

Simple Procedure for Probability Calculation of Tensile Crack Occurring in Rigid Pavement – Case Study

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Abstract—Formation of tensile cracks in concrete slabs of rigid pavement can be (among others) the initiation point of the other, more serious failures which can ultimately lead to complete degradation of the concrete slab and thus the whole pavement. Two measures can be used for reliability assessment of this phenomenon - the probability of failure and/or the reliability index. Different methods can be used for their calculation. The simple ones are called moment methods and simulation techniques. Two methods - FOSM Method and Simple Random Sampling Method - are verified and their comparison is performed. The influence of information about the probability distribution and the statistical parameters of input variables as well as of the limit state function on the calculated reliability index and failure probability are studied in three points on the lower surface of concrete slabs of the older type of rigid pavement formerly used in the Czech Republic.

Keywords—Failure, pavement, probability, reliability index, simulation, tensile crack.

I. INTRODUCTION

ALTHOUGH the calculation and measurement of deflections on the pavement has the most important role in today's engineering practice, the calculation of stresses seems to be in fact much more important. Principal stress represents the extreme normal stress in a given point of the structure and thus it is the crucial characteristic which would be used in dimensioning process. If the principal stress exceeds the material tensile strength, a local tensile crack is created. Formation of tensile cracks in concrete slabs of rigid pavement generally does not mean any problems or hazards in the pavement behavior. However, it can be (among others) the initiation point of the other, more serious failures which can ultimately lead to complete degradation of the concrete slab and the whole pavement. Thus the knowledge of probability of tensile crack occurring may be a significant indicator of serviceability of rigid pavements.

II. PROCEDURES FOR PROBABILITY CALCULATION

The tensile cracks occurring due to exceeding the concrete tensile strength in given points of concrete slab can be considered as a structural failure. To evaluate the failure probability p_f it is necessary to define the relevant basic input

variables X_i and the functional relationship (the limit state function) $g(X)$ among them [1], [2]. Mathematically, this function is given

$$Z = g(X_1, X_2, \dots, X_K) \quad (1)$$

Failure occurs when $Z < 0$. Therefore, the failure probability p_f is given

$$p_f = P(Z < 0) = \int f_z(Z) dz \quad (2)$$

where f_z is the joint probability density function and the integration is performed over the region in which $g < 0$.

Different methods can be used for calculation of p_f [3]. The simple ones are called moment methods. They are based on expanding the limit state function $g(X)$ in a Taylor series about the mean values μ_i of input variables X_i

$$Z = \mu_z + \frac{1}{1!} \sum_n \left(\frac{\partial^2 z}{\partial X_n^2} \xi_n^2 \right) + \frac{1}{2!} \left[\sum_n \left(\frac{\partial^2 z}{\partial X_n^2} \xi_n^2 + 2 \sum_{i < n} \left(\frac{\partial^2 z}{\partial X_n \partial X_i} \xi_n \xi_i \right) \right) \right] + \dots \quad (3)$$

where

$$\mu_z = g(\mu_1, \mu_2, \dots, \mu_K) \quad (4)$$

and

$$\xi_i = X_i - \mu_i \quad (5)$$

Alternative risk measure called reliability index β [1] can be defined

$$\beta = \mu_z / \sigma_z \quad (6)$$

where μ_z is the mean value and σ_z is the standard deviation of limit state function. Then the failure probability is given

$$p_f = \Phi(-\beta) \quad (7)$$

where Φ is the cumulative distribution function.

The moment methods allow determining estimates of statistical parameters of limit state function based only on the knowledge of statistical parameters of input variables and the explicit knowledge of limit state function. The probability distribution of input variables is not taken into account.

Depending on how much terms of the Taylor series and

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what statistical parameters of input variables are taken into account, there is a number of different variants of moment methods. It is obvious that the higher is the number of terms or taking into account the higher order statistical parameters, the more complicated is the method. In practice, the most frequently used variants are based on the mean value and the standard deviation and on the first order or the second order terms only. The variant taking into account the first order terms is called First Order Second Moment (FOSM) Method, the second one taking into account the second order terms is called Second Order Second Moment (SOSM) Method.

However, the above formulation has some important shortcomings. The limit state function is linearized at the mean values. When the function is complex and nonlinear, significant error may be introduced. Moreover, the moment methods completely ignore the information on the probability distribution of input variables taking into account only the mean value and the standard deviation. As a result, the moment methods give correct results only when the input variables are normally distributed and the limit state function is linear.

Another group of methods useful for failure probability calculation are simulation techniques [3], [4]. They are based on random samples of input variables that are used for calculation of limit state function $g(X)$. After performing N simulation runs the statistical set of data $\langle Z_1, Z_2, \dots, Z_N \rangle$ is obtained and statistically assessed. It can be used for failure probability calculation, because estimates of statistical parameters of Z are evaluated.

There are a number of different variants of simulation techniques depending on the method of drawing random samples. Classical method is called Monte Carlo Method (Simple Random Sampling) [4]. Improved strategies of drawing samples are used in Latin Hypercube Sampling [5] and Updated Latin Hypercube Sampling [6]. Especially for calculation of failure probability the special class of simulation techniques called Advanced Simulation Techniques was developed [3].

III. PARAMETRIC STUDY

Two different methods - FOSM Method (Method 1) and Simple Random Sampling Method (Method 2) - are used for calculation of reliability index and probability of tensile cracks occurring in concrete slabs of rigid pavement. The limit state function (1) is very simple in this case

$$Z = g(R, S) = R - S \quad (8)$$

where R is the resistance (the concrete tensile strength) and S is the action (the tensile principal stress) in a given point of the concrete slab of rigid pavement.

To evaluate the failure probability p_f , in our case the probability that the tensile principal stress in a given point is greater than the concrete tensile strength, (2) becomes

$$p_f = P(Z < 0) = P(R - S < 0) \quad (9)$$

The FOSM Method is based on the theoretical assumption of normal (Gauss) probability distribution of resistance R and action S as well as the normal probability distribution of the limit state function. If the limit state function is linear (as (8)), the basic statistical parameters of the limit state function are

$$\mu_z = \mu_R - \mu_S \quad (10)$$

$$\sigma_z = (\sigma_R^2 + \sigma_S^2)^{1/2} \quad (11)$$

where μ_z is the mean value and σ_z is the standard deviation of limit state function, μ_R is the mean value and σ_R is the standard deviation of resistance R (the concrete tensile strength), and μ_S is the mean value and σ_S is the standard deviation of action S (the tensile principal stress).

The reliability index β can be calculated as

$$\beta = \mu_z / \sigma_z \quad (12)$$

and the probability of tensile cracks occurring is

$$p_f = \Phi_{NORM}(-\beta) \quad (13)$$

where Φ_{NORM} is the normal cumulative distribution function.

Simple Random Sampling Method is the classical simulation technique. The samples of input variables that are used for calculation of limit state function (8) are obtained randomly (or pseudo-randomly). As a main advantage of Simple Random Sampling Method it can be emphasized that it allows to obtain the failure probability even when the input variables or the limit state function show a non-normal probability distribution. The disadvantage, however, is the fact that to achieve sufficiently accurate and reliable results we need thousands of simulations. In the presented study 10 000 simulations is used, i.e. the limit state function (8) is calculated 10 000 times with a randomly generated realizations of variables R and S which satisfy the requirements of the given probability distributions. Number of simulations is chosen empirically to be completely satisfactory for the problem with a very simple limit state function. From 10 000 simulations obtained data set $\{z\}$ is statistically evaluated, and in particular we obtain the following estimates of statistical parameters – mean value, standard deviation, skewness, and best suitable probability distribution.

TABLE I
STATISTICAL PARAMETERS AND SUITABLE PROBABILITY DISTRIBUTION
FUNCTION OF TENSILE PRINCIPAL STRESSES [MPa]

Point	Mean Value	Standard Deviation	Skewness	PDF
A	0,7045	0,8385	0,76	W
B	1,8767	1,1313	1,13	W
C	1,2062	1,0671	2,31	P3

The reliability index β is calculated using (12) and the probability of tensile cracks occurring is

$$p_f = \Phi_{BEST}(-\beta) \quad (14)$$

where Φ_{BEST} is the best suitable cumulative distribution function.

The concrete tensile strength (resistance R) and the tensile principal stresses (action S) occurring in the concrete slabs are random variables described by the appropriate probability distribution and the relevant statistical parameters. While for the concrete tensile strength the available statistical information can be considered as adequate due to the relatively extensive experimental research, for the tensile principal stresses in the concrete slabs of rigid pavements it is just the opposite. There is no credible information as a whole and if some experimental research is conducted, it is always a big question, what should be measured and what is measured in reality. The only way how to obtain the appropriate statistical information about tensile principal stresses seems to be the reliability analyses [7], [8].

IV. RESULTS

The aim of parametric study is to verify both proposed methods for failure probability calculation and to perform their comparison. The influence of information about the probability distribution and the statistical parameters of input variables as well as of the limit state function on the calculated reliability index β and failure probability p_f are studied in three points (point A, B, C) on the lower surface of concrete slabs of the older type of rigid pavement formerly used in the Czech Republic [7], [8]. This type of pavement is made from plain concrete, no dowels are used, and joints are made during laying of concrete. The structure is loaded by the self-weight of concrete slabs, by the thermal loading due to the temperature difference between the upper and lower surface of slabs, and by the external load of intensity 50 kN at a distance of 0,25 m from the transverse edge of the loaded slab. Contrary to the original pavement design, in the reliability analysis [8] the base layer is supposed to be made from a recycled material instead of a natural one. The statistical parameters and suitable probability distribution function (three-parametric Weibull (W) and three-parametric Pearson III (P3)) of tensile principal stresses in these points are taken from [8] and are shown in Table I.

TABLE II
STATISTICAL PARAMETERS OF CONCRETE TENSILE STRENGTH FOR DIFFERENT QUALITY LEVELS [MPa]

Concrete Quality	Mean Value	Standard Deviation	Skewness
Consistent	3,75	0,5625	0
Average	3,35	0,6700	-0,5
Low	2,95	0,7965	-0,85

Three different concrete quality levels (consistent, average and low) are considered for the concrete tensile strength. The basic statistical parameters of concrete tensile strength for each supposed quality level are shown in Table II.

A. Standard Deviation of Concrete Tensile Strength

Because the number of simulations for Simple Random Sampling Method is chosen empirically, it is advisable to verify if this number provides sufficiently correct results. The reliability index and the failure probability are calculated using both methods. It is supposed that the concrete tensile strength is described by a normal probability distribution, the mean value is 3,75 MPa and there are three different values of the standard deviation - 0,5625, 0,7500 and 1,0125 MPa. Three values of the standard deviation are chosen to examine whether the increased random variability affects the accuracy of the obtained results. Also they can serve as a first rough estimate how the concrete quality influences the failure probability. It is well known that the concrete of lower quality has, among others, higher random variability of strength.

It is also assumed that the tensile principal stresses in points A, B, C are described by a normal probability distribution. Their mean values and standard deviations are shown in the respective columns of Table I. The reliability index and the failure probability are determined by assuming that the limit state function is described by a normal probability distribution too. It can be theoretically proved, that under above simplified assumptions (linear limit state function and normal probability distribution of input variables as well as limit state function) results obtained by FOSM Method are completely accurate. Thus they can be used to verify results obtained by Simple Random Sampling Method.

Illustrative results of calculated reliability index and failure probability in points A, B, C on the lower surface of concrete slabs are shown in Table III. The results show that both methods give quite comparable results regardless of the value of standard deviation of concrete tensile strength. Therefore the chosen number of simulations for Simple Random Sampling Method can be considered to be sufficient. As result, for all calculations in the future studies Simple Random Sampling Method will be considered to be the crucial method to determine reliability index and failure probability. The reason is simple - at the moment of leaving the assumption about the normal probability distribution for the input variables and limit state function it generally gives more accurate results than FOSM Method.

TABLE III
RELIABILITY INDEX AND FAILURE PROBABILITY FOR DIFFERENT CONCRETE QUALITY LEVELS

Point	Consistent Quality				Average Quality				Low Quality			
	Method 1		Method 2		Method 1		Method 2		Method 1		Method 2	
	β	P_f	β	P_f	β	P_f	β	P_f	β	P_f	β	P_f
A	3,016	0,001280	2,996	0,001369	2,707	0,003393	2,685	0,003623	2,317	0,010262	2,297	0,010817
B	1,483	0,069075	1,488	0,068432	1,380	0,083772	1,384	0,083206	1,234	0,108624	1,236	0,108267
C	2,109	0,017481	2,098	0,017955	1,950	0,025569	1,940	0,026187	1,729	0,041878	1,721	0,042662

Also it is shown that the lower is the standard deviation, i.e. the better is the concrete quality level, the lower is the probability of tensile cracks occurring.

B. Mean Value of Concrete Tensile Strength

In the previous section the influence of different random variability of concrete tensile strength (i.e. different concrete quality level) on the failure probability is shown. However, the concrete quality has influence not only on the standard deviation but also on the mean value of concrete tensile strength. In the following study we still maintain all the assumptions introduced in the previous section about the normal probability distribution of input variables and limit state function as well as about statistical parameters of tensile principal stresses. For each supposed concrete quality level (consistent, average and low) we now take the mean value and standard deviation from Table II.

Illustrative results of calculated reliability index and failure probability obtained from Simple Random Sampling Method are shown in Table IV. When results from both sections are compared, significant influence of the mean value on the failure probability could be seen compared to influence of the standard deviation of concrete tensile strength. The higher is the mean value, i.e. the better is the concrete quality, the lower is the probability of tensile cracks occurring.

C. Non-normal Distribution of Input Variables

Next refinement in the failure probability calculation is incorporating the real probability distribution (generally three-parametric non-normal) and thus non-zero skewness for tensile principal stresses as well as for the concrete tensile strength.

TABLE IV
RELIABILITY INDEX AND FAILURE PROBABILITY FOR DIFFERENT CONCRETE QUALITY LEVELS

Point	Consistent Quality		Average Quality		Low Quality	
	β	P_f	β	P_f	β	P_f
A	2,996	0,001369	2,446	0,007225	1,925	0,027084
B	1,488	0,068432	1,123	0,130797	0,775	0,219130
C	2,098	0,017955	1,692	0,045279	1,303	0,096352

We leave the assumption of normal probability distribution of these variables and replace it with the assumption of three-parametric lognormal probability distribution for concrete tensile strength and the best probability distribution for tensile principal stresses obtained from reliability analysis [8], see Table I. For each supposed concrete quality level (consistent, average and low) the basic statistical parameters of concrete tensile strength are shown in Table II. However, we continue to keep the assumption of normal probability distribution of limit state function.

Illustrative results of calculated reliability index and failure probability obtained from Simple Random Sampling Method are shown Table V. When we compare obtained results with results in previous section, there is no greater difference visible. If any differences do exist, they rather result from partial inaccuracy of methods used for calculation of the

failure probability. Therefore the conclusion could be drawn that non-zero skewness and non-normal probability distribution of input variables do not affect the failure probability. This may be partially justified for the reliability index because this quantity depends only on the mean value and the standard deviation of limit state function and it is well known that these statistics particularly in the case of a very simple limit state function are not too sensitive on the skewness of input variables. Thus, the reliability index is not usually too dependent on the skewness of input variables.

Quite different it is in the case of the failure probability calculation, where the influence of skewness is generally accepted and usually also significantly presented in results. The explanation is relatively simple. We still assume the normal probability distribution of limit state function and thus the influence of non-zero skewness of input variables is more or less neglected, because only the mean value and standard deviation of limit state function are taken into account. In other words, the more accurate results of the failure probability calculation can be achieved only if we respect the generally non-zero skewness and the real probability distribution of limit state function.

TABLE V
RELIABILITY INDEX AND FAILURE PROBABILITY FOR DIFFERENT CONCRETE QUALITY LEVELS

Point	Consistent Quality		Average Quality		Low Quality	
	β	P_f	β	P_f	β	P_f
A	2,996	0,001369	2,448	0,007183	1,928	0,026920
B	1,488	0,068432	1,127	0,129903	0,778	0,218288
C	2,098	0,017955	1,687	0,045836	1,298	0,097205

TABLE VI
RELIABILITY INDEX AND FAILURE PROBABILITY FOR DIFFERENT CONCRETE QUALITY LEVELS

Point	Average Quality				Low Quality			
	Method 1		Method 2		Method 1		Method 2	
	β	P_f	β	P_f	β	P_f	β	P_f
A	2,465	0,006854	2,448	0,015596	1,942	0,026091	1,928	0,040274
B	1,121	0,131242	1,127	0,129222	0,776	0,218948	0,778	0,201878
C	1,701	0,044431	1,687	0,062978	1,310	0,095171	1,298	0,099143

D. Non-normal Distribution of Limit State Function

Last refinement in the failure probability calculation is incorporating the real probability distribution (generally three-parametric non-normal) and thus non-zero skewness for all input variables (tensile principal stresses, concrete tensile strength) as well as for limit state function. We leave the assumption of normal probability distribution of these variables and replace it with the assumption of three-parametric lognormal probability distribution for concrete tensile strength, the probability distribution of tensile principal stresses obtained from reliability analysis, see Table I, and the best probability distribution for the limit state function. The best probability distribution is obtained with the help of the comparative tests from a set of competing distributions [9]. In our study this set includes normal (N), three-parametric lognormal (LN), truncated normal (TN), three-parametric

Weibull (W) and three-parametric Pearson III (P3) probability distribution. For each supposed concrete quality level (consistent, average and low) the basic statistical parameters of concrete tensile strength are shown in Table II.

The probability calculation is performed using both methods - FOSM Method and Simple Random Sampling Method - and for average and low concrete quality levels only. We remind that the assumption of non-normal probability distribution can be taken into account only by Simple Random Sampling Method as FOSM Method does not respect non-zero skewness and non-normal probability distribution. Thus the results obtained from Simple Random Sampling Method can be supposed to be exact, while results obtained from FOSM Method must be supposed to be approximate.

Illustrative results of calculated reliability index and failure probability in some points on the lower surface of concrete slabs are shown Table VI. Comparing first the results concerning reliability index, there are no major differences for both methods. This may be justified because this quantity depends dominantly only on the mean value and the standard deviation of limit state function. They are not too sensitive on the skewness of input variables. Quite different it is in the case of failure probability, where differences in the values obtained by both methods are quite high.

V.CONCLUSION

Comparing the results obtained using the two proposed methods in the case of reliability index calculation there are no major differences in obtained values. This is due to the reliability index dependence only on the mean value and the standard deviation of limit state function. These statistics particularly in the case of a very simple limit state function are not too sensitive on the skewness of input variables. So FOSM Method provides fully comparable results with Simple Random Sampling Method.

In the case of failure probability calculation differences in the values obtained from both methods can be quite high. It is clear how crucial is the information about the skewness and the real probability distribution when calculating the failure probability and how important is the ability of the method used for the probability calculation to respect non-zero skewness and non-normal probability distribution. The results obtained from Simple Random Sampling Method can be supposed to be the exact values, while results obtained from FOSM Method must be supposed to be approximate. Thus Simple Random Sampling Method seems to be more accurate and more general method, because it is able to provide accurate results for problems described by input variables with non-zero skewness and non-normal probability distribution.

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