International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:13, No:12, 2019

# Several Spectrally Non-Arbitrary Ray Patterns of Order 4

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***Abstract*—**A matrix is called a ray pattern matrix if its entries are either 0 or a ray in complex plane which originates from 0. A ray pattern *A* of order *n* is called spectrally arbitrary if the complex matrices in the ray pattern class of *A* give rise to all possible *n*th degree complex polynomial. Otherwise, it is said to be spectrally non-arbitrary ray pattern. We call that a spectrally arbitrary ray pattern *A* of order *n* is minimally spectrally arbitrary if any nonzero entry of *A* is replaced, then *A* is not spectrally arbitrary. In this paper, we find that is not spectrally arbitrary when n equals to 4 for any θ which is greater than or equal to 0 and less than or equal to n. In this article, we give several ray patterns A(θ) of order n that are not spectrally arbitrary for some θ which is greater than or equal to 0 and less than or equal to n. by using the nilpotent-Jacobi method. One example is given in our paper.

***Keywords*—**Spectrally arbitrary, Nilpotent matrix, Ray patterns, sign patterns.

## I. Introduction

AN $n \times n$ ray pattern *A* is a matrix with entries $a_{ij}$ from

$$\{re^{i\theta}: r > 0\} \cup \{0\}$$

For brevity, we denote a ray $re^{i\theta}$ simply by $e^{i\theta}$. It is easy to verify that for two rays $e^{i\theta_1}$ and $e^{i\theta_2}$, if $\theta_1 - \theta_2 = 2k\pi$ where *k* is an integer number, then

$$e^{i\theta_1} = e^{i\theta_2}; \tag{1}$$

otherwise, $e^{i\theta_1} \neq e^{i\theta_2}$. For two rays $e^{i\theta_1}$ and $e^{i\theta_2}$, multiplication, division and addition are performed obviously. The product and quotient are given as:

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \tag{2}$$

and

$$e^{i\theta_1} / e^{i\theta_2} = e^{i(\theta_1-\theta_2)}. \tag{3}$$

$\theta_1$ and $\theta_2$ differ by a multiple of $2\pi$, then $e^{i\theta_1} + e^{i\theta_2} = e^{i\theta_1}$.

The sum of two distinct rays $e^{i\theta_1}$ and $e^{i\theta_2}$ is either a straight line through the origin or an open sector in the complex plane with vertex at the origin (when the two rays are opposite in direction). We denote # by any sum of rays where at least two of the rays are distinct. It is easy to verify that

$$e^{i\theta} + \# = \#, \quad e^{i\theta}\# = \#$$
$$0 + \# = \#, \quad 0\# = 0, \quad \# + \# = \#0, \quad \#\# = \#.$$

Let $z = x + iy$ be a non-zero complex number and $r = |z| = \sqrt{x^2 + y^2}$, then we get $x = r\cos\theta$, $y = r\sin\theta$, where $\theta$ is the angle made by z with the positive x-axis. Therefore, $\theta$ is unique up to the addition of a multiple of $2\pi$ radians. We call the number $\theta$ satisfying the above pair of equations and argument of z and denote it by arg z. The ray pattern class of an $n \times n$ ray pattern *A*, denoted by $Q(A)$, is the set of $n \times n$ complex matrices given by

$$\{B : [b_{pq}] \in M_n(\mathbb{C}) : b_{pq} = 0 \quad if \quad a_{pq} = 0;$$
$$\arg b_{pq} = \arg a_{pq} \quad otherwise\}.$$

An $n \times n$ ray pattern *A* is said to be spectrally arbitrary if given any monic *n*th degree polynomial $f(x)$ with coefficients from ℂ, there exists a matrix $B \in Q(A)$ having characteristic polynomial $f(x)$. A spectrally arbitrary ray pattern *A* is said to be minimally spectrally arbitrary if any nonzero entry of *A* is replaced by zero, then it is not spectrally arbitrary.

The question of the existence of spectrally arbitrary sign patterns, that is, sign patterns that allow the realization of every self-conjugate spectrum, arose in [1]. In this paper, the nilpotent-Jacobi method for showing that a sign pattern was developed and all its superpatterns are spectrally arbitrary and a conjunction that a particular family of tridiagonal patterns is spectrally arbitrary was given. Since that time there have been many papers on this topic (see, for example, [2]-[7]) and several families of spectrally arbitrary patterns have been presented and general properties of spectrally arbitrary patterns have been studied ([8-11]). In [12], Britz et al. showed that every irreducible, spectrally arbitrary sign pattern of order n must have at least 2*n*-1 nonzeros and they also gave families of patterns that have exactly 2*n* nonzeros. This result is easily extended to zero–nonzero patterns over ℝ and ℂ. In [13], the problem of classifying the spectrally arbitrary zero–nonzero patterns over ℝ is studied and all $n \times n$ spectrally arbitrary zero–nonzero patterns are classified when $n \geq 4$. This article

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This project is supported by the Educational Council Foundation of Chongqing (KJ1500517 and KJ1600512).

presented the idea that identifies the maximum number of nonzero entries such that a zero-nonzero pattern with maximum number of nonzeros is spectrally arbitrary. In [14], DeAlba et al. studied properties of reducible, spectrally arbitrary sign and zero-nonzero patterns over ℝ. Recently, McDonald and Stuart [19] described a method for proving an irreducible ray pattern with exactly $3n$ non-zeros and its superpatterns are spectrally. From that time there have many articles on this topic (see, for example, [15]-[18]).

## II. The Nilpotent-Jacobi Method

In [9], Drew et al. gave a method of establishing that a sign pattern and every of its superpatterns are spectrally arbitrary. This method worked for a sign pattern in whose class certain types of nilpotent matrices could be found. McDonald and Stuart [19] extended their method to the ray pattern case in the following manner:

The nilpotent-Jacobi method [19]:

1. Find a nilpotent matrix in the given ray pattern class.
2. Change $2n$ of the positive coefficients (denoted by $r_1, r_2, \cdots r_{2n}$) of the $e^{ij}$ in this nilpotent matrix to variables $t_1, t_2, \cdots t_{2n}$.
3. Denote the characteristic polynomial of the resulting matrix as:

$$\begin{aligned} &x^n + (f_1(t_1, \dots t_{2n}) + ig_1(t_1, \dots t_{2n}))x^{n-1} + \dots \\ &+ (f_{n-1}(t_1, \dots t_{2n}) + ig_{n-1}(t_1, \dots t_{2n}))x \\ &+ (f_n(t_1, \dots t_{2n}) + ig_n(t_1, \dots t_{2n})) \end{aligned}$$

4. Find the Jacobi matrix

$$J = \frac{\partial(f_1, g_1, \dots, f_n, g_n)}{t_1, t_2, \dots t_{2n}}$$

If the determinant of $J$ evaluated at $(t_1, t_2, \cdots t_{2n}) = (r_1, r_2, \cdots r_{2n})$ is nonzero, then by continuity of the determinant in the entries of a matrix, there exists a neighborhood $U$ of $(r_1, r_2, \cdots r_{2n})$ such that all the vectors in $U$ are strictly positive and the determinant of $J$ evaluated at any of these vectors is nonzero. Moreover, by the Implicit Function Theorem there is a neighborhood $V \subseteq U$ of $(r_1, r_2, \cdots r_{2n}) \subseteq \mathrm{R}^{2n}$, a neighborhood $W$ of $(0, 0, \dots, 0) \subseteq \mathbb{R}$ and a function $(h_1, h_2, \cdots h_{2n})$ from $W$ into $V$ such that for any $(a_1, b_1, \dots a_n, b_n) \in W$ there exists a strictly positive vector

$$(s_1, s_2 \dots s_{2n}) = (h_1, h_2 \dots h_{2n})(a_1, b_1, \dots a_{2n}, b_{2n}) \in V$$

where $f_j(s_1, s_2 \dots s_{2n}) = a_j$, $g_j(s_1, s_2 \dots s_{2n}) = b_j$. If we take positive scalar multiples of the corresponding matrices, then we have that each monic $n$th degree polynomial over ℝ is the characteristic polynomial of some matrix in this ray pattern class.

5. Consider a superpattern of our pattern. Let $c_1 e^{\theta_{i_1}\theta_{j_1}}$ be the new nonzero entries. Denote the new functions in characteristic polynomial by $\hat{F}(x)$. Let

$$\begin{aligned} \hat{F}(x) = x^n + \sum_{j=1}^{n} (&\hat{f}_j(t_1, t_2, \dots, t_{2n}, c_1, \dots, c_k) \\ &+ \hat{g}_j(t_1, t_2, \dots, t_{2n}, c_1, \dots, c_k))x^{n-j} \end{aligned}$$

where $\hat{f}_j(t_1, t_2, \dots, t_{2n}, c_1, \dots, c_k)$ and $\hat{g}_j(t_1, t_2, \dots, t_{2n}, c_1, \dots, c_k)$ represent the real and complex parts of the coefficient of $x^{n-j}$. Let

$$\hat{J} = \frac{\partial(\hat{f}_1, \hat{g}_1, \dots, \hat{f}_n, \hat{g}_n)}{t_1, t_2, \dots t_{2n}},$$

be the new Jacobi matrix. As above, let $(a_1, b_1, \dots a_n, b_n) \in W$ and $(s_1, s_2 \dots s_{2n}) = (h_1, h_2 \dots h_{2n})(a_1, b_1, \dots a_{2n}, b_{2n}) \in V$. Then

$$\begin{aligned} a_j &= f_j(s_1, s_2, \dots, s_{2n}) = \hat{f}(s_1, s_2, \dots, s_{2n}, 0, \dots, 0) \\ b_j &= g_j(s_1, s_2, \dots, s_{2n}) = \hat{g}(s_1, s_2, \dots, s_{2n}, 0, \dots, 0) \end{aligned}$$

and the determinant of evaluated at

$$(t_1, t_2, \dots, t_{2n}, c_1, c_2, \dots, c_k) = (s_1, s_2, \dots, s_{2n}, 0, 0, \dots, 0)$$

is equal to the determinant of $J$ evaluated at

$$(t_1, t_2, \dots, t_{2n}) = (s_1, s_2, \dots, s_{2n})$$

and hence is nonzero. By the implicit function theorem, there exists a neighborhood $\hat{V} \subseteq V$ of $(s_1, s_2, \dots, s_{2n})$, a neighborhood $T$ of $(0, 0, \dots, 0) \in \mathbb{R}^k$ and a function $(q_1, q_2, \dots, q_{2n})$ from $T$ into $\hat{V}$ such that for any vector $(d_1, d_2, \dots, d_k) \in T$, there exists a strictly positive vector

$$(e_1, e_2, \dots, e_{2n}) = (q_1, q_2, \dots, q_{2n})(c_1, c_2, \dots, c_k) \in \hat{V}$$

where

$$\hat{f}_j(e_1, e_2, \dots, e_{2n}, c_1, c_2, \dots, c_k) = a_j$$

and

$$\hat{g}_j(e_1, e_2, \dots, e_{2n}, c_1, c_2, \dots, c_k) = b_j.$$

Taking $(d_1, d_2, \dots, d_k) \in T$ strictly positive, we have that there also exist matrices in the superpattern's class with every characteristic polynomial corresponding to a vector in $W$. If we choose positive scalar multiples of the corresponding matrices, then we get that each monic $n$th degree polynomial over $\mathbb{C}$ is the characteristic polynomial of some matrix in this superpattern's class.

## III. Main Results

In [18], McDonald and Stuart defined the $n \times n$ ray sign patterns of the following form.

$$A_n(\theta) = \begin{pmatrix} -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 1 & e^{i\theta} & 1 & 0 & \cdots & \cdots & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \\ -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & -i & -i & \cdots & \cdots & \cdots & -i & -i \end{pmatrix} \tag{4}$$

where $0 \le \theta \le 2\pi$ and $n \ge 4$ and gave the following theorem.

**Theorem1.** [18] For $n \ge 4$, there exist infinitely many choices for $\theta$ with $0 \le \theta \le 2\pi$, so that $A_n(\theta)$ and all of its superpatterns are spectrally arbitrary ray patterns.

McDonald and Stuart [15] have proved the theorem. According to the definition of $A_n$.

$$A_4 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & e^{i\theta} & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & -i & -i & -i \end{pmatrix} \tag{5}$$

Unfortunately, we find that $A_4$ is not spectrally arbitrary for any $\theta$ with $0 \le \theta \le 2\pi$. Now we prove that $A_4$ is not spectrally arbitrary for any $\theta$ with $0 \le \theta \le 2\pi$. For convenience, we restrict $\theta$ to $0 \le \theta \le \frac{\pi}{2}$. Let $q = \cos\theta$. Suppose $B_4 \in Q(A_4)$, then by scaling and positive diagonally similarity we can assume

$$B_4 = \begin{pmatrix} a_1 & 1 & 0 & 0 \\ a_2 & q + i\sqrt{1-q^2} & 1 & 0 \\ a_3 & b_4 & 0 & 1 \\ a_4 & ib_3 & ib_2 & ib_1 \end{pmatrix} \tag{6}$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are negative and $a_4, b_4$ are positive. From [19], the characteristic polynomial of $B_4$ is as follows:

$$\begin{aligned} & x^4 + [(-a_1 - q) + i(-b_1 - \sqrt{1-q^2})]x^3 \\ & + [(-a_2 - b_1\sqrt{1-q^2} + a_1 q - b_4) \\ & \quad + i(-b_2 + a_1 b_1 + a_1\sqrt{1-q^2} + b_1)]x^2 \\ & + [(-a_3 + a_1 b_4 + a_1 b_1\sqrt{1-q^2} - b_2\sqrt{1-q^2}) \\ & \quad + i(-b_3 - a_1 b_1 q + b_2 q + \sum_{k=1}^{2} a_k b_{3-k} + b_1 b_4)]x \\ & + [-a_4 + a_1 b_2\sqrt{1-q^2} \\ & \quad + i(-a_1 b_1 b_4 - a_1 b_2 q + \sum_{k=1}^{3} a_k b_{4-k})] \end{aligned} \tag{4}$$

Suppose that $B_4$ is nilpotent. Setting the coefficient of $x^{n-j}$ equal to zero for $j = 1,2,3,4$, then solving for $a_j$ and $b_j$, we get that

$$\begin{aligned} a_1 &= -q, \\ b_1 &= -\sqrt{1-q^2}, \\ a_2 &= b_1\sqrt{1-q^2} + a_1 q - b_4 \\ &= q\sqrt{1-q^2}, \\ a_3 &= a_1 b_4 + a_1 b_1\sqrt{1-q^2} - b_2\sqrt{1-q^2} \\ &= -q b_4 + 2q(1-q^2), \\ b_3 &= -a_1 b_1 q + b_2 q + \sum_{k=1}^{2} a_k b_{3-k} + b_1 b_4 \\ &= -(1-q^2)\sqrt{1-q^2}, \\ a_4 &= a_1 b_2\sqrt{1-q^2} = q^2(1-q^2), \\ b_4 &= \frac{-a_1 b_2 q + a_1 b_3 + (1-2q^2)b_2 + 2q(1-q^2)b_1}{b_2} \\ &= 2(1-q^2) \end{aligned} \tag{8}$$

Substituting $b_4$ into the equation

$$a_3 = q b_4 + 2q(1-q^2) \tag{9}$$

we obtain

$$a_3 = -2q(1-q^2) + 2q(1-q^2) = 0 \tag{10}$$

which contradicts the fact that $B_4 \in Q(A_4)$. Thus, $A_4$ is not potentially nilpotent, which implies that $A_4$ is not spectrally arbitrary for any $\theta$ with $0 \le \theta \le 2\pi$.

## IV. Examples

**Example1.** The $4 \times 4$ ray pattern

$$A=\begin{pmatrix}-1 & 1 & 0 & 0\\ 1 & \frac{1+\sqrt{3}i}{2} & 1 & 0\\ -1 & 1 & 0 & 1\\ 1 & -i & -i & -i\end{pmatrix} \quad (11)$$

is not spectrally arbitrary. The matrix

$$B=\begin{pmatrix}a_1 & 1 & 0 & 0\\ a_2 & \frac{1+\sqrt{3}i}{2} & 1 & 0\\ a_3 & b_4 & 0 & 1\\ a_4 & ib_3 & ib_2 & ib_1\end{pmatrix} \quad (12)$$

is in the pattern class $B \in Q(A)$ whenever $a_2, a_4$ are negative and $a_1, a_3, b_1, b_2, b_3$ are positive. The characteristic polynomial of $B$ is

$$\begin{aligned} & x^4+[(a_1-\frac{1}{2})+i(-b_1-\frac{\sqrt{3}}{2})]x^3 \\ & +[(-a_2-\frac{\sqrt{3}b_1}{2}+\frac{a_1}{2}-b_4) \\ & +i(-b_2+a_1b_1+\frac{\sqrt{3}a_1}{2}-\frac{b_1}{2})]x^2 \\ & +[(-a_3+a_1b_4+\frac{\sqrt{3}a_1b_1}{2}-\frac{\sqrt{3}b_2}{2})) \\ & +i(-b_3-\frac{a_1b_1}{2}+\frac{b_2}{2}+\sum_{k=1}^{2}a_kb_{3-k}+b_1b_4)]x \\ & +[(-a_4+\frac{\sqrt{3}a_1b_2}{2})+i(-a_1b_1b_4-\frac{a_1b_2}{2}+\sum_{k=1}^{3}a_kb_{4-k})] \end{aligned} \quad (13)$$

Suppose that $B$ is nilpotent. Setting the coefficient of $x^{n-j}$ equal to zero for $j=1,2,3,4$, then solving for $a_j$ and $b_j$, we get that

$$\begin{aligned} & a_1=-\frac{1}{2}, a_2=-1, a_3=0, a_4=\frac{3}{16}, \\ & b_1=-\frac{\sqrt{3}}{2}, b_2=-\frac{\sqrt{3}}{4}, b_3=-\frac{3\sqrt{3}}{8}, b_4=\frac{3}{16} \end{aligned} \quad (14)$$

which contradicts the fact that $B \in Q(A)$. Thus, $A$ is not potentially nilpotent, which implies that $A$ is not spectrally arbitrary.

## References

[1] J. H. Drew, C. R. Johnson, D. D. Olesky, P. van den Driessche, “Spectrally arbitrary patterns,” *Linear Algebra Appl.,* vol. 308, pp. 121-137, 2000.

[2] M.S. Cavers and S.M. Fallat, “Allow problems concerning spectral properties of patterns,” *Electron. J. Linear Algebra.,* vol. 23, pp. 731-754 , 2012.

[3] M. Catral, D.D. Olesky, and P. van den Driessche, “Allow problems concerning spectral properties of sign pattern matrices: A survey,” *Linear Algebra Appl.,* vol. 430, pp. 3080-3094 , 2009.

[4] M. Cavers, C. Garnett, I.-J. Kim, D.D. Olesky, P. van den Driessche and K. Vander Meulen, “Techniques for identifying inertially arbitrary patterns,” *Electron. J. Linear Algebra.,* vol. 26, pp. 71-89 , 2013.

[5] L. Elsner, D. Hershkowitz, “On the spectra of close-to-Schwarz matrices,” *Linear Algebra Appl.,* vol. 363, pp.81-88, 2003.

[6] In-JaeKim, Bryan L.Shader, Kevin N. Vander Meulen and Matthew West, “Spectrally arbitrary pattern extensions,” *Linear Algebra Appl.,* vol. 517, pp. 120-128, 2017.

[7] J. J. McDonald, D. D. Olesky, M. J. Tsatsomeros, P. van den Driessche, “On the spectra of striped sign patterns,” *Linear and Multilinear Algebra,* vol. 51, pp. 39-48, 2003.

[8] A. Behn, K.R. Driessel, I.R. Hentzel, K. Vander Velden and J. Wilson, “ Some nilpotent, tridiagonal matrices with a special sign pattern,” *Linear Algebra Appl.*, vol. 36, no. 12, pp. 4446–4450 , 2012.

[9] M. S. Cavers and K. N. Vander Meulen, “Spectrally and inertially arbitrary sign patterns,” *Linear Algebra Appl.,* vol. 394, pp. 53-72 , 2005.

[10] M. S. Cavers, I. J. Kim, B. L. Shader and K. N. Vander Meulen, “On determining minimal spectrally arbitrary patterns,” *Electron. J. Linear Algebra.,* vol. 13, pp. 240-248 , 2005.

[11] G. MacGillivray, R. M. Tifenbach, P. van den Driessche, “Spectrally arbitrary star sign patterns,” *Linear Algebra Appl.,* vol. 400, pp. 99-119, 2005.

[12] T. Britz, J. J. McDonald, D. D. Olesky and P. van den Driessche, “Minimal spectrally arbitrary sign patterns,” *SIAM J. Matrix. Anal. Appl.*, vol. 36, pp. 257–271, 2004.

[13] L. Corpuz and J.J. McDonald, “Spectrally arbitrary zero nonzero patterns of order 4,” *Linear and Multilinear Algebra,* vol. 55, pp. 249-273, 2007.

[14] L. M. DeAlba, I. R. Hentzel, L. Hogben, J. J. McDonald, R. Mikkelson and O. Pryporova, “Spectrally arbitrary patterns: Reducibility and the 2n conjecture for n = 5,” *Linear Algebra Appl.,* vol. 423, pp. 262-276, 2007.

[15] Yubin Gao and Yanling Shao, “New classes of spectrally arbitrary ray patterns,” *Linear Algebra Appl.,* vol. 434, pp. 2140-2148, 2011. .

[16] Yinzhen Mei, Yubin Gao, Yan Ling Shao and Peng Wang, “A new family of spectrally arbitrary ray patterns,” *Czechoslovak Mathematical Journal.,* vol. 66, pp. 1049–10589, 2016.

[17] Y. Mei, Y. Gao, Y. Shao and P. Wang, “The minimum number of nonzeros in a spectrally arbitrary ray pattern,” *Linear Algebra Appl.,* vol. 453, pp. 99–109, 2014

[18] Ling Zhang, Ting-Zhu Huang, Zhongshan Li and Jing-Yue Zhang, “Several spectrally arbitrary ray patterns,” *Linear and Multilinear Algebra,* vol.61, pp. 543-564, 2013.

[19] L. Corpuz and J.J. McDonald, “Spectrally arbitrary zero nonzero patterns of order 4,” *Linear and Multilinear Algebra,* vol. 55, pp. 249-273 , 2007.
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