - \right.\\ \left. k_{nmr}e_{33}U_{nmr}^{(3)+} + (\epsilon_{33}^{S}k_{nmr} \mp j\epsilon_M|\xi_{nm}|)\varphi_{nmr}^{\pm}\right)e^{\pm jk_{nmr}h}\\ +\sum_r \hat{a}_{nmr}^{i-}\left(e_{31}\left(r_nU_{nmr}^{(1)-} + s_mU_{nmr}^{(2)-}\right) + \right.\\ \left. k_{nmr}e_{33}U_{nmr}^{(3)-} - (\epsilon_{33}^{S}k_{nmr} \pm j\epsilon_M|\xi_{nm}|)\varphi_{nmr}^{\pm}\right)e^{\mp jk_{nmr}h}\end{gathered} \tag{39}$$

From (37) and (38) the relationship between the tangential electric field and normal electric induction on the surfaces of piezoelectric layer can be obtained immediately:

$$(E_{\xi}) = \xi_{nm}(G)\,(\Delta D_3), \tag{40}$$

where $(E_\xi) = (E^+_{\xi nm}\, E^-_{\xi nm})^T$, $(\Delta D_3) = (\Delta D^+_{3nm}\, \Delta D^-_{3nm})$ and

$$(G) = (L)(M)^{-1}. \quad (41)$$

In (41) the elements of corresponding matrices L and M are defined in (36) and (39), respectively.

V. BIS-Expansion Approximation

To satisfy the electric boundary conditions (16) and find the unknowns A_i, =1,2 in (30) the BIS-expansion method is used [8]. For this purpose, the electric field and normal electric induction on the plane $x_3 = h$ can be expanded into the series:

$$E_1^+ = j\sum_{l,n,m} \alpha_l^m\, S_{n-l} P_{n-l}(\Delta)\Psi_{nm},$$
$$E_{2,1}^+ = j\sum_{l,n,m} \tilde{\alpha}_l^m\, S_{n-l} P_{n-l}(\Delta)\Psi_{nm}, \quad (42)$$
$$\Delta D_3^+ = \sum_{l,n,m} \beta_l^m\, P_{n-l}(\Delta)\Psi_{nm},$$

where $\Delta = \cos(\pi w/\Lambda)$, $P_k(\cdot)$ - is the Legendre polynomials; $S_\nu = 0$ for $\nu < 0$ and $S_\nu = 1$ otherwise; w - is the strip's width. The above expansions yield the electric field satisfying boundary conditions and the edge condition. Specifically, E_1 and D_3 components are inverse square-root singular functions at the strip edges. As regard the E_2 components, it is not singular but its spatial derivative with respect to x_1 has inverse square-root singularity at the strip edges as well. Therefore, in (42) the corresponding series expansion of $E_{2,1}^+$ is defined. In a similar manner on the plane $x_3 = -h$ electric field components can be expanded in the series as follows:

$$E_2^- = j\sum_{l',n,m} \gamma_{l'}^n\, S_{m-l'} P_{m-l'}(\Delta)\Psi_{nm},$$
$$E_{1,2}^- = j\sum_{l',n,m} \tilde{\gamma}_{l'}^n\, S_{m-l'} P_{m-l'}(\Delta)\Psi_{nm}, \quad (43)$$
$$\Delta D_3^- = \sum_{l',n,m} \eta_{l'}^n\, P_{m-l'}(\Delta)\Psi_{nm}.$$

Using (25) it can be shown that the following relationships between coefficients exist:

$$\tilde{\alpha}_l^m = -js_m\alpha_l^m,\ \ \tilde{\gamma}_{l'}^n = -jr_n\gamma_{l'}^n, \quad (44)$$

The unknown coefficients α_l^m, β_l^m and $\gamma_{l'}^n$, $\eta_{l'}^n$ can be evaluated using the relationship between tangential electric field and normal electric induction on the upper and bottom boundaries given by (40). It should be noted, that the $(n,m)^{\text{th}}$ tangential electric field can be expressed as follows:

$$E^+_{\xi nm} = \frac{\xi_{nm}}{r_n} E^+_{1nm}\,, E^-_{\xi nm} = \frac{\xi_{nm}}{s_m} E^-_{2nm}\,, \quad (45)$$

where the (42) and (42) were used. Consequently, the expansions for tangential field components can be written as follows:

$$E_\xi^+ = j\sum_{l,n,m} \frac{\xi_{nm}}{r_n}\alpha_l^m\, S_{n-l}P_{n-l}(\Delta)\Psi_{nm},$$
$$E_\xi^- = j\sum_{l',n,m} \frac{\xi_{nm}}{s_m}\gamma_{l'}^n\, S_{m-l'}P_{m-l'}(\Delta)\Psi_{nm}. \quad (46)$$

To proceed further, let's consider the matrix G in (41) for large indexes (n,m) corresponding to imaginary k_{nmr} and the spatial harmonics being well-localized at a given surface of piezoelectric layer which represent the evanescent modes. In this case the following approximation of the matrix G for sufficiently large indices (n,m) holds [8]:

$$\begin{pmatrix} E^+_{\xi nm} \\ E^-_{\xi nm}\end{pmatrix} \approx jS_{nm}\begin{pmatrix} \dfrac{1}{\epsilon^+_\infty} & 0 \\ 0 & -\dfrac{1}{\epsilon^-_\infty}\end{pmatrix}\begin{pmatrix}\Delta D^+_{3nm} \\ \Delta D^-_{3nm}\end{pmatrix}, \quad (47)$$

where $S_{nm} \equiv S_nS_m$; ϵ^+_∞ and ϵ^-_∞ can be obtained from (41) and (36), (39) upon substituting the approximation $k_{nmr} \approx -j|p_{nm}|$ which can be applied for sufficiently large (n,m):

$$\epsilon^\pm_\infty = (\varepsilon_{33} + \epsilon_M) - \alpha_3^\pm e_{33} \pm je_{31}\big(\alpha_1^\pm S_n + \alpha_2^\pm S_m\big). \quad (48)$$

In (48) the corresponding constants $\alpha_i^\pm$ results from the asymptotic analysis of the corresponding relations between partial wave amplitudes $U^{(i)\pm}_{nmr}$ and $\varphi^\pm_{nmr}$:

$$\alpha_i^\pm = \lim_{n,m\to\infty}\left(\sum_r U^{(i)\pm}_{nmr}\Big/\sum_r \varphi^\pm_{nmr}\right) \quad (49)$$

and can be only obtained numerically for specified material constants of the piezoelectric layer.

For certain large indices $n>N$ and $m>M$ for which (47) holds, the following relationships between α_l^m, β_l^m and $\gamma_{l'}^n$, $\eta_{l'}^n$ results immediately:

$$\beta_l^m = \frac{\xi_{nm}}{r_n}\epsilon^+_\infty S_{n-l}\alpha_l^m, \qquad \eta_{l'}^n = -\frac{\xi_{nm}}{s_m}\epsilon^-_\infty S_{m-l'}\gamma_{l'}^n\,, \quad (50)$$

which is a generalization of the so-called a BIS-expansion approximation [8]. Taking into account (50) and substituting the spatial harmonics with indices ($n<N$, $m<M$) from (42), (43) and (46) into (40) the following conditions for unknown coefficients α_l^m, $-N \le l \le N$ and $\gamma_{l'}^n$, $-M \le l' \le M$ for any n, m separately within these domains:

$$\begin{gathered}\alpha_l^m\left[j\frac{\xi_{nm}}{|\xi_{nm}|}-G_{11}\epsilon_\infty^+\right]S_{n-l}P_{n-l}(\Delta)+\\ \gamma_{l'}^nG_{12}\left(\frac{r_n}{s_m}\right)\epsilon_\infty^-S_{m-l'}P_{m-l'}(\Delta)=0,\\ -\alpha_l^mG_{21}\left(\frac{s_m}{r_n}\right)\epsilon_\infty^+S_{n-l}P_{n-l}(\Delta)+\\ \gamma_{l'}^n\left[j\frac{\xi_{nm}}{|\xi_{nm}|}+G_{22}\epsilon_\infty^-\right]S_{m-l'}P_{m-l'}(\Delta)=0.\end{gathered}\tag{51}$$

Applying (*n, m*) outside the chosen domains results in a trivial solution for the additionally included unknowns [8]. Strips potentials can now be evaluated by integration of the tangential electric field, their integration being performed for each spatial harmonic separately. For a strip placed at x_1=0 (on the upper surface) x_2=0 (at the bottom surface) they are:

$$\begin{aligned}V^+&=-\int E_1^+dx_1=\alpha_l^m\frac{\sum_n S_{n-l}P_{n-l}(\Delta)}{jr_n}=V(r)\\ V^-&=-\int E_2^-dx_2=\gamma_{l'}^n\frac{\sum_m S_{m-l'}P_{m-l'}(\Delta)}{js_m}=0\end{aligned},\tag{52}$$

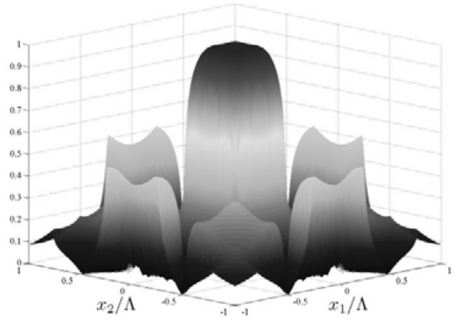
(the bottom strips assumed grounded). The corresponding summation over *n* and *m* in (52) can be evaluated explicitly using the Dougall identity [9]:

$$\begin{gathered}(-1)^l\alpha_l^mP_{-l-r/K}(-\Delta)=j\delta_{m0}\frac{K}{\pi}e^{jrl\Lambda}\sin\pi r/K\\ (-1)^{l'}\gamma_{l'}^nP_{-l'-s/K}(-\Delta)=0\end{gathered}\tag{53}$$

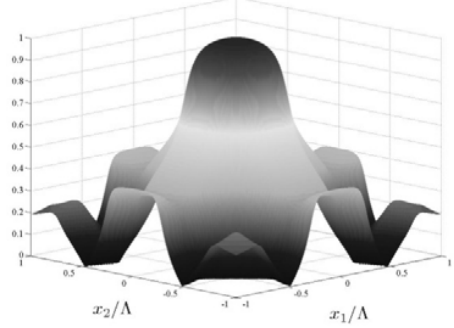
for any (*n,m*) accounted for in (51). In (53) where δ_{ij} - is the Kronecker delta. Therefore, (51), (53) yields the closed system of linear equations for unknown expansion coefficients $\alpha_l^m,\gamma_{l'}^n$. Specifically, there are (2*N*+1)(2*M*+1) equations altogether and the same number of unknowns $\alpha_l^m,\gamma_{l'}^n$, where $-N\le l\le N$ and $-M\le l'\le M$ is assumed. Once the system of equations for $\alpha_l^m,\gamma_{l'}^n$, is solved, the unknown constants $A_{nm}^i, i=1{,}2$ defined in (30) can be obtained from (37) accounting for (46), which explicitly yields for (*n,m*)^th spatial harmonics:

$$\begin{pmatrix}A_{nm}^1\\A_{nm}^2\end{pmatrix}=\begin{pmatrix}L_{nm}^{1+}&L_{nm}^{2+}\\L_{nm}^{1-}&L_{nm}^{2-}\end{pmatrix}^{-1}\begin{pmatrix}-\frac{1}{r_n}\sum_l\alpha_l^m S_{n-l}P_{n-l}(\Delta)\\-\frac{1}{s}\sum_{l'}\gamma_{l'}^n S_{m-l'}P_{m-l'}(\Delta)\end{pmatrix}\tag{54}$$

Solving (54) yields the constants $A_{nm}^i, i=1{,}2$ and therefore, all unknown constants $C_{nmr}^{\pm}$ for each spatial harmonics from (32). This completes the solution of the problem in general case, when arbitrary potentials of the strips are specified.



(a)



(b)

Fig. 2 Magnitude of the normal electric field in the $\Lambda\times\Lambda$ domain of the layer at the plane $x_3=0$ for different thickness (a) $h/\Lambda=0.1$ and (b) $h/\Lambda=0.5$; the strip's width $w/\Lambda=0.5$

Some numerical examples are shown in Fig. 2. Specifically, normal component of the electrostatic field in the layer middle plane $x_3=0$ is shown in relative scale for fixed strip's width $w/\Lambda=0.5$ and different thickness of the layer.

The example corresponds to the case when a single cell of the transducer is excited by uniform voltage applied to one upper strip and all bottom strips grounded. As is seen from Fig. 2, the electric field distribution at the middle plane of the layer significantly departs from uniform and spans somewhat outside the cell covered by the supplied strips.

VI. Conclusion

Summarizing, the extension of the BIS-expansion method, originally developed for electrostatic analysis of 1-D periodic planar systems of strips, was presented for modeling of 2-D periodic structure comprised of crossed arrays of strips placed on the opposite surfaces of the dielectric piezoelectric layer. It is an example of novel 2-D array transducer architecture with potential application in 3-D ultrasound imaging. Without loss of generality the same strip width an period on the opposite surfaces was assumed. The method can be generalized for different strip period and width straightforwardly. Numerical examples show the resulting nonuniform electrostatic field induced in the area of a single matrix cell excited by a uniform voltage applied to one upper strip and all bottom strips grounded.

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