

Signal Reconstruction using Cepstrum of Higher Order Statistics

Adnan Al-Smadi and Mahmoud Smadi

Abstract—This paper presents an algorithm for reconstructing phase and magnitude responses of the impulse response when only the output data are available. The system is driven by a zero-mean independent identically distributed (i.i.d) non-Gaussian sequence that is not observed. The additive noise is assumed to be Gaussian. This is an important and essential problem in many practical applications of various science and engineering areas such as biomedical, seismic, and speech processing signals. The method is based on evaluating the bicepstrum of the third-order statistics of the observed output data. Simulations results are presented that demonstrate the performance of this method.

Keywords—Cepstrum, bicepstrum, third order statistics

I. INTRODUCTION

HISTORICALLY, the cepstrum is concerned with the problem of deconvolution [8, 15, 6]. Bogert *et al.* [6] observed that the logarithm of the power spectrum of a signal containing an echo has an additive periodic component due to the echo. Hence, the Fourier transform of the logarithm of the power spectrum exhibits a peak at the echo delay. That is, the effect in the log spectrum turns out to be a ripple. The frequency of this ripple is determined by calculating the spectrum of the logarithm spectrum wherein this frequency will appear as a peak. It should be mentioned that the units of frequency of this ripple in the logarithm spectrum are in units of time. To avoid confusion, Bogert *et al.* [6] introduced new terms according to a syllabic interchange rule. The term that has been widely used is the *cepstrum* that was obtained by interchanging letters in the word *spectrum*. In defining the complex cepstrum, any base can be used for the logarithm [13]. Typically, the natural logarithm (i.e., base e) is used.

The transformation of a signal into its cepstrum function is known as a homomorphic transformation [13]. Homomorphic transformation is an important property of the cepstrum in which the output is a superposition of the input signals; i.e., the input signals and their corresponding responses are superimposed by an operation that has the algebraic characteristics of addition. Under a cepstral transformation, the convolution of two signals $x_1[n] * x_2[n]$ becomes

equivalent to the sum of the cepstra of the signals $\hat{x}_1[n] + \hat{x}_2[n]$. Since introduction of the cepstrum, the concepts of the cepstrum have proven to be useful in signal analysis and have been applied in geophysics, sonar, radar, communications, and biomedical signal analysis problems to decompose superimposed signals [16, 11, 5]. For example, if the wave shape of the original signal is known, then cepstrum techniques can be used to reconstruct the number of echoes, the amplitude, and the number of occurrence. Bogert and Ossana [7] showed that the power cepstrum approach can detect echoes when the autocorrelation of the received signal could not do so. In speech signal processing, the cepstrum has been used to estimate the spectral content of the speech from the pitch frequency of the speech [17]. For voiced speech, the vocal tract spectral envelope multiplies the frequency spectrum in which the discrete line spectrum of the periodic excitation. Cepstral truncation is used to remove the pitch ripple.

The use of second-order statistics (i.e., autocorrelation) is motivated by the implicit assumption that the processes are Gaussian. For modeling time series data, second order statistics are almost exclusively used because they are usually based on least-squares optimization criteria. However, most real world signals are non-Gaussian [3]. The information contained in the power spectrum is that which is present in the autocorrelation sequence. Since autocorrelation-based techniques are phase-blind, all phase information is suppressed in the autocorrelation domain. Therefore, a nonminimum phase system will be identified as being minimum phase. Hence, these techniques often have serious difficulties. However, there are practical applications in which one must look beyond the second order statistics and extract signal phase information.

With the expansion of basic digital signal processing theory and the rapid growth in applications due to the development of fast and inexpensive digital signal processors, there is a growing interest in higher order statistics. The field of higher order statistics (HOS) has been used widely for analyzing non-Gaussian processes [3, 4]. There are several motivations behind the interest of HOS [11]. The emphasis of this paper is based on the property that HOS preserve the true phase character of parametric signals. Hence, cumulants are useful for identifying nonminimum phase systems or for reconstructing nonminimum phase signals if the signals are non-Gaussian.

This paper presents a method for extracting signal phase and magnitude responses of the impulse response from the

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observed corrupted system. The proposed algorithm uses higher order cumulants and higher order cepstra of the distorted signal. The organization of the paper is as follows. Section 2 presents the proposed method. Illustrative examples and simulations are presented in section 3. Section 4 presents some conclusion remarks.

II. PROBLEM FORMULATION

Consider the general causal linear time-invariant (LTI) recursive system described by the linear difference equation

$$\sum_{i=0}^p a_i s(k-i) = \sum_{i=0}^q b_i x(k-i) \quad ; \quad a_0=1 \quad (1)$$

where $x(k)$ is the input signal and $s(k)$ is the noiseless output signal. The system in (1) produces an autoregressive moving average (ARMA) process. The a_i and b_i are the coefficients of the ARMA model with order (p, q) . The system is assumed causal, stable, and generally nonminimum phase. The input signal $s(k)$ is a sequence of independent identically distributed (i.i.d), zero-mean, non-Gaussian random process. The signal $s(n)$ is observed in additive noise

$$y(n) = s(n) + d(n) \quad (2)$$

where $d(n)$ is additive Gaussian noise. The transfer function of the ARMA system is given by

$$H(w) = \frac{\sum_{k=0}^q b_k \exp(-jwk)}{\sum_{k=0}^p a_k \exp(-jwk)} \quad (3)$$

or equivalently,

$$H(w) = |H(w)| \exp[j\theta(w)] \quad (4)$$

where $|H(w)|$ is the magnitude and $\theta(w)$ is the phase of the Fourier transform. If the system is minimum phase, then autocorrelation-based methods will correctly identify both magnitude and phase. If $H(w)$ is nonminimum phase, then the autocorrelation-based methods will correctly identify the magnitude but not the phase. However, both $|H(w)|$ and $\theta(w)$ can be correctly estimated by exploring the use of higher order spectra. Namely, bispectrum that is by definition, the Fourier transform of the third order cumulant sequence. Assuming that $s(k)$ is stationary, then the third order cumulant is [1]

$$R(m, n) = E\{s(k)s(k+m)s(k+n)\} \quad (5)$$

Then its bispectrum is defined as [2]

$$S(w_1, w_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(m, n) \exp[-j(w_1 m + w_2 n)] \quad (6)$$

which is the third order cumulant spectrum.

Now, the cepstrum is defined as the inverse Fourier transform of the logarithm of $H(w)$ in Equation (4); that is [9]

$$h_c(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln[H(w)] \exp(jwn) dw \quad (7)$$

or

$$h_c(n) = F^{-1}\{\ln[H(w)]\} \quad (8)$$

where $F^{-1}[\cdot]$ denotes the inverse Fourier transform. Substituting (4) into (7)

$$\begin{aligned} h_c(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln[|H(w)| \exp[j\theta(w)]] \exp(jwn) dw \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\ln|H(w)| + j\theta(w)] \exp(jwn) dw \end{aligned} \quad (9)$$

Hence, the cepstrum is the spectrum of $\ln[H(w)]$. We can separate the inverse Fourier transform in (9) to magnitude and phase as follows.

$$h_{mag}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|H(w)| \exp[jwn] dw \quad (10)$$

$$h_{\theta}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(w) \exp[jwn] dw \quad (11)$$

In some applications such as speech signal processing, we are interested in magnitude only; hence, only the component $h_{mag}(n)$ is computed. In such case, the phase of the signal is ignored. Therefore, the signal cannot be reconstructed from the magnitude only.

The transfer function $H(z)$ in Equation (3) may also be written in terms of poles and zeros [6]

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h_c(n) z^{-n} = \\ &= \frac{Lz^r \prod_{k=1}^{M_i} (1 - a_k z^{-1}) \prod_{k=1}^{M_o} (1 - b_k z)}{\prod_{k=1}^{N_i} (1 - c_k z^{-1}) \prod_{k=1}^{N_o} (1 - d_k z)} \end{aligned} \quad (12)$$

where $|a_k|$, $|b_k|$, $|c_k|$, and $|d_k|$ are all less than unity. The parameter L is a constant and r is a positive integer. The term z^r corresponds to a delay or advance of the sequence $h(n)$. If $r=0$, then this term vanishes. The expression in (11) is called the minimum-phase maximum-phase decomposition. The factors of the form $(1 - a_k z^{-1})$ and $(1 - c_k z^{-1})$ correspond to M_i zeros and N_i poles inside the unit circle. The factors $(1 - b_k z)$ and $(1 - d_k z)$ correspond to the M_o zeros and N_o poles outside

the unit circle. That is, $H(z)$ can be decomposed into the product of its minimum and maximum delay components, as follows.

$$H(z) = H_{\min}(z)H_{\max}(z) \quad (13)$$

So that

$$h_c(n) = h_{\min}(n) * h_{\max}(n) \quad (14)$$

Now,

$$\ln[H(z)] = \ln |L| + \ln\left(\frac{\prod (1 - a_k z^{-1})}{\prod (1 - c_k z^{-1})}\right) + \ln\left(\frac{\prod (1 - b_k z)}{\prod (1 - d_k z)}\right) \quad (15)$$

It is shown as follows using the initial value theorem that $\ln|L|$ is the cepstrum at $n=0$; i.e., $\ln|L| = h_c(0)$. For a causal sequence $h_c(n)$, we can write Equation (12) as

$$H_c(z) = h_c(0) + h_c(1)z^{-1} + h_c(2)z^{-2} + \dots + h_c(n)z^{-n} + \dots \quad (16)$$

It is seen as $z \rightarrow \infty$, the term $z^{-n} \rightarrow 0$ for each $n > 0$, so that

$$\lim_{z \rightarrow \infty} H_c(z) = h_c(0) \quad (17)$$

The second term in Equation (15) is the causal part and the third term is the anticausal part. Hence,

$$\lim_{z \rightarrow \infty} \ln[H_c(z)] = \lim_{z \rightarrow \infty} \ln |L| = h_c(0) \quad (18)$$

Therefore, $\ln|L| = h_c(0)$.

Now, the cepstrum, $p_s(m)$, of the power spectrum $\Phi(w)$ is defined as

$$p_s(m) = F^{-1}\{\ln \Phi(w)\} \quad (19)$$

which is equal to [2]

$$p_s(m) = \begin{cases} \ln |L| & m = 0 \\ \frac{-1}{m} [A(m) + B(m)] & m > 0 \\ \frac{1}{m} [A(-m) + B(-m)] & m < 0 \end{cases} \quad (20)$$

where $A(k)$ and $B(k)$ are the cepstral coefficients containing the minimum and maximum phase information, respectively. Recall that

$$\Phi(w) = |H(w)|^2 \quad (21)$$

The cepstral coefficients are

$$A(k) = \sum_{i=1}^{M_i} a_i^k - \sum_{i=1}^{N_i} c_i^k \quad (22)$$

$$B(k) = \sum_{i=1}^{M_o} b_i^k - \sum_{i=1}^{N_o} d_i^k \quad (23)$$

The bicepstrum can be computed directly from the observed data as follows [15].

$$b_s(m, n) = \frac{1}{m} F_2^{-1} \left[\frac{F_2 \{m \bullet R(m, n)\}}{F_2 \{R(m, n)\}} \right] \quad (24)$$

where $F_2 \{.\}$ and $F_2^{-1} \{.\}$ denote two-dimensional Fourier and inverse Fourier transforms, respectively, $R(m, n)$ is the third order cumulants and is estimated as follows.

$$R(m, n) = \frac{1}{N} \sum_k s(k)s(k+m)s(k+n) \quad (25)$$

In this equation, N denotes the length of the sequence. It should be stated that the bicepstrum can also be obtained from the cepstral coefficients $A(k)$ and $B(k)$ as follows [14].

$$b_s(m, n) = F_2^{-1}[S(w_1, w_2)] = \begin{cases} \ln |L| & m = 0, n = 0 \\ \frac{-1}{m} A(n) & m = 0, n > 0 \\ \frac{-1}{n} A(m) & n = 0, m > 0 \\ \frac{-1}{m} B(-m) & n = 0, m < 0 \\ \frac{1}{n} B(-n) & m = 0, n < 0 \\ \frac{-1}{n} B(n) & m = n > 0 \\ \frac{1}{n} A(-n) & m = n < 0 \\ 0 & otherwise \end{cases} \quad (26)$$

Pan and Nikias [14] showed that the bicepstrum of $s(k)$ is zero everywhere except on the axes $m=0, n=0$, and on the diagonal (i.e., $m=n$). We estimate the phase of the system as follows. We first calculate the third order cumulants of the system using Equation (5). Then, the bicepstrum is obtained from the observed data using Equation (24). After that, the cepstral coefficients are calculated from the bicepstrum using Equation (26). Having the cepstral coefficients, we can extract the phase response as follows.

$$\theta(w) = F \left\{ \frac{j}{2m} [A(m) - B(m)] \right\} \quad (27)$$

Now, substituting Equation (21) into Equation (19)

$$P_s(m) = F^{-1} \{ \ln |H(w)|^2 \} \quad (28)$$

Taking Fourier transforms of both sides of (28),

$$F \{ p_s(m) \} = \ln |H(w)|^2$$

Raising both sides to the base e

$$\exp(\ln |H(w)|^2) = \exp(F\{p_s(m)\}) \quad (29)$$

Solving for $|H(w)|$, we obtain

$$|H(w)| = \sqrt{\exp(F\{p_s(m)\})} \quad (30)$$

That is, the magnitude response is obtained using the Fourier transform of the cepstrum of power spectrum of the available distorted signal.

III. NUMERICAL EXAMPLES

To demonstrate the effectiveness of the proposed method, several numerical examples were considered. The simulations were carried out using MATLAB.

Example 1: The system under consideration was

$$h = [0.7 \ 1 \ 0.8 \ -0.7 \ 0.6 \ -0.5 \ 0.4 \ -0.3 \ 0.2 \ -0.1] \quad (31)$$

A Gaussian noise of 25 dB signal-to-noise ratio (SNR) was added to the system. The third order cumulants of the corrupted signal were computed. Then, the bicepstrum was obtained using two-dimensional Fourier and inverse Fourier transforms of the cumulants. Next, the cepstral coefficients were obtained. Finally, the phase and the magnitude responses were recovered. The simulation with noise with different seeds was performed 100 times. Then, these simulations were averaged to estimate the desired phase and magnitude of the system. Figure 1 shows the average result of 100 Monte Carlo simulations that estimated the phase versus the true phase responses of the system. Figure 2 shows the average result of 100 Monte Carlo simulations of the estimates of the magnitude versus the true magnitude response of the system. Table 1 shows the first 10 by 10 entries of the bicepstrum matrix.

Example 2: The system under consideration was

$$h = [1 \ 0.9 \ 1.9 \ -1 \ 1 \ -0.4 \ 1.5 \ -0.4 \ 0.5 \ -1] \quad (32)$$

As in example 1, the system was corrupted with additive Gaussian noise with SNR of 25 dB on the output signal. The third order cumulants of the corrupted signal were computed. The phase and magnitude responses were recovered. The true phase versus the estimated phase responses (an average result of 100 Monte Carlo simulations) of the system are shown in Figure 3. The true magnitude versus the estimated magnitude (an average result of 100 Monte Carlo simulations) is shown in Figure 4. Table 2 shows the first 10 by 10 entries of the bicepstrum matrix.

In these examples, the phase and magnitude responses are reconstructed and the estimated values appear almost on the top of the true values. The superior performance of this algorithm is because higher order cumulants are blind to all kinds of Gaussian noise. That is, higher order cumulants for a Gaussian process are identically zero.

IV. CONCLUSION

In this paper, a cepstrum based method for recovering the phase and magnitude responses of the impulse response from a distorted with additive Gaussian noise output data was described. The proposed algorithm uses third order cumulants and higher order cepstra coefficients of the corrupted signal. Examples show that the reconstructed phase and magnitude are in close agreement with the true values.

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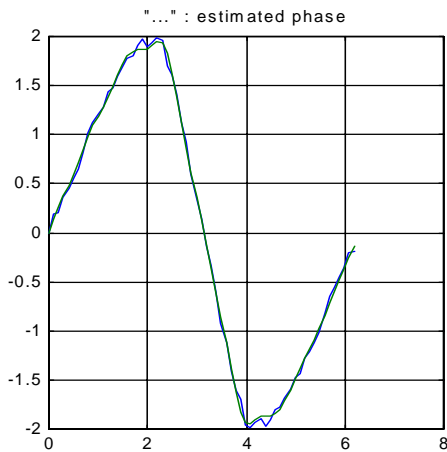


Fig. 1 True VS estimated phase for Example 1

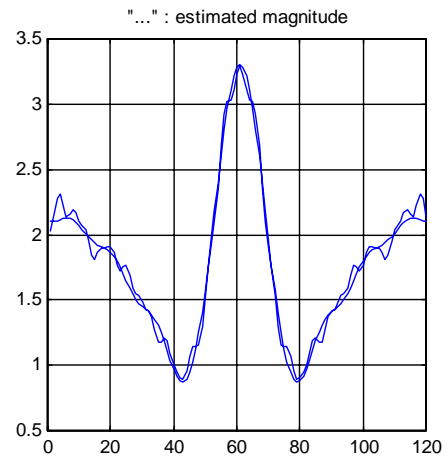


Fig. 2 True VS estimated magnitude for Example 1

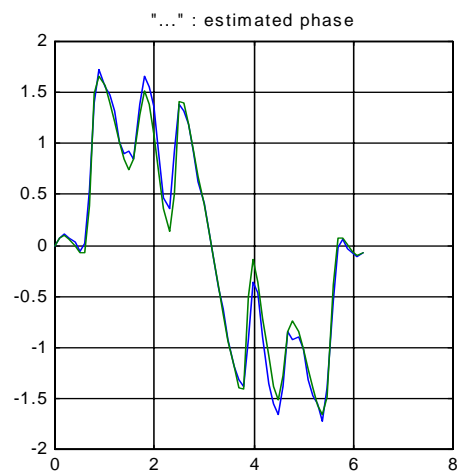


Fig. 3 True VS estimated phase for Example 2

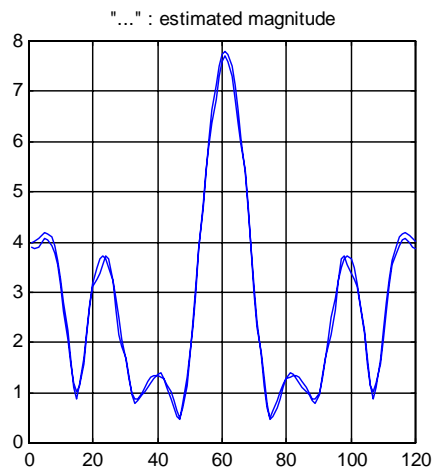


Fig. 4 True VS estimated magnitude for Example 2

TABLE I
THE FIRST 10 BY 10 ENTRIES OF THE BICEPSTRUM MATRIX

	m									
n	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	-0.883	0.965	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.372	0.000	-0.046	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	-0.196	0.000	0.00	-0.106	0.000	0.000	0.000	0.000	0.000	0.000
	0.071	0.000	0.000	0.000	0.086	0.000	0.000	0.000	0.000	0.000
	-0.007	0.000	0.000	0.000	0.000	-0.040	0.000	0.000	0.000	0.000
	-0.034	0.000	0.000	0.000	0.000	0.000	0.008	0.000	0.000	0.000
	0.041	0.000	0.000	0.000	0.000	0.000	0.000	0.007	0.000	0.000
	-0.014	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.007	0.000
	-0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.004

TABLE II
THE FIRST 10 BY 10 ENTRIES OF THE BICEPSTRUM MATRIX

	m									
n	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	-0.211	0.324	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.372	0.000	-0.518	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	-0.196	0.000	0.00	-0.127	0.000	0.000	0.000	0.000	0.000	0.000
	0.071	0.000	0.000	0.000	0.036	0.000	0.000	0.000	0.000	0.000
	-0.007	0.000	0.000	0.000	0.000	0.0115	0.000	0.000	0.000	0.000
	-0.034	0.000	0.000	0.000	0.000	0.000	-0.138	0.000	0.000	0.000
	0.041	0.000	0.000	0.000	0.000	0.000	0.000	-0.011	0.000	0.000
	-0.014	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.012	0.000
	-0.020	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.057