Ruin probability for a Markovian risk model with two-type claims

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Abstract—In this paper, a Markovian risk model with two-type claims is considered. In such a risk model, the occurrences of the two type claims are described by two point processes $\{N_i(t), t \ge 0\}$, i = 1, 2, where $\{N_i(t), t \ge 0\}$ is the number of jumps during the interval (0, t] for the Markov jump process $\{X_i(t), t \ge 0\}$. The run probability $\Psi(u)$ of a company facing such a risk model is mainly discussed. An integral equation satisfied by the run probability $\Psi(u)$ is obtained and the bounds for the convergence rate of the run probability $\Psi(u)$ are given by using key-renewal theorem.

Keywords-Risk model; Ruin probability; Markov jump process; Integral equation

I. INTRODUCTION

The classical risk model has been extensively studied since the work of Cramer[3], and has been generalized to various Markovian risk model which have been studied extensively [1, 2, 4, 7]. Recently, many authors have studied continuoustime risk models involving two classes of claims. Yuen et al. [9] consider the non-ruin probability for a correlated risk process involving two dependent classes of insurance risks, with exponential claims, which can be transformed into a surplus process with two independent classes of insurance risks, for which one claim number process is Poisson and the other is a renewal process with Erlang(2) claim inter-arrival times. Li and Garrido [5] consider a risk process with two classes of independent risks, namely the compound Poisson process and the renewal process with generalized Erlang(2) inter-arrivals times. A further extension was given by Li and Lu [6]. They derive a system of integro-differential equations for the Gerber-Shiu expected discounted penalty functions, when the ruin is caused by a claim belonging either to the first or to the second class and obtained explicit results when the claim sizes are exponentially distributed. Zhang et al. [8] extended the model of Li and Lu [6], by considering the claim number process of the second class to be a renewal process with generalized Erlang(n) inter-arrival times.

In this paper, we mainly consider a Markovian risk model with two-type claims. Integral equation for the ruin probability is found and the bounds for the convergence rate of the ruin probability are given.

Let (Ω, \mathcal{F}, P) be a complete probability space containing all objects defined in the following, $(S_i, B_i)(i = 1, 2)$ be two measurable spaces where S_i is a subset of real line R and B_i

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is a Borel σ -algebra on S_i . Consider the risk model

$$U(t) = u + ct - \sum_{k=1}^{N_1(t)} Y_k - \sum_{k=1}^{N_2(t)} Z_k,$$
(1)

where $u = U(0) \ge 0$ is the initial surplus, c > 0 is the premium income rate, $\{Y_k, k \ge 1\}$ are i.i.d nonnegative random sequence with common distribution function F_1 and mean value μ_1 ; $\{Z_k, k \ge 1\}$ are also i.i.d nonnegative random sequence but with common distribution function F_2 and mean value μ_2 , $Y = \{Y_k, k \ge 1\}$ and $Z = \{Z_k, k \ge 1\}$ denote the two-type claim processes; $N_i(t)$ is the number of jumps during the interval (0, t] for the Markov jump process $X_i = \{X_i(t), t \ge 0\}$ on space S_i with bounded intensity function $\lambda_i(x)$ and jumping measure $Q_i(x, B)$. Throughout this paper, we always assume that X_i is stationary ergodic with initial stationary distribution $q_i(\cdot)$, i.e., $\int_B \lambda_i(x)q_i(dx) =$ $\int_{S_i} \lambda_i(x)Q_i(x, B)q_i(dx)$ and X_1, X_2, Y, Z are mutually independent.

Let

$$\begin{split} T &= \inf\{t \geq 0 : U(t) < 0\}, (\inf \Phi = \infty) \\ \Psi(u) &= \mathbf{P} \left(T < \infty | U(0) = u\right), \\ R(u) &= 1 - \Psi(u), \\ \Psi_x(u) &= \mathbf{P} \left(T < \infty | U(0) = u, X_1(0) = x\right), \\ R_x(u) &= 1 - \Psi_x(u), x \in \mathcal{S}_1, \\ \tilde{\Psi}_y(u) &= \mathbf{P} \left(T < \infty | U(0) = u, X_2(0) = y\right), \\ \tilde{R}_y(u) &= 1 - \tilde{\Psi}_y(u), y \in \mathcal{S}_2, \\ \Psi_{xy}(u) &= \mathbf{P} \left(T < \infty | U(0) = u, X_1(0) = x, X_2(0) = y\right), \\ R_{xy} &= 1 - \Psi_{xy}(u), x \in \mathcal{S}_1, y \in \mathcal{S}_2. \end{split}$$

We call T the time of ruin, $\Psi(u)$ the ruin probability, R(u) the survival probability. Obviously, we have

$$\Psi(u) = \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \Psi_{xy}(u) q_2(\mathrm{d}y) q_1(\mathrm{d}x)$$
$$= \int_{\mathcal{S}_1} \Psi_x(u) q_1(\mathrm{d}x)$$
$$= \int_{\mathcal{S}_2} \widetilde{\Psi}_y(u) q_2(\mathrm{d}y).$$

Let

$$\rho = \frac{c - \mu_1 \int_{\mathcal{S}_1} \lambda_1(x) q_1(\mathrm{d}x) - \mu_2 \int_{\mathcal{S}_2} \lambda_2(x) q_2(\mathrm{d}x)}{\mu_1 \int_{\mathcal{S}_1} \lambda_1(x) q_1(\mathrm{d}x) + \mu_2 \int_{\mathcal{S}_2} \lambda_2(x) q_2(\mathrm{d}x)}$$

be the relative security loading. Throughout the paper, we always assume that $\rho > 0$.

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II. INTEGRAL EQUATION OF RUIN PROBABILITY

Lemma 2.1 Under the assumption that $\rho > 0$, we have $\lim_{u \to 0} \Psi(u) = 0$.

Proof Put Y(t) = U(t) - u, since $\lim_{t\to\infty} \frac{N_i(t)}{t} = \int_{S_i} \lambda_i(x) q_i(\mathrm{d}x), i = 1, 2$, then

$$\lim_{t \to \infty} \frac{Y_t}{t} = \lim_{t \to \infty} \left(c - \frac{1}{t} \sum_{k=1}^{N_1(t)} Y_k - \frac{1}{t} \sum_{k=1}^{N_2(t)} Z_k \right)$$
$$= c - \lim_{t \to \infty} \frac{1}{N_1(t)} \sum_{k=1}^{N_1(t)} Y_k \cdot \frac{N_1(t)}{t}$$
$$- \lim_{t \to \infty} \frac{1}{N_2(t)} \sum_{k=1}^{N_2(t)} Z_k \cdot \frac{N_2(t)}{t}$$
$$= c - \mu_1 \int_{\mathcal{S}_1} \lambda_1(x) q_1(\mathrm{d}x) - \mu_2 \int_{\mathcal{S}_2} \lambda_2(x) q_2(\mathrm{d}x).$$

By the theory of Markov process and the assumption that $\lambda_i(x)$ is bounded, it is clear that X_i has only finite jumps during the interval $(0, \tau]$, thus $\inf_{t \ge 0} Y_t$ is finite with probability one and thus

$$\lim_{u\to\infty}\Psi(u)=\lim_{u\to\infty}\mathsf{P}(\inf_{t\geq 0}(u+Y(t)<0)=0,$$

then Lemma 2.1 is proved.

Corollary 2.1 For $\Psi_x(u), x \in S_1$, $\widetilde{\Psi}_y(u), y \in S_2$ we have

$$\lim_{u \to \infty} \Psi_x(u) = 0, \quad q_1(\cdot) - a.e. \quad x \in \mathcal{S}_1;$$
$$\lim_{u \to \infty} \widetilde{\Psi}_y(u) = 0, \quad q_2(\cdot) - a.e. \quad y \in \mathcal{S}_2.$$

Proof Since $\Psi(u) = \int_{S_1} \Psi_x(u) q_1(dx) = \int_{S_2} \widetilde{\Psi}_y(u) q_2(dy)$, by the dominated convergence theorem, we can get

$$0 = \lim_{u \to \infty} \Psi(u) = \int_{\mathcal{S}_1} \lim_{u \to \infty} \Psi_x(u) q_1(\mathrm{d}x)$$
$$= \int_{\mathcal{S}_2} \lim_{u \to \infty} \widetilde{\Psi}_y(u) q_2(\mathrm{d}y).$$

Obviously, it is that $\lim_{u\to\infty}\tilde{\Psi}_x(u)>0, q_1(\cdot)-a.e. \ x\in\mathcal{S}_1 \text{ and}$ $\lim_{u\to\infty}\tilde{\Psi}_y(u)>0, q_2(\cdot)-a.e. \ y\in\mathcal{S}_2, \text{ thus}$

$$\lim_{u \to \infty} \Psi_x(u) = 0, \quad q_1(\cdot) - a.e. \quad x \in \mathcal{S}_1;$$
$$\lim_{u \to \infty} \widetilde{\Psi}_y(u) = 0, \quad q_2(\cdot) - a.e. \quad y \in \mathcal{S}_2,$$

the proof of Corollary 2.1 is completed.

In the following, by using the backward differential technique, we give an integral equation satisfied by the ruin probability $\Psi(u)$.

Theorem 2.1 If the relative security loading $\rho > 0$, then

$$\begin{split} \Psi(0) &= \frac{1}{c} \left(\mu_1 \int_{\mathcal{S}_1} \lambda_1(x) q_1(\mathrm{d}x) + \mu_2 \int_{\mathcal{S}_2} \lambda_2(x) q_2(\mathrm{d}x) \right), \\ \Psi(u) &= \frac{1}{c} \int_{\mathcal{S}_1} \lambda_1(x) q_1(\mathrm{d}x) \int_u^\infty \overline{F}_1(z) \mathrm{d}z \\ &+ \frac{1}{c} \int_0^u \left[\int_{\mathcal{S}_1} \lambda_1(x) \Psi_x(u-z) q_1(\mathrm{d}x) \right] \overline{F}_1(z) \mathrm{d}z \\ &+ \frac{1}{c} \int_{\mathcal{S}_2} \lambda_2(x) q_2(\mathrm{d}x) \int_u^\infty \overline{F}_2(z) \mathrm{d}z \\ &+ \frac{1}{c} \int_0^u \left[\int_{\mathcal{S}_2} \lambda_2(y) \widetilde{\Psi}_y(u-z) q_2(\mathrm{d}y) \right] \overline{F}_2(z) \mathrm{d}z, \end{split}$$

where $\overline{F}_i(z) = 1 - F_i(z), i = 1, 2.$

Proof Using the backward differential technique, we have

$$R_{xy}(u) = (1 - \lambda_1(x)\Delta)(1 - \lambda_2(y)\Delta)R_{xy}(u + c\Delta) + \lambda_1(x)\Delta(1 - \lambda_2(y)\Delta) \int_{\mathcal{S}_1} Q_1(x, \mathrm{d}x_1) \times \int_0^{u+c\Delta} R_{x_1y}(u + c\Delta - z)\mathrm{d}F_1(z) + \lambda_2(y)\Delta(1 - \lambda_1(x)\Delta) \int_{\mathcal{S}_2} Q_2(y, \mathrm{d}y_1) \times \int_0^{u+c\Delta} R_{xy_1}(u + c\Delta - z)\mathrm{d}F_2(z) + \circ(\Delta).$$
(2)

Thus

$$cR'_{xy}(u) = (\lambda_1(x) + \lambda_2(y))R_{xy}(u) -\lambda_1(x) \int_{\mathcal{S}_1} Q_1(x, \mathrm{d}x_1) \int_0^u R_{x_1y}(u-z)\mathrm{d}F_1(z) -\lambda_2(y) \int_{\mathcal{S}_2} Q_2(y, \mathrm{d}y_1) \int_0^u R_{xy_1}(u-z)\mathrm{d}F_2(z).$$

Replacing u by t and integrating from t = 0 to t = u, we obtain

$$c(R_{xy}(u) - R_{xy}(0)) = \lambda_{1}(x) \int_{0}^{u} R_{xy}(t) dt -\lambda_{1}(x) \int_{0}^{u} \int_{S_{1}} Q_{1}(x, dx_{1}) \int_{0}^{t} R_{x_{1}y}(t-z) dF_{1}(z) dt +\lambda_{2}(y) \int_{0}^{u} R_{xy}(t) dt -\lambda_{2}(y) \int_{0}^{u} \int_{S_{2}} Q_{2}(y, dy_{1}) \int_{0}^{t} R_{xy_{1}}(t-z) dF_{2}(z) dt =\lambda_{1}(x) \int_{0}^{u} R_{xy}(t) dt -\lambda_{1}(x) \int_{S_{1}} Q_{1}(x, dx_{1}) \int_{0}^{u} R_{x_{1}y}(t) dt +\lambda_{1}(x) \int_{S_{1}} Q_{1}(x, dx_{1}) \int_{0}^{u} \overline{F}_{1}(z) R_{x_{1}y}(u-z) dz +\lambda_{2}(y) \int_{0}^{u} R_{xy}(t) dt -\lambda_{2}(y) \int_{S_{2}} Q_{2}(y, dy_{1}) \int_{0}^{u} R_{xy_{1}}(t) dt +\lambda_{2}(y) \int_{S_{2}} Q_{2}(y, dy_{1}) \int_{0}^{u} \overline{F}_{2}(z) R_{xy_{1}}(u-z) dz.$$
(3)

Integrating both sides of Eq.(3) about $q_1(\cdot)$ and $q_2(\cdot)$, we get

$$c[R(u) - R(0)] = \int_0^u \left[\int_{\mathcal{S}_1} \lambda_1(x) R_x(u-z) q_1(\mathrm{d}x) \right] \overline{F}_1(z) \mathrm{d}z + \int_0^u \left[\int_{\mathcal{S}_2} \lambda_2(y) \widetilde{R}_y(u-z) q_2(\mathrm{d}y) \right] \overline{F}_2(z) \mathrm{d}z$$

Let $t \to \infty$ in the above equation, by the dominated convergence theorem, then

$$c[R(\infty) - R(0)] = \int_0^\infty \left[\int_{\mathcal{S}_1} \lambda_1(x) R_x(\infty) q_1(\mathrm{d}x) \right] \overline{F}_1(z) \mathrm{d}z \\ + \int_0^\infty \left[\int_{\mathcal{S}_2} \lambda_2(y) \widetilde{R}_y(\infty) q_2(\mathrm{d}y) \right] \overline{F}_2(z) \mathrm{d}z.$$

It follows from Corollary2.1 that

$$c\Psi(0) = \mu_1 \int_{\mathcal{S}_1} \lambda_1(x)q_1(\mathrm{d}y) + \mu_2 \int_{\mathcal{S}_2} \lambda_2(y)q_2(\mathrm{d}y),$$

then

$$\Psi(0) = \frac{1}{c} \left(\mu_1 \int_{\mathcal{S}_1} \lambda_1(x) q_1(\mathrm{d}x) + \mu_2 \int_{\mathcal{S}_2} \lambda_2(x) q_2(\mathrm{d}x) \right),$$

and

$$\begin{split} \Psi(u) &= \Psi(0) - \frac{1}{c} \times \\ & \int_0^u \left[\int_{\mathcal{S}_1} \lambda_1(x) (1 - \Psi_x(u - z)) q_1(\mathrm{d}x) \right] \overline{F}_1(z) \mathrm{d}z \\ & - \frac{1}{c} \int_0^u \left[\int_{\mathcal{S}_2} \lambda_2(y) (1 - \widetilde{\Psi}_y(u - z) q_2(\mathrm{d}y) \right] \overline{F}_2(z) \mathrm{d}z \\ &= \frac{1}{c} \int_{\mathcal{S}_1} \lambda_1(x) q_1(\mathrm{d}x) \int_u^\infty \overline{F}_1(z) \mathrm{d}z \\ & + \frac{1}{c} \int_0^u \left[\int_{\mathcal{S}_1} \lambda_1(x) \Psi_x(u - z) q_1(\mathrm{d}x) \right] \overline{F}_1(z) \mathrm{d}z \\ & + \frac{1}{c} \int_{\mathcal{S}_2} \lambda_2(x) q_2(\mathrm{d}x) \int_u^\infty \overline{F}_2(z) \mathrm{d}z \\ & + \frac{1}{c} \int_0^u \left[\int_{\mathcal{S}_2} \lambda_2(y) \widetilde{\Psi}_y(u - z) q_2(\mathrm{d}y) \right] \overline{F}_2(z) \mathrm{d}z. \end{split}$$

Thus the theorem is completed.

III. BOUNDS FOR CONVERGENCE RATE OF RUIN PROBABILITY

Let $h_i(r) = \int_0^{+\infty} e^{rx} dF_i(x) - 1, \widetilde{\lambda}_i = \sup_{x \in S_i} \{\lambda_i(x)\}, \widehat{\lambda} = \inf_{x \in S_i} \{\lambda_i(x)\}, i = 1, 2$. In the following, we assume that

$$\widetilde{\rho} = \frac{c}{\widetilde{\lambda}_1 \mu_1 + \widetilde{\lambda}_2 \mu_2} - 1 > 0, \\ \widehat{\rho} = \frac{c}{\widehat{\lambda}_1 \mu_1 + \widehat{\lambda}_2 \mu_2} - 1 > 0,$$

and assume that there exists a real number $r_{\infty} > 0$ such that $h_i(r) \to \infty$ when $r \to \infty$ (we allow for the possibility $r_{\infty} = \infty$).

Lemma 3.1 Under the above assumptions, there exist $\widetilde{R}, \widehat{R}$ such that

$$\frac{\widetilde{\lambda}_1}{c}h_1(\widetilde{R}) + \frac{\widetilde{\lambda}_2}{c}h_2(\widetilde{R}) = \widetilde{R}, \qquad \frac{\widehat{\lambda}_1}{c}h_1(\widehat{R}) + \frac{\widehat{\lambda}_2}{c}h_2(\widehat{R}) = \widehat{R}.$$

The proof of Lemma 3.1 is omitted.

Theorem 3.1 For the probability $\Psi(u)$, we have

$$\limsup_{u \to \infty} e^{\widetilde{R}u} \Psi(u) \le \frac{1 + \widetilde{\rho}}{\left(1 + \widetilde{\rho}\right) \left(\frac{\widetilde{\lambda}_1}{c} h_1'(\widetilde{R}) + + \frac{\widetilde{\lambda}_2}{c} h_2'(\widetilde{R}) - 1\right)},\tag{4}$$

$$\liminf_{u \to \infty} e^{\widehat{R}u} \Psi(u) \ge \frac{1+\widehat{\rho}}{\left(1+\widehat{\rho}\right) \left(\frac{\widehat{\lambda}_1}{c} h_1'(\widehat{R}) + +\frac{\widehat{\lambda}_2}{c} h_2'(\widehat{R}) - 1\right)}.$$
(5)

Proof By theorem 2.1, we have

$$\begin{split} \Psi(u) &\leq \frac{\widetilde{\lambda}_1}{c} \int_u^\infty \overline{F}_1(z) \mathrm{d}z + \frac{\widetilde{\lambda}_2}{c} \int_u^\infty \overline{F}_2(z) \mathrm{d}z \\ &+ \frac{\widetilde{\lambda}_1}{c} \int_0^u \Psi(u-z) \overline{F}_1(z) \mathrm{d}z + \frac{\widetilde{\lambda}_2}{c} \int_0^u \Psi(u-z) \overline{F}_2(z) \mathrm{d}z \\ &= \frac{\widetilde{\lambda}_1}{c} \int_u^\infty \overline{F}_1(z) \mathrm{d}z + \frac{\widetilde{\lambda}_2}{c} \int_u^\infty \overline{F}_2(z) \mathrm{d}z \\ &+ \int_0^u \Psi(u-z) \left(\frac{\widetilde{\lambda}_1}{c} \overline{F}_1(z) + \frac{\widetilde{\lambda}_2}{c} \overline{F}_2(z) \right) \mathrm{d}z. \end{split}$$

Multiplying the above inequality by e^{Ru} , we have

$$e^{\tilde{R}u}\Psi(u)$$

$$\leq \frac{\tilde{\lambda}_1}{c}e^{\tilde{R}u}\int_u^{\infty}\overline{F}_1(z)\mathrm{d}z + \frac{\tilde{\lambda}_2}{c}e^{\tilde{R}u}\int_u^{\infty}\overline{F}_2(z)\mathrm{d}z$$

$$+ \int_0^u e^{\tilde{R}(u-z)}\Psi(u-z)e^{\tilde{R}z}\left(\frac{\tilde{\lambda}_1}{c}\overline{F}_1(z) + \frac{\tilde{\lambda}_2}{c}\overline{F}_2(z)\right)\mathrm{d}z$$

Thus, by lemma 3.1, we have that

$$\int_0^\infty e^{\widetilde{R}z} \left(\frac{\widetilde{\lambda}_1}{c} \overline{F}_1(z) + \frac{\widetilde{\lambda}_2}{c} \overline{F}_2(z) \right) \mathrm{d}z = 1,$$

and then

$$0 \leq \lim_{u \to \infty} \frac{\widetilde{\lambda}_1}{c} e^{\widetilde{R}u} \int_u^{\infty} \overline{F}_1(z) dz + \frac{\widetilde{\lambda}_2}{c} e^{\widetilde{R}u} \int_u^{\infty} \overline{F}_2(z) dz$$
$$\leq \lim_{u \to \infty} \int_u^{\infty} e^{\widetilde{R}z} \left(\frac{\widetilde{\lambda}_1}{c} \overline{F}_1(z) + \frac{\widetilde{\lambda}_2}{c} \overline{F}_2(z) \right) dz = 0,$$

so by the key-renewal theorem, we obtain

$$\limsup_{u \to \infty} e^{\tilde{R}u} \Psi(u) \le \frac{c_1}{c_2}$$

where

$$c_{1} = \int_{0}^{\infty} e^{\tilde{R}u} \int_{u}^{\infty} \left(\frac{\tilde{\lambda}_{1}}{c} \overline{F}_{1}(z) + \frac{\tilde{\lambda}_{2}}{c} \int_{u}^{\infty} \overline{F}_{2}(z) \right) dz du,$$
$$c_{2} = \int_{0}^{\infty} z e^{\tilde{R}z} \left(\frac{\tilde{\lambda}_{1}}{c} \overline{F}_{1}(z) + \frac{\tilde{\lambda}_{2}}{c} \int_{u}^{\infty} \overline{F}_{2}(z) \right) dz.$$

So from the two above equations, we can get

$$c_1 = \frac{\widetilde{\rho}}{\widetilde{R}(1+\widetilde{\rho})}, c_2 = \frac{1}{\widetilde{R}} \left(\frac{\widetilde{\lambda}_1}{c} h_1'(\widetilde{R}) + \frac{\widetilde{\lambda}_2}{c} h_2'(\widetilde{R}) - 1 \right).$$

Then the proof of (4) is completed.

We can get the proof of (5) by imitating the above proof of (4). \Box

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