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S-Fuzzy Left h-Ideal of Hemirings

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***Abstract*—** The notion of S-fuzzy left h-ideals in a hemiring is introduced and it's basic properties are investigated.We also study the homomorphic image and preimage of S-fuzzy left h-ideal of hemirings.Using a collection of left h-ideals of a hemiring, S-fuzzy left h-ideal of hemirings are established.The notion of a finite-valued S-fuzzy left h-ideal is introduced,and its characterization is given.S-fuzzy relations on hemirings are discussed.The notion of direct product and S-product are introduced and some properties of the direct product and S-product of S-fuzzy left h-ideal of hemiring are also discussed.

***Keywords*—** hemiring,left h-ideal,anti fuzzy h-ideal,S-fuzzy left h-ideal,t-conorm , homomorphism.

I. Introduction

THE concept of fuzzy subset was introduced by L.A.Zadeh [8]. Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. B.Schweizer and A.Sklar [5,6] introduced the notions of Triangular norm (t-norm) and Triangular conorm (t-conorm).Triangular norm (t-norm) and Triangular conorm (t-conorm or s-norm) are the most general families of binary operations that satisfy the requirement of the conjunction and disjunction operators respectively.The ideal theory plays an important role in algebraic structure.La Torre [7] studied the notion of h-ideals and k-ideals in hemirings.Then Y.B Jun et. al[4] introduced the notion of fuzzy h-ideal of hemirings and discussed related properties.First,Abu Osman [1] introduced the notion of fuzzy subgroup with respect to t-norm. Following this,J.Zhan [9] introduced the notion of T-fuzzy left h-ideal of hemirings.Then,J.Zhan [10] introduced the notion of fuzzy hyper ideals in hyper near-rings with respect to t-norm.Recently,Y.U Cho et. al[3] introduced the notion of fuzzy subalgebras with respect to t-conorm of BCK-algebras and M.Akram et. al.[2] introduced the notion of sensible fuzzy ideal with respect to t-conorm in BCK-algebras.Using the idea of [2] and [3],In this paper we introduce the notion of S-fuzzy left h-ideal of hemirings and investigate it is related properties.Also,we review several results described in [9] using t-conorm.

II. Preliminaries

An algebra $(R;+,.)$ is said to be a *semiring* if $(R;+)$ and $(R;.)$ are semigroups satisfying $a.(b+c) = a.b+a.c$ and $(b+c).a = b.a+c.a$ for all $a,b,c \in R$.A semiring R is said to be *additively commutative* if $a+b = b+a$ for all $a,b,c \in S$. A semiring R may have an identity 1, defined by $1.a = a = a.1$ and a zero 0, defined by $0+a = a = a+0$ and $a.0 = 0 = 0.a$ for all $a \in R$.A semiring R is said to be a *hemiring* if it is an additively commutative with zero.A non-empty subset I of R is said to be a *left (resp., right ideal)* if $x,y \in I$ and $r \in R$ imply that $x+y \in I$ and $rx \in I$ $(resp., xr \in I)$. If I is both left and right ideal of R , we say I is a *two-sided ideal, or simply ideal,* of R. A left ideal I of a semiring R is said to be a *k-ideal* if $a \in I$ and $x \in R$, and if $x+a \in I$ or $a+x \in I$ then $x \in I$. Right k-ideal is defined dually, and two-sided k-ideal or simply a k-ideal is both a left and a right k-ideal.A left ideal I of a hemiring R is called a *left h-ideal* if $x+a+z = b+z$ implies that $x \in I$ for all $x,y \in R$ and $a,b \in R$.Right h-ideals are defined similarly.

Definition 2.1: Let X be a non-empty set. A fuzzy subset of X is a function $\mu : X \to [0,1]$. Let μ be the fuzzy subset of a set X. For a fixed $0 \leq t \leq 1$, the set

$$L(\mu;t) = \{x \in X : \mu(x) \leq t\}$$

is called a *lower level set* or simply *level set* of μ.

Definition 2.2: A fuzzy subset μ of a hemiring R is said to be *fuzzy left (resp., right) ideal* of R if
$(FI1)\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and
$(FI2)\mu(xy) \geq \mu(y) \quad (resp., \mu(xy) > \mu(x))$
for all $x,y \in R$.
If μ is a *fuzzy ideal* of R if it is both fuzzy left and a fuzzy right ideal of R.

Definition 2.3: A fuzzy subset μ of a hemiring R is said to be an *anti fuzzy left (resp., right) ideal* of R if
$(FI1)\mu(x+y) \leq \max\{\mu(x), \mu(y)\}$ and
$(FI2)\mu(xy) \leq \mu(y) \quad (resp., \mu(xy) \leq \mu(x))$
for all $x,y \in R$.
If μ is an *anti fuzzy ideal* of R if it is both an anti fuzzy left and anti fuzzy right ideal of R.

Definition 2.4: Let R and R' be hemirings. A mapping $f : R \to R'$ is said to be a *homomorphism* if
$f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$
for all $x,y \in R$.

Definition 2.5: A fuzzy subset μ of a hemiring R is said to be a *fuzzy left (resp., right) h-ideal* of R if
$(AFI1)\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and
$(AFI2)\mu(xy) \geq \mu(y) \quad (resp., \mu(xy) \geq \mu(x))$
for all $x,y \in R$.
$(AFI3)$ If $x+a+z = b+z$ implies that
$\mu(x) \geq min\{\mu(a), \mu(b)\}$, for all $a,b,x,z \in S$.
If μ is *fuzzy h-ideal* of R if it is both a fuzzy left and fuzzy right h-ideal of R.

Definition 2.6: A fuzzy subset μ of a hemiring R is said to be an *anti fuzzy left (resp., right) h-ideal* of R if
$(AFI1)\mu(x+y) \leq \max\{\mu(x), \mu(y)\}$ and
$(AFI2)\mu(xy) \leq \mu(y) \quad (resp., \mu(xy) \leq \mu(x))$

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for all $x, y \in R$.
$(AFI3)$ If $x + a + z = b + z$ implies that
$\mu(x) \leq max\{\mu(a), \mu(b)\}$, for all $a, b, x, z \in S$.
If μ is an *anti fuzzy h-ideal* of R it is both an anti fuzzy left h-ideal and anti fuzzy right h-ideal of R.

Definition 2.7: A triangular conorm (t-conorm) is a mapping $S : [0,1] \times [0,1] \longrightarrow [0,1]$ that satisfies the following conditions:
$(S1)$ $S(x, 0) = x$,
$(S2)$ $S(x, y) = S(y, x)$,
$(S3)$ $S(x, S(y, z)) = S(S(x, y), z)$,
$(S4)$ $S(x, y) \leq S(x, z)$ whenever $y \leq z$,
for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition S, we obtain the concept of t-norm T.

Proposition 2.8: For a t-conorm S.Then the following statement holds $S(x, y) \geq max(x, y)$, for all $x, y \in [0, 1]$.

Definition 2.9: Let S be a t-conorm. A fuzzy subset μ in a hemiring R is called *sensible* with respect to S
if $Im\mu \subseteq \triangle_S$,where $\triangle_S = \{t \in [0, 1] | S(t, t) = t\}$.

III. S-FUZZY LEFT H-IDEALS IN HEMIRINGS

In what follows,R and S denotes a hemiring and t-conorm respectively, unless otherwise specified.

Definition 3.1: A fuzzy subset μ of R is called a *S-fuzzy left ideal* of a hemiring R (briefly, fuzzy left ideal with respect to t-conorm) if it satisfies the following conditions:
$(SFI1)\mu(x + y) \leq S(\mu(x), \mu(y))$,
$(SFI2)\mu(xy) \leq \mu(y)$,for all $x, y \in S$.
S-fuzzy right ideals are defined similarly.

Definition 3.2: A S-fuzzy ideal μ of R is said to be a *S-fuzzy left h-ideal* if it satisfies the following condition:
$(SFI3)x+a+z = b+z$ implies that $\mu(x) \leq S(\mu(a), \mu(b))$, for all $a, b, x, z \in S$.
S-fuzzy right h-ideals are defined similarly.

Definition 3.3: A S-fuzzy left h-ideal μ of R is said to be a *sensible* if it satisfies the sensible property.

Example 3.4: Let R be the set of natural numbers including 0, and R is a hemiring with usual addition and multiplication .Define a fuzzy subset $\mu : R \longrightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} 0 & if\ x\ \ is\ even\ or\ 0, \\ 1 & otherwise. \end{cases}$$

and let $S_m : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ be a function defined by $S_m(\alpha, \beta) = min\{x + y, 1\}$ for all $x, y \in [0, 1]$.Then, S_m is a t-conorm.By routine calculation, we know that μ is a sensible S-fuzzy left h-ideal of R.

Proposition 3.5: Let S be a t-conorm .Then, every sensible S-fuzzy left h-ideal μ of a hemiring R is a anti fuzzy left h-ideal of R.

Proof: The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [9]. ■

Corollary 3.6: If μ is a sensible S-fuzzy left h-ideal of R, then each non-empty level subset $L(\mu; t)$ of μ is a left h-ideal of R.

Proof: Assume that μ is a sensible S-fuzzy left h-ideal of R and $L(\mu; t)$ is a non-empty level subset of μ in R.
(i) Since $L(\mu; t)$ is a non-empty level subset of μ, there exists $x, y \in L(\mu; t)$, $\mu(x + y) \leq S(\mu(x), \mu(y)) = t$.
Thus $x + y \in L(\mu; t)$.
(ii) Let $x, y \in L(\mu; t)$, such that $\mu(xy) \leq \mu(y) \leq t$.
Thus $xy \in L(\mu; t)$.
(iii) Let $a, b, x, z \in L(\mu; t)$, If $x + a + z = b + z$ implies that $\mu(x) \leq S(\mu(a), \mu(b)) = t$.Thus $x \in L(\mu; t)$
Hence,$L(\mu; t)$ is a left h-ideal of R.

■

The following example shows that there exists a t-conorm S such that an anti fuzzy h-ideal of R may not be an sensible S-fuzzy left h-ideal of R.

Example 3.7: Let R be a hemiring in $Example$[3.4].Define a fuzzy subset $\mu : R \longrightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} \frac{1}{5} & if\ x\ \ is\ even\ or\ 0, \\ \\ \frac{1}{3} & otherwise. \end{cases}$$

is an anti fuzzy h-ideal of R.
Let $\nu = (0, 1)$ and define the binary operation S_ν on $(0, 1)$ as follows

$$S_\nu(\alpha, \beta) = \begin{cases} \max\{\alpha, \beta\} & if\ \ \min\{\alpha, \beta\} =\ 0, \\ 0 & \max\{\alpha, \beta\} > 0,\ \alpha + \beta \geq 1 + \nu \\ \nu & otherwise. \end{cases}$$

Then,S_ν is a t-conorm.It is easy to check that μ is a S-fuzzy left h-ideal of R, but

$$S_\nu\left(\mu(0), \mu(0)\right) = S_\nu\left(\frac{1}{5}, \frac{1}{5}\right) = \nu \neq \mu(0).$$

Hence,μ is not a sensible S-fuzzy left h-ideal R.

Now, we consider the following theorem.

Theorem 3.8: Let S be a t-conorm and let μ be a sensible fuzzy subset in a hemiring R, then μ is a sensible S-fuzzy left h-ideal of R if and only if each non-empty level subset $L(\mu; t)$ of μ is a left h-ideal of R.

Proof: The necessary condition can be given by $corollary$[3.6].Coversely,assume that each non-empty level subset $L(\mu; t)$ is a left h-ideal of R.
(i) Let $x, y \in R$.Let if possible,$\mu(x + y) > S(\mu(x), \mu(y))$.Set $t_0 := \frac{1}{2}\{\mu(x + y) + S(\mu(x), \mu(y))\}$,we have $x \in L(\mu; t_0)$ and $y \in L(\mu; t_0)$,since $L(\mu; t)$ is a left h-ideal of R.Then $x + y \in L(\mu; t_0)$ and $\mu(x + y) \leq t_0$,a contradiction.Thus $\mu(x + y) \leq S(\mu(x), \mu(y))$.
(ii) If $x, y \in L(\mu; t)$ then $xy \in L(\mu; t)$.Then
$\mu(xy) \leq \mu(y) \leq t$.Thus $\mu(xy) \leq \mu(y)$.
(iii) Let $a, b, x, z \in R$.If $x + a + z = b + z$ implies that $x \in L(\mu; t)\}$.Define $t = min\{\mu(a), \mu(b)\}$.Then $\mu(x) \leq t = min\{\mu(a), \mu(b)\}$. Thus $\mu(x) =\leq max\{\mu(a), \mu(b)\}$.
Hence,μ is a sensible S-fuzzy left h-ideal of R. ■

Definition 3.9: Let R be a hemiring and a family of fuzzy sets $\{\mu_i | i \in I\}$ in R.Then the union $\left(\bigvee_{i \in I} \mu_i\right)$ of $\{\mu_i | i \in I\}$ is defined by

$$\left(\bigvee_{i \in I} \mu_i\right)(x) = \sup\{\mu_i(x) | i \in I\}$$

Theorem 3.10: If $\{\mu_i | i \in I\}$ is a family of S-fuzzy left h-ideal of R,then $(\bigvee_{i \in I} \mu_i)(x)$ is a S-fuzzy left h-ideal of R.

Proof: Let $\{\mu_i | i \in I\}$ be a family of S-fuzzy left h-ideal of R.

(i)For all $x, y \in R$,we have

$$\left(\bigvee_{i\in I} \mu_i\right)(x+y) = \sup\{\mu_i(x+y)|i \in I\}$$

$$\leq \sup\{S(\mu_i(x), \mu_i(y))\, | i \in I\}$$

$$= S(\sup(\mu_i(x)|i \in I), \sup(\mu_i(y)|i \in I))$$

$$= S\left(\left(\bigvee_{i\in I} \mu_i\right)(x), \left(\vee_{i\in I} \mu_i\right)(y)\right)$$

(ii) For all $x, y \in R$,we have

$$\left(\bigvee_{i\in I} \mu_i\right)(xy) = \sup\{\mu_i(xy)|i \in I\}$$

$$\leq \sup\{S(\mu_i(x))\, | i \in I\}$$

$$= S\left(\left(\bigvee_{i\in I} \mu_i\right)(x)\right)$$

(iii) For all $a, b, x, z \in R$ and if $x + a + z = b + z$ then

$$\left(\bigvee_{i\in I} \mu_i\right)(x) = \sup\{\mu_i(x)|i \in I\}$$

$$\leq \sup\{S(\mu_i(a), \mu_i(b))\, | i \in I\}$$

$$= S(\sup(\mu_i(a)|i \in I), \sup(\mu_i(b)|i \in I))$$

$$= S\left(\left(\bigvee_{i\in I} \mu_i\right)(a), \left(\bigvee_{i\in I} \mu_i\right)(b)\right)$$

Hence $\left(\bigvee_{i\in I} \mu_i\right)$ is a S-fuzzy left h-ideal of R. ■

Definition 3.11: Let $f : R \longrightarrow R'$ be a mapping ,where R and R' are non-empty sets and μ is a fuzzy subset of R.The *preimage* of μ under f written μ^f,is a fuzzy subset of R defined by $\mu^f = \mu(f(x))$,for all $x \in R$.

Theorem 3.12: Let $f : R \longrightarrow R'$ be a homomorphism of hemirings. If μ is a S-fuzzy left h-ideal of R',then μ^f is S-fuzzy left h-ideal of R.

Proof: Suppose μ is a S-fuzzy left h-ideal of R',then

(i) For all $x, y \in R$,we have

$$\begin{aligned}\mu^f(x+y) &= \mu(f(x+y)) = \mu(f(x) + f(y)) \\ &\leq S(\mu(f(x)), \mu(f(y))) \\ &= S(\mu^f(x), \mu^f(y))\end{aligned}$$

(ii)For all $x, y \in R$,we have

$$\begin{aligned}\mu^f(xy) &= \mu(f(xy)) = \mu(f(x)f(y)) \\ &\leq \mu(f(y)) = \mu^f(y)\end{aligned}$$

(iii)For all $a, b, x, z \in R$ and if $x + a + z = b + z$ then

$$\begin{aligned}\mu^f(x) &= \mu(f(x)) \\ &\leq S(\mu(f(a)), \mu(f(b))) \\ &= S(\mu^f(a), \mu^f(b))\end{aligned}$$

■

Hence μ^f is a S-fuzzy left h-ideal of R.

Theorem 3.13: Let $f : R \longrightarrow R'$ be a homomorphism of hemirings. If μ^f is a S-fuzzy left h-ideal of R ,then μ is S-fuzzy left h-ideal of R'.

Proof: Suppose μ is a S-fuzzy left h-ideal of R',then

(i)Let $x', y' \in R'$,there exists $x, y \in R$ such that $f(x) = x'$ and $f(y) = y'$,we have

$$\begin{aligned}\mu(x'+y') &= \mu(f(x) + f(y)) \\ &= \mu(f(x+y)) \\ &= \mu^f(x+y) \\ &\leq S(\mu^f(x), \mu^f(y)) \\ &= S(\mu(f(x)), \mu(f(y))) \\ &= S(\mu(x'), \mu(y'))\end{aligned}$$

(ii)Let $x', y' \in R'$,there exists $x, y \in R$ such that $f(x) = x'$ and $f(y) = y'$,we have

$$\begin{aligned}\mu(x'y') &= \mu(f(x)f(y)) = \mu(f(xy)) \\ &= \mu^f(xy) \\ &\leq \mu^f(y) \\ &= \mu(f(y)) \\ &= \mu(y')\end{aligned}$$

(iii)Let $a', b', x', z' \in R'$,there exists $a, b, x, z \in R$ such that $f(a) = a', f(b) = b', f(x) = x', f(z) = z'$.If $x' + a' + z' = b' + z'$.Then $f(x+a+z) = f(b+z)$ and so $f(x)+f(a)+f(z) = f(b) + f(z)$.It follows that

$$\begin{aligned}\mu(x') &= \mu(f(x)) \\ &= \mu^f(x) \\ &\leq S(\mu^f(a), \mu^f(b)) \\ &= S(\mu(f(a)), \mu(f(b))) \\ &= S(\mu(a'), \mu(b'))\end{aligned}$$

Hence μ is a S-fuzzy left h-ideal of R'. ■

Definition 3.14: Let f be a mapping defined on R.If ν is a fuzzy subset in $f(R)$,then the fuzzy subset $\mu = \nu \circ f$ in R(i.e., the fuzzy subset defined by $\mu(x) = \nu(f(x))$ for all $x \in R$) is called the *preimage* of ν under f.

Proposition 3.15: An onto homomorphic preimage of a S-fuzzy left h-ideal R is S-fuzzy left h-ideal.

Proof: The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [9 ,Proposition 3.10]. ■

Let μ be a fuzzy subset in a hemiring R and f be a mapping defined on R.Then the fuzzy subset μ^f in $f(R)$ defined by $\mu^f(y) = \inf_{x\in f^{-1}(y)} \mu(x)$ for all $y \in f(R)$ is called the *image* of μ under f.A fuzzy subset μ in R is said to have an *inf property* if for every subset $H \subseteq R$,there exists $h_0 \in H$ such that $\mu(h_0) = \inf_{h\in H} \mu(h)$.

Proposition 3.16: An onto homomorphic image of S-fuzzy left h-ideal with inf property is S-fuzzy left h-ideal.

Proof: Let $f : R \longrightarrow R'$ be an onto homomorphism of semirings and let μ be a S-fuzzy left h-ideal of R with the inf property.

(i)Given $x', y' \in R'$,we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \inf_{h\in f^{-1}(x')} \mu(h), \;\; \mu(y_0) = \inf_{h\in f^{-1}(y')} \mu(h)$$

respectively.Then , we have

$$\mu^f(x'+y') = \inf_{z\in f^{-1}(x'+y')} \mu(z) \le \max\{\mu(x_0), \mu(y_0)\}$$

$$\le S(\mu(x_0), \mu(y_0))$$

$$= S\left(\inf_{h\in f^{-1}(x')} \mu(h), \inf_{h\in f^{-1}(y')} \mu(h)\right)$$

$$= S\left(\mu^f(x'), \mu^f(y')\right)$$

(ii)Given $x', y' \in R'$,we let $x_0 \in f^{-1}(x')$ and $y_0 \in f^{-1}(y')$ be such that

$$\mu(x_0) = \inf_{h\in f^{-1}(x')} \mu(h), \ \ \mu(y_0) = \inf_{h\in f^{-1}(y')} \mu(h)$$

respectively.Then , we have

$$\begin{aligned} \mu^f(x'y') &= \inf_{z\in f^{-1}(x'y')} \mu(z) \le \mu(y_0) \\ &= \inf_{h\in f^{-1}(y')} \mu(h) = \mu^f(y') \end{aligned}$$

(ii) Given $a', b', x', y' \in R'$, we let $a_0 \in f^{-1}(a')$, $b_0 \in f^{-1}(b')$, $x_0 \in f^{-1}(x')$, $z_0 \in f^{-1}(z')$ be such that

$$\begin{aligned} \mu(a_0) &= \inf_{h\in f^{-1}(a')} \mu(h), \ \ \mu(b_0) = \inf_{h\in f^{-1}(b')} \mu(h) \\ \mu(x_0) &= \inf_{h\in f^{-1}(x')} \mu(h), \ \ \mu(z_0) = \inf_{h\in f^{-1}(z')} \mu(h) \end{aligned}$$

respectively.If $x'+a'+z' = b'+z'$ then $x_0+a_0+z_0 = b_0+z_0$, where $(x_0+a_0+z_0) \in f^{-1}(x'+a'+z')$ and $(b_0+z_0) \in f^{-1}(b'+z')$,we have

$$\mu^f(x') = \inf_{z\in f^{-1}(x')} \mu(z) \le \max\{\mu(a_0), \mu(b_0)\}$$

$$= S\left(\inf_{h\in f^{-1}(a')} \mu(h), \inf_{h\in f^{-1}(b')} \mu(h)\right)$$

$$= S\left(\mu^f(a'), \mu^f(b')\right)$$

■

Hence,μ^f is a S-fuzzy left h-ideal of R'.

Definition 3.17: A $t-$conorm S on $[0,1]$ is called a continuous $t-$conorm if S is a continuous function from $[0,1] \times [0,1] \longrightarrow [0,1]$ with respect to usual topology.

We observe that the function " max " is always a continuous t-conorm

Proposition 3.18: Let S be a continuous $t-$conorm and let f be a homomorphism on a hemiring R.If μ is a S-fuzzy left h-ideal of R,then μ^f is a S-fuzzy left h-ideal of $f(R)$.

Proof: Let $A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1+y_2)$,where $y_1+y_2 \in f(R)$. Consider the set

$$A_1+A_2 = \{x \in R | x = a_1 + a_2 \ for \ some \ a_1 \in A_1, a_2 \in A_2\}$$

If $x \in A_1 + A_2$,then $x = x_1 + x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that we have $f(x) = f(x_1+x_2) = f(x_1) + f(x_2) = y_1 + y_2$,that is , $x \in f^{-1}(y_1+y_2) = A_{12}$.Thus , $A_1 + A_2 \subseteq A_{12}$.It follows that

$$\begin{aligned} \mu^f(y_1+y_2) &= \inf\left\{\mu(x) | x \in f^{-1}(x_1+x_2)\right\} \\ &= \inf\{\mu(x) | x \in A_{12}\} \\ &\le \inf\{\mu(x) | x \in A_1 + A_2\} \\ &\le \inf\{\mu(x_1+x_2) | x_1 \in A_1, x_2 \in A_2\} \\ &\le \inf\{S(\mu(x_1), \mu(x_2)) | x_1 \in A_1, x_2 \in A_2\} \end{aligned}$$

Since S is continuous for every $\epsilon > 0$,we see that if

$$inf\{\mu(x_1)|x_1 \in A_1\} - x_1^\star \le \delta \text{ and}$$
$$inf\{\mu(x_2)|x_2 \in A_2\} - x_2^\star \le \delta\text{,then}$$

$$S(\inf\{\mu(x_1)|x_1 \in A_1\}, \inf\{\mu(x_2)|x_2 \in A_2\}) - S(x_1^*, x_2^*) \le \epsilon$$

Choose $a_1 \in A_1$ and $a_2 \in A_2$,such that

$$inf\{\mu(x_1)|x_1 \in A_1\} - \mu(a_1) \le \delta \text{ and}$$
$$inf\{\mu(x_2)|x_2 \in A_2\} - \mu(a_2) \le \delta\text{,then}$$

$$S(\inf\{\mu(x_1)|x_1 \in A_1\}, \inf\{\mu(x_2)|x_2 \in A_2\})$$
$$-S(\mu(a_1), \mu(a_2)) \le \varepsilon$$

Thus,we have

$$\begin{aligned} (i)\mu^f(y_1+y_2) &\le \inf\{S(\mu(x_1), \mu(x_2)) | x_1 \in A_1, x_2 \in A_2\} \\ &= S(\inf\{\mu(x_1)|x_1 \in A_1\}, \inf\{\mu(x_2)|x_2 \in A_2\}) \\ &= S\left(\mu^f(y_1), \mu^f(y_2)\right) \end{aligned}$$

(ii)Similarly, we can prove that

$$\mu^f(y_1y_2) \le \mu^f(y_2)$$

(iii) Now , let $a_1, b_1, x_1, z_1 \in f(R)$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$.we can prove that

$$\mu^f(x_1) \le S\left(\mu^f(a_1), \mu^f(b_1)\right)$$

Hence,μ^f is a S-fuzzy left h-ideal of $f(R)$.

■

Lemma 3.19: Let T be a t-norm.Then t-conorm S can be defined as

$$S(x,y) = 1 - T(1-x, 1-y).$$

Proof: Straightforward. ■

Theorem 3.20: A fuzzy subset μ of R is a T-fuzzy left h-ideal if and only if its complement μ^c is a S-fuzzy left h-ideal of R.

Proof: Let μ be a T-fuzzy left h-ideal of R.

(i) For all $x, y \in R$,we have

$$\begin{aligned} \mu^c(x+y) &= 1 - \mu(x+y) \\ &\le 1 - T(\mu(x), \mu(y)) \\ &= 1 - T(1-\mu^c(x), 1-\mu^c(y)) \\ &= S(\mu^c(x), \mu^c(y)) \end{aligned}$$

(ii)For all $x, y \in R$,we have

$$\mu^c(xy) = 1 - \mu(xy) \le 1 - \mu(y) = \mu^c(y)$$

(iii)For all $a, b, x, z \in R$,If $x + a + z = b + z$ such that

$$\begin{aligned} \mu^c(x) &= 1 - \mu(x) \\ &\le 1 - T(\mu(a), \mu(b)) \\ &= 1 - T(1-\mu^c(a), 1-\mu^c(b)) \\ &= S(\mu^c(a), \mu^c(b)) \end{aligned}$$

Hence $\mu^c(x)$ is a S-fuzzy left h-ideal of R.
The converse is proved similarly. ■

IV. Chain conditions

Let μ and ν be a fuzzy subset in a hemiring R.Then the $S{-}h$-product of μ and ν is defined by

$$\mu\circ_h\nu\,(x)=\begin{cases}\inf\left(S\left(\mu(a_i),\mu(b_i)\right)\mid i=1,2\right)\\ \qquad if\ x\ can\ be\ expressed\ as\\ \qquad x+a_1b_1+z=a_2b_2+z,\\ 0\qquad otherwise.\end{cases}$$

Proposition 4.1: Let μ and ν be a fuzzy subset of R.If they are S-fuzzy left h-ideal of R,then so $\mu\cup\nu$,where $\mu\cup\nu$ is defined by $(\mu\cup\nu)(x)=S(\mu(x),\nu(x))$ for all $x,\in R$.Moreover,If μ and ν are a S-fuzzy right h-ideal and a S-fuzzy left h-ideal respectively,then $\mu\circ_h\nu\subseteq\mu\cup\nu$

Proof: The proof is obtained dually by using the notion of t-conorm S instead of t-norm T in [9,proposition 4.2]. ■

Theorem 4.2: Let μ be a fuzzy subset in R and $Im(\mu)=\{\alpha_0,\alpha_1,...,\alpha_k\}$,where $\alpha_i<\alpha_j$ whenever $i>j$.Let $\{A_n|n=0,1,...,k\}$ be a family of ideals of R such that
(i) $A_0\subset A_1\subset...\subset A_k=R$,
(ii) $\mu(A^*)=\alpha_n$,where $A_n^*=A_n\setminus A_{n-1},A_{-1}=\phi$ for $n=0,1,...,k$.
Then μ is a S-fuzzy left h-ideal of R.

Proof: Suppose $\{A_n|n=0,1,...,k\}$ be a family of ideals of R.
(i) For all $x,y\in R$,Then we discuss the following cases:If $x+y\in A_n$ and $y\in A_n$ such that $x\in A_n$,since A_n is an ideal of R.thus

$$\mu(x+y)\le\alpha_n=S(\mu(x),\mu(y)).$$

If $x+y\notin A_n^*$ and $y\notin A_n^*$,then the following four cases arise:

1) $x+y\in R\setminus A_n$ and $y\in R\setminus A_n$
2) $x+y\in A_{n-1}$ and $y\in A_{n-1}$
3) $x+y\in R\setminus A_n$ and $y\in A_{n-1}$
4) $x+y\in A_{n-1}$ and $y\in R\setminus A_n$

But,in either cases,we know that

$$\mu(x+y)\le S(\mu(x),\mu(y)).$$

If $x+y\in R\setminus A_n^*$ and $y\notin A_n^*$ then either $y\in A_{n-1}$ or $y\in R\setminus A_n$. It follows that either $x\in A_n$ or $x\in R\setminus A_n$.Thus

$$\mu(x+y)\le S(\mu(x),\mu(y)).$$

If $x+y\notin R\setminus A_n^*$ and $y\in A_n^*$ then by similar process we have

$$\mu(x+y)\le S(\mu(x),\mu(y)).$$

(ii) Similarly, for $x,y\in R$,we have

$$\mu(xy)\le\mu(y).$$

(iii)For all $a,b,x,z\in R$,If $x+a+z=b+z$ such that $a\in A_n$ and $b\in A_n$ then $x\in A_n$.By the above process it is easy to show that

$$\mu(x)\le S(\mu(a),\mu(b)).$$

Hence μ is a S-fuzzy left h-ideal of R. ■

Theorem 4.3: Let $\{A_n|n\in N\}$ be a family of h-ideals of a hemiring R which is nested,that is,$R=A_1\supset A_2\supset$ ....Let μ be a fuzzy subset in R defined by

$$\mu\left(x\right)=\begin{cases}\frac{1}{n+1} & if\ x\in A_n\backslash A_{n+1},n=1,2,3...,\\ 0 & if\ x\in\bigcap\limits_{n=1}^{\infty}A_n\,.\end{cases}$$

for all$x\in R$.Then μ is a S-fuzzy left h-ideal of R.

Proof: Let $x,y\in R$.
(i)Suppose that $x\in A_k\setminus A_{k+1}$ and $y\in A_r\setminus A_{r+1}$ for $k=1,2,...;r=1,2,...$ .Without loss of generality,we may assume that $k\le r$.Then $x+y\in A_k$ and so

$$\mu\left(x+y\right)\le\frac{1}{k+1}=\max\left\{\mu\left(x\right),\mu\left(y\right)\right\}\le S\left(\mu\left(x\right),\mu\left(y\right)\right)$$

If $x,y\in\bigcap\limits_{n=1}^{\infty}A_n$ then $x+y\in\bigcap\limits_{n=1}^{\infty}A_n$ and thus

$$\mu\left(x+y\right)=0=S\left(\mu\left(x\right),\mu\left(y\right)\right)$$

If $x\in\bigcap\limits_{n=1}^{\infty}A_n$ then $y\notin\bigcap\limits_{n=1}^{\infty}A_n$,then there exists $i\in N$ such that $y\in A_i\setminus A_{i+1}$.It follows that $x+y\in A_i$ so that

$$\mu\left(x+y\right)\le\frac{1}{i+1}=\max\left\{\mu\left(x\right),\mu\left(y\right)\right\}\le S\left(\mu\left(x\right),\mu\left(y\right)\right)$$

Similarly,we know that

$$\mu\left(x+y\right)\le S\left(\mu\left(x\right),\mu\left(y\right)\right)$$

for all $x\notin\bigcap\limits_{n=1}^{\infty}A_n$ then $y\in\bigcap\limits_{n=1}^{\infty}A_n$.

(ii)Now,if $y\in A_r\backslash A_{r+1}$ for some $k=1,2,...$,then $xy\in A_k$ for all $x\in R$.Thus

$$\mu\left(x+y\right)\le\frac{1}{k+1}=\mu\left(y\right)$$

If $y\in\bigcap\limits_{n=1}^{\infty}A_n$ then $xy\in\bigcap\limits_{n=1}^{\infty}A_n$ for all $x\in R$.Thus

$$\mu\left(xy\right)=0=\mu\left(y\right)$$

(iii) Now,let $a,b,x,z\in R$ be such that $x+a+z=b+z$.If $a,b\in A_r\setminus A_{r+1}$ for some $r=1,2,3...$, then $x\in A_r$ as A_r is a left h-ideal of R.Thus

$$\mu\left(x\right)\le\frac{1}{k+1}=\max\left\{\mu\left(a\right),\mu\left(b\right)\right\}\le S\left(\mu\left(a\right),\mu\left(b\right)\right)$$

If $a,b\in\bigcap\limits_{n=1}^{\infty}A_n$ then $x\in\bigcap\limits_{n=1}^{\infty}A_n$ and so

$$\mu\left(x\right)=0=S\left(\mu\left(a\right),\mu\left(b\right)\right)$$

Assume that $a\in A_r\setminus A_{r+1}$ for some $r=1,2,3,...$,and $b\in\bigcap\limits_{n=1}^{\infty}A_n$(or , $a\in\bigcap\limits_{n=1}^{\infty}A_n$ and $b\in A_r\setminus A_{r+1}$ for some $r=1,2,3...$).Then $x\in A_r$ and so

$$\mu\left(x\right)\le\frac{1}{r+1}=\max\left\{\mu\left(a\right),\mu\left(b\right)\right\}\le S\left(\mu\left(a\right),\mu\left(b\right)\right)$$

Hence, μ is a S-fuzzy left h-ideal of R. ■

Let $\mu : R \longrightarrow [0,1]$ be a fuzzy subset of R.The smallest S-fuzzy left h-ideal containing μ is called the S-fuzzy left h-ideal generated by μ, and μ is said to be n-valued if $\mu(R)$ is a finite set of n elements.When no specific n is intended, we call μ a finite-valued fuzzy subset.

Theorem 4.4: A S-fuzzy left h-ideal ν of R is finite valued if and only if a finite-valued fuzzy subset μ of R is generated by ν .

Proof: If $\nu : R \longrightarrow [0,1]$ is a finite-valued S-fuzzy left h-ideal of R,then one may choose $\mu = \nu$.Consequently, assume that $\mu : R \longrightarrow [0,1]$ is a n-valued fuzzy subset with n distinct values $t_1, t_2, ..., t_n$,where $t_1 < t_2 < ... < t_n$.Let G^i be the inverse image of t_i under μ, that is,$G^i = \mu^{-1}(t_i)$.Obviously, $\bigcup\limits_{i=1}^{j} G^i \subseteq \bigcup\limits_{i=1}^{r} G^i$ when $j < r$.We denote by A^j the left h-ideal of R generated by the set $\bigcup\limits_{i=1}^{j} G^i$.Then we have the following chain of left h-ideals:

$$A^1 \supseteq A^2 \supseteq ... \supseteq A^n = R$$

Define a fuzzy $\nu : R \longrightarrow [0,1]$ by

$$\nu(x) = \begin{cases} t_n & if \in A^n, \\ t_j & if \ \in A^j \backslash A^{j-1}; j = 1, 2, ..., n-1 \end{cases}$$

We claim that ν is a S-fuzzy left h-ideal of R and μ is generated by ν.Let $x, y \in R$ and let i and j be the largest integer such that $x \in A^i$ and $y \in A^j$.we may assume that $i < j$ without loss of generality.Then $x+y \in A^i$ and $xy \in A^i$ and so

$$\begin{aligned} \nu(x+y) &\leq t_j = \max\{t_i, t_j\} = \max\{\nu(x), \nu(y)\} \\ &\leq S(\nu(x), \nu(y)) \end{aligned}$$

and

$$\nu(xy) \leq t_j = \nu(y)$$

Now,let $a, b, x, z \in R$ be such that $x+a+z = b+z$.If $a \in A^i$ and $b \in A^j$ for some $i < j$,then $a, b \in A^i$ and so $x \in A^i$ as A^i is a h-ideal of R.Thus

$$\begin{aligned} \nu(x) &\leq t_j = \max\{t_i, t_j\} = \max\{\nu(a), \nu(b)\} \\ &\leq S(\nu(a), \nu(b)) \end{aligned}$$

Hence, μ is a S-fuzzy left h-ideal of R.

If $x \in R$ and $\mu(x) = t_j$,then $x \in G^j$ and so $x \in A^j$.But we get $\nu(x) \leq t_j = \mu(x)$.Consequently, $\nu \subseteq \mu$.Let γ be any S-fuzzy left h-ideal of R which is a subset of μ.Then, $\bigcup\limits_{i=1}^{j} G^i = L(\gamma; t_j) \subseteq L(\mu; t_j)$, and thus $A^j \subseteq L(\gamma; t_j)$.Hence,$\gamma \subseteq \mu$ and μ is generated by ν.Note that $\mid Im\mu \mid = n = \mid Im\nu \mid$.Thus completing the proof. ■

A semiring R is a said to be *left h-artinian* (see [9]) if it satisfies the descending chain condition on left h-ideals of R.

Theorem 4.5: If R is a h-artinian hemiring, then every S-fuzzy left h-ideal of R is finite valued.

Proof: Let $\mu : R \longrightarrow [0,1]$ be a S-fuzzy left h-ideal of R which is not finite valued.Then,there exists sequence of distinct numbers $\mu(0) = t_1 > t_2 > ... > t_n$,where $t_1 = \mu(x_i)$ for some $x_i \in R$. This sequence induces an infinite sequence of distinct left h-ideals of R:

$$L(\mu; t_1) \supset L(\mu; t_2) \supset ... \supset L(\mu; t_n) \supset ... \ .$$

This is a contradiction.

■

Combining Theorem 8 and Theorem 9,we have the following corollary.

Corollary 4.6: If R is a h-artinian hemiring, then every S-fuzzy left h-ideal of R is generated by a finite fuzzy subset in R.

V. S-PRODUCT OF S-FUZZY LEFT h-IDEALS

Definition 5.1: (see [2]) A fuzzy relation on any set R is a fuzzy subset $\mu : R \times R$.

Definition 5.2: Let S be a t-conorm . If μ is a fuzzy relation on a set R and ν is a fuzzy set in R,then μ is a S-fuzzy relation on ν if $\mu_\nu(x,y) \geq S(\nu(x), \nu(y))$, $for\ all\ x, y \in R$

Definition 5.3: Let S be a t-conorm . Let μ and ν be a fuzzy subset of R . Then direct S-product of μ and ν is defined by $(\mu \times \nu)(x,y) = S(\mu(x), \nu(y))$, $for\ all\ x, y \in R$

Lemma 5.4: Let S be a t-conorm .Let μ and ν be a fuzzy subset of R .Then,

(i) $\mu \times \nu$ is a S-fuzzy relation on S.

(ii) $L(\mu \times \nu; t) = L(\mu; t) \times L(\nu; t)$,for all $t \in [0,1]$

Proof: The proof is obvious. ■

Definition 5.5: Let S be a t-conorm .Let μ be a fuzzy subset of R,then μ is said to be the strongest S-fuzzy relation on R if $\mu_\nu(x,y) \geq S(\nu(x), \nu(y))$, $for\ all\ x, y \in R$

Lemma 5.6: For given fuzzy subset ν in a set R,let μ_ν be the strongest S-fuzzy relation on R.Then

$L(\mu_\nu; t) = L(\mu; t) \times L(\nu; t)$, for all $t \in [0,1]$.

Proof: The proof is obvious. ■

Proposition 5.7: For given fuzzy subset ν in a set R,let μ_ν be the strongest S-fuzzy relation on R. If μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$,then $\nu(a) \geq \nu(0)$ for $a \in R$.

Proof: If μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$,then $\mu_\nu(a,a) \geq \mu_\nu(0,0)$ for $a \in R$.This means $S(\nu(a), \nu(a)) \geq S(\nu(0), \nu(0))$ for $a \in R$.Since μ is sensible,then $\nu(a) \geq \nu(0)$ for $a \in R$. ■

The following proposition is an immediate consequence of lemma 5.6.

Proposition 5.8: Let μ and ν be S-fuzzy left h-ideal of R,then the level left h-ideals of μ_ν are given by $L(\mu_\nu; t) = L(\mu; t) \times L(\nu; t)$, for all $t \in R$.

Theorem 5.9: Let S be a t-conorm. Let μ and ν be S-fuzzy left h-ideal of R,then $\mu \times \nu$ is a S-fuzzy left h-idealof $R \times R$.

Proof: Suppose μ and ν be S-fuzzy left h-ideal of R.Let $\mu \times \nu$ is a S-fuzzy left h-idealof $R \times R$.Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any element of $R \times R$.Then,

(i)

$$\begin{aligned} (\mu \times \nu)(x+y) &= (\mu \times \nu)((x_1, x_2) + (y_1, y_2)) \\ &= (\mu \times \nu)((x_1 + y_1, x_2 + y_2)) \\ &= S(\mu(x_1 + y_1), \nu(x_2 + y_2)) \\ &\leq S(S(\mu(x_1), \mu(y_1))), S(\nu(x_2), \nu(y_2))) \\ &= S(S(\mu(x_1), \nu(x_2))), S(\mu(y_1), \nu(y_2))) \\ &= S((\mu \times \nu)(x_1, x_2), (\mu \times \nu)(y_1, y_2)) \\ &= S((\mu \times \nu)(x), (\mu \times \nu)(y)) \end{aligned}$$

(ii)

$$\begin{aligned}(\mu\times\nu)(xy) &= (\mu\times\nu)((x_1,x_2)(y_1,y_2))\\ &= (\mu\times\nu)((x_1y_1,x_2y_2))\\ &= S(\mu(y_1),\nu(y_2))\\ &= (\mu\times\nu)(y_1,y_2)\\ &= (\mu\times\nu)(y)\end{aligned}$$

(iii) Let $x=(x_1,x_2)$,$z=(z_1,z_2)$, $a=(a_1,a_2)$ and $b=(b_1,b_2)$ be such that $x_1+a_1+z_1=b_1+z_1$ and $x_2+a_2+z_2=b_2+z_2$.Then,

$$\begin{aligned}(\mu\times\nu)(x) &= (\mu\times\nu)((x_1,x_2))\\ &= S(\mu(x_1),\nu(x_2))\\ &\le S(S(\mu(a_1),\mu(b_1)),S(\nu(a_2),\nu(b_2)))\\ &= S(S(\mu(a_1),\nu(a_2)),S(\mu(b_1),\nu(b_2)))\\ &= S((\mu\times\nu)(a_1,a_2),(\mu\times\nu)(b_1,b_2))\\ &= S((\mu\times\nu)(a),(\mu\times\nu)(b))\end{aligned}$$

Thus, $\mu\times\nu$ is a S-fuzzy left h-ideal of $R\times R$. ■

Corollary 5.10: Let S be a t-conorm. Let μ and ν be a sensible S-fuzzy left h-ideal of R,then $\mu\times\nu$ is a sensible S-fuzzy left h-ideal of $R\times R$.

Proof: By Theorem 5.9,we have $\mu\times\nu$ is a S-fuzzy left h-ideal of $R\times R$.Let $x=(x_1,x_2)$ be any element in $R\times R$,then

$$\begin{aligned}(\mu\times\nu)(x) &= (\mu\times\nu)((x_1,x_2))\\ &= S(\mu(x_1),\nu(x_2))\\ &= S(S(\mu(x_1),\mu(x_1)),S(\nu(x_2),\nu(x_2)))\\ &= S(S(\mu(x_1),\nu(x_2)),S(\mu(x_1),\nu(x_2)))\\ &= S((\mu\times\nu)(x_1,x_2),(\mu\times\nu)(x_1,x_2))\\ &= S((\mu\times\nu)(x),(\mu\times\nu)(x))\end{aligned}$$

Hence, $\mu\times\nu$ is a sensible S-fuzzy left h-ideal of $R\times R$. ■

As the converse of Corollary 5.10,we have a following question: If $\mu\times\nu$ is a sensible S-fuzzy left h-ideal of $R\times R$,then are both μ and ν sensible S-fuzzy left h-ideal of R? The following example gives a negative answer.

Example 5.11: Let R be a hemiring with $|R|\ge 2$ and let $t\in[0,1]$.Define a sensible fuzzy subset μ and ν in R by $\mu(x)=1$ and

$$\nu(x)=\begin{cases}1 & if\, x=0,\\ t & otherwise.\end{cases}$$

for all $x\in R$,respectively.

If $x=0$,then $\nu(x)=1$, and thus

$$(\mu\times\nu)(x,x)=S(\mu(x),\nu(x))=S(1,1)=1$$

If $x\neq 0$,then $\nu(x)=t$, and thus

$$(\mu\times\nu)(x,x)=S(\mu(x),\nu(x))=S(1,t)=1$$

That is,$\mu\times\nu$ is a constant function,and so $\mu\times\nu$ is a sensible S-fuzzy left h-ideal of $R\times R$.Now,μ is a sensible S-fuzzy left h-ideal of R,but ν is not a sensible S-fuzzy left h-ideal of R,since for $x\neq 0$,we have $\nu(0)=1>t=\nu(x)$.

Now,we generalize the product of two S-fuzzy left h-ideal of R to the product of n S-fuzzy left h-ideal.we first need to generalize the domain of t-conorm R to $\prod_{i=1}^{n}[0,1]$ as follows.

Definition 5.12: The function $S_n:\prod_{i=1}^{n}[0,1]\to[0,1]$ is defined by

$S_n(\alpha_1,\alpha_2,\dots,\alpha_n)=$

$$S(\alpha_i,S_{n-1}(\alpha_1,\alpha_2,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n))$$

for all $1\le i\le n$,where $n\ge 2$ $S_2=S$ and $S_1=identity$.

Lemma 5.13: For a t-conorm S and every $\alpha_i,\beta_i\in[0,1]$, where $1\le i\le n$,$n\ge 2$,we have

$$\begin{aligned}&S_n(S(\alpha_1,\beta_1),S(\alpha_2,\beta_2),\dots,S(\alpha_n,\beta_n))\\ &\quad = S(S_n(\alpha_1,\alpha_2,\dots,\alpha_n),S_n(\beta_1,\beta_2,\dots,\beta_n)).\end{aligned}$$

Proposition 5.14: Let S be a t-conorm. Let $\{R_i\}_{i=1}^{n}$ be the finite collection of hemirings and $R=\prod_{n=1}^{n}R_i$ the S-product of S_i.Let μ_i be a S-fuzzy left h-ideal of S_i,where $1\le i\le n$. Then, $\mu=\prod_{n=1}^{n}\mu_i$ defined by

$$\begin{aligned}\mu(x_1,x_2,\dots,x_n) &= \prod_{i=1}^{n}\mu_i(x_1,x_2,\dots,x_n)\\ &= S_n(\mu_1(x_1),\mu_2(x_2),\dots,\mu_n(x_n))\end{aligned}$$

for all $x_1,x_2,\dots,x_n\in R$ is a S-fuzzy left h-ideal of R.

Proof: The proof is similar to the proof of Theorem 10. ■

Definition 5.15: Let μ and ν be fuzzy subset in R.Then,the S-product of μ and ν,written as

$$[\mu\,.\,\nu]_S(x)=S(\mu(x),\nu(x))$$

for all $x\in R$.

Theorem 5.16: Let μ and ν be S-fuzzy left h-idealof R.If S^* is a t-conorm which dominates S,that is,

$$S^*(S(\alpha,\beta),S(\gamma,\delta))\ge S(S^*(\alpha,\beta),S^*(\gamma,\delta))$$

for all $\alpha,\beta,\gamma,\delta\in R$.Then S^*-product of μ and ν,$[\mu\,.\,\nu]_{S^*}$ is a S-fuzzy left h-ideal of S.

Proof: Let $x,y\in R$,then we have

(i)

$$\begin{aligned}[\mu\,.\,\nu]_{S^*}(x+y) &= S^*(\mu(x+y),\nu(x+y))\\ &\le S^*(S(\mu(x),\mu(y)),S(\nu(x),\nu(y)))\\ &\le S^*(S(\mu(x),\nu(x)),S(\mu(y),\nu(y)))\\ &= S([\mu\,.\,\nu]_{S^*}(x),[\mu\,.\,\nu]_{S^*}(y))\end{aligned}$$

(ii)

$$\begin{aligned}[\mu\,.\,\nu]_{S^*}(xy) &= S^*(\mu(xy),\nu(xy))\\ &\le S^*(\mu(y),\nu(y))\\ &= [\mu\,.\,\nu]_{S^*}(y)\end{aligned}$$

(iii) Now,let $a,b,x,z\in R$ be such that $x+a+z=b+z$.Then

$$\begin{aligned}[\mu\,.\,\nu]_{S^*}(x) &= S^*(\mu(x),\nu(x))\\ &\le S^*(S(\mu(a),\mu(b)),S(\nu(a),\nu(b)))\\ &\le S^*(S(\mu(a),\nu(a)),S(\mu(b),\nu(b)))\\ &= S([\mu\,.\,\nu]_{S^*}(a),[\mu\,.\,\nu]_{S^*}(b))\end{aligned}$$

Hence,$[\mu\,.\,\nu]_{S^*}$ is a S-fuzzy left h-ideal of R. ■

Theorem 5.17: Let $R\longrightarrow R'$ be an onto homomorphism of hemirings.Let S^* be a t-conorm such that S^* dominates S.Let μ and ν be S-fuzzy left h-ideal of S'.If $[\mu\,.\,\nu]_{S^*}$ is the S^*-product of μ and ν, and $\left[f^{-1}(\mu)\,.\,f^{-1}(\nu)\right]_{S^*}$ is the S^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$,then

$$f^{-1}([\mu\,.\,\nu]_{S^*})=\left[f^{-1}(\mu)\,.\,f^{-1}(\nu)\right]_{S^*}$$

Proof: Let $x \in R$,then we have

$$\begin{aligned} f^{-1}\left([\mu \,.\, \nu]_{S^*}\right)(x) &= [\mu \,.\, \nu]_{S^*}(f(x)) \\ &= S^*\left(\mu\left(f(x)\right), \nu\left(f(x)\right)\right) \\ &= S^*\left(f^{-1}\left(\mu(x)\right), f^{-1}\left(\nu(x)\right)\right) \\ &= \left[f^{-1}(\mu) \,.\, f^{-1}(\nu)\right]_{S^*}(x) \end{aligned}$$

■

Theorem 5.18: Let ν be a sensible fuzzy subset of R. Let μ_ν be the strongest S-fuzzy relation on R.Then ν is a sensible S-fuzzy left h-ideal of R if and only if μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$.

Proof: Suppose that ν is a sensible S-fuzzy left h-ideal of R.Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $R \times R$.Then,

(i)

$$\begin{aligned} \mu_\nu(x+y) &= \mu_\nu\left((x_1,x_2)+(y_1,y_2)\right) \\ &= \mu_\nu\left((x_1+y_1),(x_2+y_2)\right) \\ &= S\left(\nu(x_1+y_1),\nu(x_2+y_2)\right) \\ &\le S\left(S\left(\nu(x_1),\nu(y_1)\right), S\left(\nu(x_2),\nu(y_2)\right)\right) \\ &= S\left(S\left(\nu(x_1),\nu(x_2)\right), S\left(\nu(y_1),\nu(y_2)\right)\right) \\ &= S\left(\mu_\nu\left(x_1,x_2\right), S\left(\mu_\nu\left(y_1,y_2\right)\right)\right) \\ &= S\left(\mu_\nu(x), S\left(\mu_\nu(y)\right)\right) \end{aligned}$$

(ii)

$$\begin{aligned} \mu_\nu(xy) &= \mu_\nu\left((x_1,x_2)(y_1,y_2)\right) \\ &= \mu_\nu\left((x_1y_1,x_2y_2)\right) \\ &= S\left(\nu(x_1y_1),\nu(x_2y_2)\right) \\ &\le \mu_\nu\left((x_1,x_2)(y_1,y_2)\right) \\ &= \mu_\nu\left(y_1,y_2\right) \\ &= \mu_\nu(y) \end{aligned}$$

(iii) Let $x = (x_1, x_2)$, $z = (z_1, z_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be such that $x_1 + a_1 + z_1 = b_1 + z_1$ and $x_2 + a_2 + z_2 = b_2 + z_2$.Then,

$$\begin{aligned} \mu_\nu(x) &= \mu_\nu\left((x_1,x_2)\right) \\ &= S\left(\nu(x_1),\nu(x_2)\right) \\ &\le S\left(S\left(\nu(a_1),\nu(b_1)\right), S\left(\nu(a_2),\nu(b_2)\right)\right) \\ &= S\left(S\left(\nu(a_1),\nu(a_2)\right), S\left(\nu(b_1),\nu(b_2)\right)\right) \\ &= S\left(\mu_\nu\left(a_1,a_2\right), S\left(\mu_\nu\left(b_1,b_2\right)\right)\right) \\ &= S\left(\mu_\nu(a), S\left(\mu_\nu(b)\right)\right) \end{aligned}$$

Thus, μ_ν is a S-fuzzy left h-ideal of $R \times R$.

(iv) For any $x = (x_1, x_2) \in R \times R$,then

$$\begin{aligned} S\left(\mu_\nu(x), \mu_\nu(x)\right) &= S\left(\mu_\nu\left(x_1,x_2\right), \mu_\nu\left(x_1,x_2\right)\right) \\ &= S\left(S\left(\nu(x_1),\nu(x_2)\right), S\left(,\nu(x_1),\nu(x_2)\right)\right) \\ &= S\left(S\left(\nu(x_1),\nu(x_1)\right), S\left(,\nu(x_2),\nu(x_2)\right)\right) \\ &= S\left(\nu(x_1),\nu(x_2)\right) \\ &= \mu_\nu\left(x_1,x_2\right) \\ &= \mu_\nu(x) \end{aligned}$$

Hence, μ_ν is a sensible S-fuzzy left h-ideal of R.

Conversely,suppose that μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$. Let $x, y \in R$, we have

(i)

$$\begin{aligned} \nu(x+y) &= S\left(\nu(x+y), \nu(x+y)\right) \\ &= \mu_\nu(x+y, x+y) \\ &= \mu_\nu\left((x,x)+(y,y)\right) \\ &\le S\left(\mu_\nu(x,x), \mu_\nu(y,y)\right) \\ &= S\left(S\left(\nu(x), \nu(x)\right), S\left(\nu(y), \nu(y)\right)\right) \\ &= S\left(\nu(x), \nu(y)\right) \end{aligned}$$

(ii)

$$\begin{aligned} \nu(xy) &= S\left(\nu(xy), \nu(xy)\right) \\ &= \mu_\nu(xy, xy) \\ &\le \mu_\nu(y,y) \\ &= S\left(\nu(y), \nu(y)\right) \\ &= \nu(y) \end{aligned}$$

(iii)Let $a, b, x, z \in R$ be such that $(x, x) + (a, a) + (z, z) = (b, b) + (z, z)$.Since μ_ν is a sensible S-fuzzy left h-ideal of $R \times R$.Then

$$\begin{aligned} \nu(x) &= S\left(\nu(x), \nu(x)\right) \\ &= \mu_\nu(x,x) \\ &= \mu_\nu\left((x,x)+(y,y)\right) \\ &\le S\left(\mu_\nu(a,a), \mu_\nu(b,b)\right) \\ &= S\left(S\left(\nu(a), \nu(a)\right), S\left(\nu(b), \nu(b)\right)\right) \\ &= S\left(\nu(a), \nu(b)\right) \end{aligned}$$

Consequently,ν is a sensible S-fuzzy left h-ideal of R.

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