

# Robust Coherent Noise Suppression by Point Estimation of the Cauchy Location Parameter

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**Abstract**—This paper introduces a new point estimation algorithm, with particular focus on coherent noise suppression, given several measurements of the device under test where it is assumed that 1) the noise is first-order stationary and 2) the device under test is linear and time-invariant. The algorithm exploits the robustness of the Pitman estimator of the Cauchy location parameter through the initial scaling of the test signal by a centred Gaussian variable of pre-determined variance. It is illustrated through mathematical derivations and simulation results that the proposed algorithm is more accurate and consistently robust to outliers for different tailed density functions than the conventional methods of sample mean (coherent averaging technique) and sample median search.

**Keywords**—Central limit theorem, Fisher-Cramer Rao, gamma function, Pitman estimator.

## I. INTRODUCTION

In this paper, we introduce a new noise suppression algorithm for efficient estimation of the linear time invariant (LTI) response of the device under test (DUT) in the presence of unknown but first-order stationary noise sources, given several DUT measurements. The aim is to have a consistent and more accurate noise suppression technique than the conventional methods utilizing the sample mean (time coherent averaging) and sample median estimation of the LTI response signals [1]-[4] for wide range of noise density functions. By limiting noise density functions of varying tail thicknesses to a Cauchy density function as opposed to a Gaussian, thus overcoming the limitations of the central limit theorem (CLT) for heavy tailed density functions [5],[6], the robust and accurate Cauchy location parameter (CLP) pitman estimator [7], [8] is used to estimate the noise free (or LTI) signals given several DUT measurements. The result is a method that is at least as accurate as any of the two conventional methods depending on the pre-determined parameters in the DUT testing phase.

In impulse response testing of DUTs, there is usually some amount of additive noise such that the signal captured by the transducer is

$$x(n) = h(n) * s(n) + \varepsilon(n), \quad n \in [0, N-1]. \quad (1)$$

where  $s(n)$  is the excitation signal to the DUT of impulse response  $h(n)$ , with  $*$  as the convolution operator. If the signal-to-noise ratio (SNR) is adequate then  $h(n) * s(n)$  is

dominant and the resulting measurement may be an acceptable reflection of the DUT's impulse response. However, this is not always the case and noise suppression methods are required to enhance the SNR so that the results relate to the system being measured and not corrupted by extraneous noise sources. The conventional method for enhancing the SNR given a LTI impulse response is time coherent averaging of several measurements [1]-[3]. Let

$$x_m(n) = h(n) * s(n) + \varepsilon_m(n), \quad n \in [0, N-1]$$

be the  $m$ -th measurement signal, for  $m \in [0, M-1]$ , where  $\varepsilon_m(n)$  is the additive noise observed for this particular run. The averaged measurement is

$$\overline{x(n)} = \frac{1}{M} \sum_{m=0}^{M-1} [h(n) * s(n) + \varepsilon_m(n)]. \quad (2)$$

If the noise signals  $\varepsilon_0(n), \varepsilon_1(n), \dots, \varepsilon_{M-1}(n)$  are independent and identically distributed (i.i.d), then it can be shown that

$$\overline{x(n)} = h(n) * s(n) + \sqrt{\frac{\varepsilon^2(n)}{M}}, \quad (3)$$

where  $\overline{\varepsilon^2(n)}$  is the mean squared value of the observed noise sequence. Thus in general, by increasing the number of measurements  $M$ , the root mean square (RMS) error to the LTI signal of interest  $h(n) * s(n)$  can be reduced. However, the underlying noise properties are usually unknown and can be non-stationary at the time of measurement, and this raises questions as to whether (2) is the most efficient noise suppression method for any noise density functions.

Let  $X_n$  be the variable of the time coherent sample (TCS) observations at the discrete time index  $n$  across all the  $M$  measurement signals of the DUT response. Then,  $\overline{x(n)}$  in (2) is the TCS-mean  $\widehat{\mu_n}$  estimation of the LTI signal of interest  $h(n) * s(n)$ , thus we may infer the efficiency of  $\overline{x(n)}$  by considering properties of the sample mean on estimating the location parameter of noise density functions with different tail thicknesses. In [4], it is illustrated via the use of power exponential density functions that the sample mean is consistently outperformed by the sample median for leptokurtic densities (thicker tails than the Gaussian density). It is also shown that the sample mean is the minimum-variance estimator for the location parameter of the Gaussian density function. These results follow directly from the CLT, where by the law of large numbers it is the i.i.d. platykurtic variables that usually average out to a Gaussian

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variable [5]. The averaged sum of leptokurtic variables does not necessarily reduce to a Gaussian variable, but usually to another leptokurtic variable especially if their density functions are heavy tailed [6].

Based on the CLT, it is often assumed that background noise, which is usually the sum of several noise sources, is approximately Gaussian and this encourages the use of the TCS-mean for estimation of the LTI signals. However, it is possible to have a single (or very few) dominant non-Gaussian noise sources which by the law of large numbers does not imply that the net density function is Gaussian. Even if there are a large number of noise sources, their variances (say around  $\sigma_\varepsilon^2$ ) must be bounded and small because by the Fisher-Cramer Rao (FCR) [4], the lower bound of the uncertainty associated with the TCS mean is such that

$$\lim_{M \rightarrow \infty} [M \cdot \text{var}(\hat{\mu}_n)] \rightarrow \sigma_\varepsilon^2 \quad (4)$$

In relation to (3),  $\sigma_\varepsilon^2$  is equivalent to  $\overline{\varepsilon^2(n)}$  which means that for noise sources with large variances the RMS error would be large for the averaging technique. One consistent location parameter estimator is the sample median [4]. Though robust to outliers, it is never really the minimum variance estimator and thus not always the most efficient noise suppression method.

The major problem here is that different density functions can have different minimum-variance estimators for their location parameter estimators, and the underlying net noise density function is usually unknown and can be non-stationary during the time of measurement. To address this problem, this paper is structured as follows: in Section II, we introduce the proposed noise suppression algorithm where different noise density functions are approximated to a Cauchy density function in the time-domain given a LTI impulse response of the DUT. Section III covers the simulation results comparing this algorithm to the TCS-mean and TCS-median. Summary remarks and discussions are presented in Section IV.

## II. THE PROPOSED ALGORITHM

There are three major steps to estimating the LTI signals  $h(n) * x(n)$ , for all  $n \in [0, N-1]$ .

### A. Gaussian Variable Scaling of the Test Signal

For every transducer captured signal  $x_m(n)$ ,  $m \in [0, M-1]$ , a realization of the Gaussian variable  $Z \sim N(0, \sigma_z^2)$  is used to scale the test signal such that

$$x_m(n) = z_m h(n) * s(n) + \varepsilon_m(n), \quad n \in [0, N-1] \quad (5)$$

It is important that  $Z$  is Gaussian and centred because the ratio of two centred Gaussians is a Cauchy variable.

### B. The Ratio Output Signals

Derive the ratio output signal  $y_m(n)$ , for  $m \in [0, M-1]$ , as

$$y_m(n) = \frac{x_m(n)}{z_m} = h(n) * s(n) + \frac{\varepsilon_m(n)}{z_m}, \quad n \in [0, N-1] \quad (6)$$

Using (6), we can form the matrix

$$\mathbf{Y} = \begin{pmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{M-1} \end{pmatrix} = \begin{pmatrix} Q_0 & Q_1 & \dots & Q_{N-1} \end{pmatrix}$$

where the row variable  $Y_m$ , for  $m \in [0, M-1]$ , is realized by sample values of the  $m$ -th signal as given by (6). The column variable  $Q_n$ , for  $n \in [0, N-1]$ , is realized by the time coherent values  $y_m(n)$ , for  $m \in [0, M-1]$ , which is the set of values at the same discrete time index  $n$  from the  $M$  output signals as given by (6). Thus we refer to the  $Q_n$  as a TCS variable.

### C. Noise Suppression using the CLP Pitman Estimator

In (6),  $h(n) * x(n)$  is invariant throughout the TCS values of  $Q_n$ . It is  $\varepsilon_m(n)$  and  $z_m$  that can change from sample value to sample value. Let  $\xi_n := \{\varepsilon_0(n), \varepsilon_1(n), \dots, \varepsilon_{M-1}(n)\}$  be the TCS noise variable at the discrete time index  $n$ , then the density function of  $Q_n$  is defined by the ratio sample  $R_n = \frac{\xi_n}{Z}$  shifted by the constant  $h(n) * x(n)$ .

The most efficient location parameter estimator for  $Q_n$  is determined by the type of density function of  $R_n$ . If the location parameter of  $R_n$  is  $\mu_n$ , then for  $Q_n$  it is  $\mu_n + h(n) * x(n)$ . For first-order stationary noise, the location parameters of the  $Q_n$  are shifted by the same amount  $\mu_n$ . Therefore, without loss of generality let  $\mu_\varepsilon = 0$  (thus  $\mu_z = 0$ ) such that the location parameter for  $Q_n$  is the LTI value  $h(n) * x(n)$ . As already pointed out, there are two types of noise density functions to consider:

1) *Platykurtic Noise Densities:* As the sum of i.i.d. platykurtic variables tend to a Gaussian variable, for this class of noise densities averaging is an effective method for noise suppression. Since  $h(n) * x(n)$  is a constant throughout the TCS variables, the efficiency of the Cauchy location parameter's (CLP) Pitman estimator is determined by its uncertainty in estimation of the location parameter of  $R_n$ , whereas that for the TCS-mean and TCS-median is determined by their estimation of the location parameter of the TCS noise variable  $\xi_n$ . Therefore, in the performance analysis the value of interest  $h(n) * x(n)$  for each TCS variable shall be left out.

*Theorem 1:* The CLP Pitman estimator  $\hat{c}_n$  for the location parameter of the TCS ratio variable  $R_n = \frac{\xi_n}{Z}$ , where  $\xi_n$  is the sum of i.i.d. platykurtic variables and  $Z \sim N(0, \sigma_z^2)$ , is at least as efficient as the sample mean  $\hat{\mu}_n$  estimation of the location parameter of  $\xi_n$  for  $\sigma_z^2 \geq 2$ .

*Proof:* By the CLT,  $\xi_n$  tends to a Gaussian variable of

mean  $\mu_\varepsilon = 0$  and variance  $\sigma_\varepsilon^2$ . The density function of the ratio sample  $R_n$  is given by

$$p(r_n) = \int_{-\infty}^{\infty} |z| \cdot p(z, \varepsilon_n) dz.$$

Since  $p(z, \varepsilon_n) = p(z)p(\varepsilon_n) = p(z)p(r_n z)$ , then

$$p(r_n) = \int_{-\infty}^{\infty} |z| \cdot p(z)p(r_n z) dz.$$

Substituting for the Gaussian densities results in

$$\begin{aligned} p(r_n) &= \frac{1}{2\pi\sigma_z\sigma_\varepsilon} \int_{-\infty}^{\infty} |z| \cdot e^{-z^2(\frac{1}{2\sigma_z^2} + \frac{r_n^2}{2\sigma_\varepsilon^2})} dz \\ &= \frac{1}{\pi\sigma_z\sigma_\varepsilon} \int_0^{\infty} z \cdot e^{-z^2(\frac{1}{2\sigma_z^2} + \frac{r_n^2}{2\sigma_\varepsilon^2})} dz \end{aligned}$$

Using the integral  $\int_0^{\infty} z \cdot e^{-az^2} dz = \frac{1}{2a}$ ,

$$p(r_n) = \frac{\frac{\sigma_\varepsilon}{\sigma_z}}{\pi[(\frac{\sigma_\varepsilon}{\sigma_z})^2 + r_n^2]}$$

which means that  $R_n$  is a centred Cauchy variable of scale parameter  $u_n = \frac{\sigma_\varepsilon}{\sigma_z} \in (0, \infty)$ . From [7], let the CLP Pitman estimator  $\hat{c}_n$  for the location parameter of  $R_n$  be defined as

$$\hat{c}_n = \sum_{m=0}^{M-1} r_{n,m} \frac{\Re(\Psi_{n,m})}{\sum_{m=0}^{M-1} \Re(\Psi_{n,m})}, \quad (7)$$

where  $\Re(\Psi_{n,m})$  denotes the real part of

$$\Psi_{n,m} = \prod_{l \neq m} \left[ \frac{1}{(r_{n,m} - r_{n,l})^2 + 4u_n^2} \right] \left[ 1 - \frac{2u_n}{(r_{n,m} - r_{n,l})} \sqrt{-1} \right], \quad (8)$$

and  $r_{n,m}$  is  $m$ -th sample value from the variable  $R_n$ , for  $m \in [0, M-1]$ . In [7], it is shown that the uncertainty associated with the CLP Pitman estimator is bounded by  $\frac{2u_n^2}{M}$ , therefore by the law of large numbers the uncertainty associated with the Pitman estimator for the Cauchy variable  $R_n$  is such that

$$\lim_{M \rightarrow \infty} [M \cdot \text{var}(\hat{c}_n)] \rightarrow 2u_n^2 = \frac{2\sigma_\varepsilon^2}{\sigma_z^2} \quad (9)$$

From (4) and (9), the relative efficiency  $\Phi(\hat{c}_n, \hat{\mu}_n)$  of  $\hat{c}_n$  with respect to the sample mean  $\hat{\mu}_n$  is

$$\Phi(\hat{c}_n, \hat{\mu}_n) = \lim_{M \rightarrow \infty} \left[ \frac{\text{var}(\hat{c}_n)}{\text{var}(\hat{\mu}_n)} \right] = \frac{2}{\sigma_z^2}. \quad (10)$$

By the result of (10), for  $\sigma_z^2 = 2$  the CLP Pitman estimator for the location parameter of  $R_n$  is as asymptotically efficient as the sample mean estimation of the location parameter of  $\xi_n$ , and for  $\sigma_z^2 \geq 2$  the Pitman estimator is more efficient than the sample mean. This completes the proof.

For certain applications such as loudspeaker testing, the advantages of  $\sigma_z^2 > 2$  can be exploited since their dynamic range (DR) is usually large. For DUTs with a narrow DR, one can always reduce the variance of the test signal  $s(n)$  such that after scaling with  $Z \sim N(0, \sigma_z^2 > 2)$ , the amplitudes of  $z_m s(n)$  do not drive the DUT into non-linear distortions.

*Corollary 1:* The CLP Pitman estimator  $\hat{c}_n$  of the location parameter of ratio sample  $R_n = \frac{\xi_n}{Z}$ , where  $\xi_n$  is the sum of centred i.i.d. platykurtic variables and  $Z \sim N(0, \sigma_z^2)$ , is more efficient than the sample median  $\hat{d}_n$  estimation of the location parameter  $\xi_n$  of for  $\sigma_z^2 \geq 2$ .

*Proof:* From [4],  $\text{var}(\hat{\mu}_n) < \text{var}(\hat{d}_n)$  for estimating the location parameter of platykurtic variables. By transitivity, it follows that  $\text{var}(\hat{c}_n) < \text{var}(\hat{d}_n)$ .

From *Theorem 1* and *Corollary 1*, the performance of the CLP Pitman estimator can be expected to be relatively better than the conventional methods, and consequently a better noise suppression method for platykurtic noise densities given  $\sigma_z^2 > 2$ .

2) *Leptokurtic Noise Densities:* For leptokurtic densities,  $\sigma_\varepsilon^2$  can be unbounded or too large which means that the uncertainty associated with the TCS-mean as given by (4) is large as well. This is because the averaged sum of i.i.d. leptokurtic variables does not necessarily reduce to a Gaussian, but usually to another leptokurtic variable [6]. Consequently,  $R_n = \frac{\xi_n}{Z}$  is not necessarily a Cauchy variable. From [4], the sample median is a more efficient estimator than the sample mean for leptokurtic variables.

*Lemma 1:* By the law of large numbers, as the scale parameter  $u_n \rightarrow 0$  then the CLP Pitman estimator  $\hat{c}_n$  approximates to the sample median for symmetric density functions, and if  $u_n$  is sufficiently large then  $\hat{c}_n$  approximates to the sample mean.

*Proof:* As  $u_n \rightarrow 0$ , then

$$\lim_{u_n \rightarrow 0} \Psi_{n,m} \rightarrow \prod_{l \neq m} \left[ \frac{1}{(r_{n,m} - r_{n,l})^2} \right].$$

If  $\hat{r}_{n,l}$  is the mode of the density function  $p(r_n)$ , then the most probable Euclidian distance is  $(\hat{r}_{n,m} - r_{n,l}) = 0$ , which means that  $\Psi_{n,m} \rightarrow \infty$ . For values around the mode, the distance  $(r_{n,m} - r_{n,l})^2 \approx 0$  and thus  $\Psi_{n,m}$  may still be significantly large around these sample values. If  $r_{n,m}$  is significantly greater or less than most other sample values (located at the tails), then  $(r_{n,m} - r_{n,l})^2$  will frequently evaluate to large values, and by the law of large numbers  $\Psi_{n,m} \rightarrow 0$ . Therefore, to a good approximation

$$\lim_{u_n \rightarrow 0} \Psi_{n,m} \rightarrow \delta(\widehat{r(n, m)}),$$

where  $\delta(\widehat{r(n, m)})$  is the Dirac-delta function located at the mode of the density function, and this is the sample value weighting function of the sample median (the sample-median is determined solely by the middle values in an ordered set). For  $u_n$  large then

$$\Psi_{n,m} \rightarrow \prod_{l \neq m} \frac{1}{4u_n^2} \cdot \left[ 1 - \frac{2u_n}{(r_{n,m} - r_{n,l})^2} \sqrt{-1} \right].$$

Substituting into (7) results in  $\hat{c}_n \approx \frac{1}{M} \sum_{m=0}^{M-1} r_{n,m}$ , which is an expression for the sample mean. This results follows

from the fact that  $4u_n^2 \gg (r_{n,m} - r_{n,l})^2$ , meaning that the Euclidian distance between the sample values is almost irrelevant and all values weigh equally towards the estimation of the location parameter.

**Lemma 2:** If the probability density function  $p(r_n)$  of the variable  $R_n$  is heavy-tailed, then the weighted function  $\Re(\Psi_{n,m})p(r_{n,m})$ , for all  $m \in [0, M-1]$ , is platykurtic for  $u_n$  sufficiently small.

**Proof:** From lemma 1, if  $u_n$  is small then the tails of the weight function  $\Re(\Psi_{n,m})$  decay to zero, which means that the heavy tailed function  $p(r_n)$  is reduced to a platykurtic function  $\Re(\Psi_{n,m})p(r_{n,m})$ , for all  $m \in [0, M-1]$ .

**Theorem 2:** The CLP Pitman estimator  $\hat{c}_n$  for the location parameter of  $R_n = \frac{\xi_n}{Z}$ , where  $\xi_n$  is a leptokurtic variable and  $Z \sim N(0, \sigma_z^2)$ , is more efficient than the sample median  $\hat{d}_n$  estimation of the location parameter of  $\xi_n$  for  $u_n$  small.

**Proof:** Let  $\Psi_{n,m}, \forall m \in [0, M-1]$ , be derived from the observations of  $R_n$ . By definition, the CLP Pitman estimator is the sample mean of the weighted values  $r_{n,m} \Re(\Psi_{n,m})$ , or the weighted density function  $\Re(\Psi_{n,m})p(r_{n,m})$ , for all  $m \in [0, M-1]$ , which by Lemma 2 is platykurtic for  $u_n$  small.

Define a new density function  $\Re(\Psi_{n,m})p(\varepsilon_{n,m})$ , for all  $m \in [0, M-1]$ , and let  $\hat{d}_{n,\Psi}$  be the sample median estimation of the this weighted noise density function of the leptokurtic noise variable  $\xi_n$ . Since  $\Re(\Psi_{n,m})p(r_{n,m})$ , for all  $m \in [0, M-1]$ , is platykurtic for  $u_n$  small, it follows that  $\Re(\Psi_{n,m})p(\varepsilon_{n,m})$ , for all  $m \in [0, M-1]$ , is platykurtic as well because the ratio sample  $R_n$  is more heavy tailed than  $\xi_n$  [9]. For  $p(\varepsilon_n)$  symmetric, then  $p(r_n)$  is symmetric and thus  $\Re(\Psi_{n,m})$ , for all  $m \in [0, M-1]$ , as well. It follows that the sample median  $\hat{d}_n$  estimation of the location parameter of  $\xi_n$  coincides with  $\hat{d}_{n,\Psi}$ . It is known that the variance of the sample median decreases with decreasing tail thickness [4], and thus  $\text{var}(\hat{d}_{n,\Psi}) \ll \text{var}(\hat{d}_n)$ .

The asymptotic variance of the sample median is fairly consistent for different platykurtic densities albeit slightly larger than that of the sample mean (or Pitman estimator of weighted samples). Therefore,  $\text{var}(\hat{c}_n) \cong \text{var}(\hat{d}_{n,\Psi})$ , which means that  $\text{var}(\hat{c}_n) < \text{var}(\hat{d}_n)$  for leptokurtic density functions. This completes the proof.

**Corollary 2:** The CLP Pitman estimator  $\hat{c}_n$  for the location parameter of  $R_n = \frac{\xi_n}{Z}$ , where  $\xi_n$  is a leptokurtic variable and  $Z \sim N(0, \sigma_z^2)$ , is more efficient than the sample mean  $\hat{\mu}_n$  estimation of the location parameter of  $\xi_n$  for  $u_n$  small.

**Proof:** From [4],  $\text{var}(\hat{d}_n) < \text{var}(\hat{\mu}_n)$  for leptokurtic variables. From Theorem 2,  $\text{var}(\hat{c}_n) < \text{var}(\hat{d}_n)$  therefore  $\text{var}(\hat{c}_n) < \text{var}(\hat{\mu}_n)$ .

Based on Theorem 2 and corollary 2, even though  $R_n$  is

not necessarily a Cauchy variable, the weight function  $\Psi_{n,m}$ , for all  $m \in [0, M-1]$ , allows the proposed algorithm to be more efficient than either the TCS-mean or TCS-median for leptokurtic noise densities as well. This means that the RMS error in estimating  $h(n) * s(n)$  can be expected to be less than that of the conventional methods.

**Remark:** The use of the Pitman estimator assumes that the scale parameter  $u_n$  is known, but this is not the case even for platykurtic densities because  $\sigma_\varepsilon^2$  is unknown. From Lemma 1, it can be deduced that  $u_n$  controls the selectivity of the Pitman estimator, which determines its robustness to outliers. While a reasonably accurate estimation of  $u_n$  is essential, minimum-variance estimation is not of primary importance due to possible computational complexities. For simplicity, one can use half the inter-quartile range (IQR) as an estimate for  $u_n$ .

### III. RESULTS AND ANALYSIS

From [4], we can write the generalized density function for the power exponential family as

$$p(\varepsilon_n|\gamma) = c(\gamma)e^{-|\varepsilon_n - \mu_\varepsilon|^\gamma} \quad (11)$$

where  $\mu_\varepsilon$  is the location parameter,  $\gamma \in (0, \infty)$  controls the thickness of the tails and  $c(\gamma)$  is a normalization constant in  $\gamma$  such that  $p(\varepsilon_n|\gamma)$  is a density function for different values of  $\gamma$ . Decreasing  $\gamma$  results in heavy-tailed density functions, while increasing it leads to thin-tailed density functions. For example, when  $\gamma = 2$  then  $p(\varepsilon_n|\gamma = 2)$  is Gaussian and if  $\gamma = 1$  we have the double exponential function. Therefore, leptokurtic densities are defined for  $\gamma < 2$  and platykurtic densities occur when  $\gamma > 2$ , making (11) ideal for robustness analysis of location parameter estimators. The noise location parameter has no influence on the variance of the different estimators, and is thus left as  $\mu_\varepsilon = 0$ . For even values of  $k \geq 0$ ,

$$\begin{aligned} E(\xi_n^k) &= \int_{-\infty}^{\infty} \varepsilon_n^k \cdot c(\gamma) \cdot e^{-|\varepsilon_n|^\gamma} d\xi_n \\ &= 2c(\gamma) \int_0^{\infty} \varepsilon_n^k \cdot e^{-\varepsilon_n^\gamma} d\varepsilon_n \end{aligned}$$

Using the Gamma function  $\Gamma(t+1) = \int_0^{\infty} y^t e^{-y} dy$ , for  $y = \varepsilon_n^\gamma$  it follows that:

$$E(\xi_n^k) = 2c(\gamma)\Gamma\left(\frac{k+1}{\gamma}\right)/\gamma.$$

For  $k = 0$  we obtain the normalization function

$$c(\gamma) = \frac{\gamma}{2\Gamma(\frac{1}{\gamma})},$$

and from  $k = 2$  and  $k = 4$  the kurtosis expression is

$$\beta = \frac{\Gamma(\frac{5}{\gamma})\Gamma(\frac{1}{\gamma})}{\Gamma^2(\frac{3}{\gamma})}.$$

Kurtosis measures the thickness of the tails of a given density function. It can be shown that for  $6 \leq \gamma < \infty$  then

$1.8 < \beta \leq 2$ , and for  $\gamma \leq 0.2$ ,  $\beta$  is very large. Therefore we shall constrain our analysis to  $0.2 \leq \gamma \leq 6$ . The FCR lower uncertainty bounds associated with the sample mean and sample median can also be evaluated using the gamma function as discussed in [4]. This is not possible for the CLP Pitman estimator because the moments of the Cauchy density function are either undefined or infinite. We resort to the use of random number generators to provide synthetic noise for comparing the three methods. Consider the cumulative distribution function of (11):

$$P(\varepsilon_n < \ell) = \int_{-\infty}^{\ell} p(\varepsilon_n | \gamma) d\varepsilon_n.$$

For  $\ell > 0$ , then

$$P(\varepsilon_n < \ell | \ell > 0) = \frac{1}{2} + \frac{1}{2\Gamma(\frac{1}{\gamma})} \int_0^{\ell} e^{-\varepsilon_n^{\gamma}} d\varepsilon_n,$$

due to symmetry of the power exponential density functions. With  $y = \varepsilon_n^{\gamma}$ , then

$$P(\varepsilon_n < \ell | \ell > 0) = \frac{1}{2} + \frac{1}{2\Gamma(\frac{1}{\gamma})} \int_0^{\ell^{\gamma}} y^{\frac{1}{\gamma}-1} \cdot e^{-y} dy$$

Noting that the integral is the lower incomplete gamma function  $\Gamma(\frac{1}{\gamma}, \ell^{\gamma})$ , it follows that

$$P(\varepsilon_n < \ell | \ell > 0) = \frac{1}{2} + \frac{1}{2} \tilde{\Gamma}\left(\frac{1}{\gamma}, \ell^{\gamma}\right) \quad (12)$$

where  $\tilde{\Gamma}\left(\frac{1}{\gamma}, \ell^{\gamma}\right)$  is the regularized lower incomplete gamma function. Since  $P(\varepsilon_n < \ell | \ell > 0) + P(\varepsilon_n < \ell | \ell < 0) = 1$ ,

$$P(\varepsilon_n < \ell | \ell < 0) = \frac{1}{2} - \frac{1}{2} \tilde{\Gamma}\left(\frac{1}{\gamma}, \ell^{\gamma}\right) \quad (13)$$

Based on the fact that  $P(\varepsilon_n < \ell) \in [0, 1]$ , if an observation  $v_m \in [0, 1]$  from the uniform density function is assumed to be a realization of  $P(\varepsilon_n < \ell)$ , then from (12) and (13) we have the power exponential random generator

$$\varepsilon_{n,m} = \begin{cases} \{\tilde{\Gamma}^{-1}[\frac{1}{\gamma}, (2v_m - 1)]\}^{\frac{1}{\gamma}}, & \text{for } v_m > \frac{1}{2} \\ \{-\tilde{\Gamma}^{-1}[\frac{1}{\gamma}, (1 - 2v_m)]\}^{\frac{1}{\gamma}}, & \text{for } v_m \leq \frac{1}{2} \end{cases} \quad (14)$$

where  $\tilde{\Gamma}^{-1}[\frac{1}{\gamma}, v]$  is the regularized lower inverse incomplete gamma function from zero to  $v$  (implemented as "gammaincinv" in MATLAB). Fig.1 is a summary of the RMS errors of the sample mean, sample median and the CLP Pitman estimator on estimating the location parameter of (11) for  $0.2 \leq \gamma \leq 6$  observed in  $10^5$  trials for a fixed number of samples per trial  $M$ . For the CLP Pitman estimator three different values of  $\sigma_z^2$  are used. When  $\sigma_z^2 = 2$  the asymptotic efficiency  $\Phi(\hat{c}_n, \hat{\mu}_n) \rightarrow 1$  for platykurtic densities, which helps to illustrate the inherent advantage of the CLP Pitman estimator over the sample mean and median for leptokurtic noise densities. As previously discussed, the only way to increase the SNR via the sample mean or median is to increase the number of measurements. The variances  $\sigma_z^2 = 4$  and 16 are used to demonstrate an alternative to increasing

TABLE I  
PERFORMANCE OF SEVERAL ESTIMATORS IN ESTIMATED POSITION OF DIFFERENT TAILED DENSITY FUNCTIONS OBSERVED IN  $10^5$  TRIALS FOR A FIXED NUMBER OF OBSERVATIONS  $M$ .

$\gamma$	$M$	rms error in estimated position				
		Sample Mean	Sample Median	CLP Pitman Estimator		
				$\sigma_z^2 = 2$	$\sigma_z^2 = 4$	$\sigma_z^2 = 16$
0.2	3	$3.15 \times 10^4$	$9.25 \times 10^3$	<b>2.71</b> $\times 10^3$	$1.33 \times 10^3$	369.71
	5	$2.47 \times 10^4$	$4.76 \times 10^3$	<b>852.86</b>	400.42	105.85
	10	$1.93 \times 10^4$	$1.63 \times 10^3$	<b>204.94</b>	111.17	25.896
	20	$1.25 \times 10^4$	622.94	<b>72.108</b>	33.581	9.4685
	40	$9.14 \times 10^3$	289.24	<b>27.163</b>	13.234	3.3635
0.5	3	6.4203	4.5103	<b>1.7245</b>	0.9928	0.3159
	5	4.8182	2.8566	<b>0.9943</b>	0.4703	0.1217
	10	3.4822	1.6091	<b>0.4192</b>	0.2015	0.0520
	20	2.4318	0.9281	<b>0.2069</b>	0.1011	0.0297
	40	1.7174	0.5813	<b>0.1122</b>	0.0581	0.0158
1	3	0.8210	0.8002	<b>0.6426</b>	0.3467	0.1326
	5	0.6351	0.5920	<b>0.3824</b>	0.1899	0.0502
	10	0.4464	0.3821	<b>0.1950</b>	0.0960	0.0280
	20	0.3165	0.2567	<b>0.0848</b>	0.0567	0.0175
	40	0.2251	0.1761	<b>0.0558</b>	0.0373	0.0097
2	3	<b>0.4096</b>	0.4778	0.4131	0.2862	0.0753
	5	0.3126	0.3730	<b>0.3110</b>	0.1603	0.0413
	10	<b>0.2238</b>	0.2641	0.2288	0.1083	0.0287
	20	0.1566	0.1898	<b>0.1501</b>	0.0873	0.0233
	40	<b>0.1109</b>	0.1366	0.1112	0.0585	0.0156
4	3	<b>0.3360</b>	0.4241	0.3376	0.1786	0.0486
	5	0.2592	0.3481	<b>0.2439</b>	0.1332	0.0355
	10	<b>0.1825</b>	0.2534	0.1840	0.0982	0.0252
	20	0.1298	0.1904	<b>0.1278</b>	0.0662	0.0174
	40	0.0916	0.1384	<b>0.0911</b>	0.0489	0.0133
6	3	0.3268	0.4202	<b>0.3266</b>	0.1823	0.0512
	5	<b>0.2513</b>	0.3527	0.2559	0.1334	0.0366
	10	<b>0.1779</b>	0.2529	0.1803	0.0925	0.0235
	20	<b>0.1260</b>	0.1942	0.1267	0.0712	0.0189
	40	<b>0.0886</b>	0.1401	0.0892	0.0487	0.0124

the SNR without increasing the number of measurements for the proposed algorithm.

With regard to the sample mean, sample median and the CLP Pitman estimator for  $\sigma_z^2 = 2$ , the least RMS error results in Table I are shown in bold between the three estimators. It is observed that the CLP algorithm is more efficient than both the sample mean and median for leptokurtic densities as predicted by *Theorem 2* and *Corollary 2*. For platykurtic variables  $\Phi(\hat{c}_n, \hat{\mu}_n) \rightarrow 1$ , which is as expected because of the CLT.

The results for  $\sigma_z^2 = 4$  and 16 show that as the variance of the Gaussian sample  $Z$  is increased, the uncertainty associated with the CLP Pitman estimator is reduced. In fact, based on the FCR lower bound of the CLP Pitman estimator  $2\sigma_z^2$ , one can expect an SNR improvement of 3dB for every doubling of  $\sigma_z^2$ . As an example, from Table I to achieve a sample mean RMS error of less than 0.17 given Gaussian noise requires  $M = 20$  measurement signals, whereas for the CLP Pitman

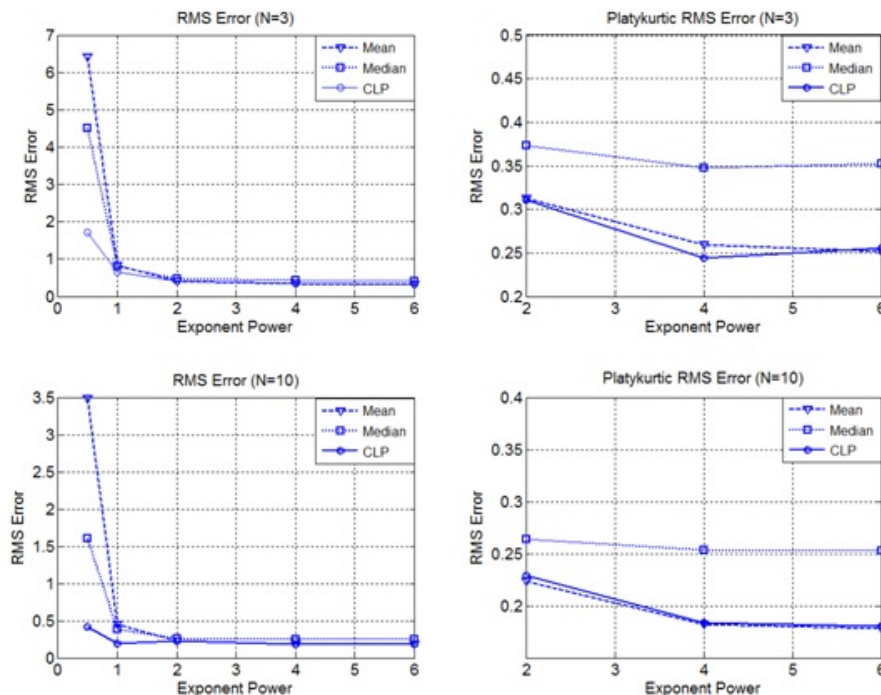


Fig. 1. The rms error curves for the CLP Pitman estimator ( $\sigma_z^2 = 2$ ) illustrate better consistency than those of the sample mean and sample median as they are more flat for different tail thicknesses. The CLP Pitman estimator performs as well as the sample mean given the same asymptotic efficiency for platykurtic density functions.

estimator one can use  $M = 5$  measurements and  $\sigma_z^2 = 4$ . The largest possible value of  $\sigma_z^2$  is application dependent and should be such that the DUT is not driven into non-linearity. In general, it can be observed that the accuracy of all three estimators increase with increasing  $\gamma$  due to reduced tail thickness.

On the left-hand side of Fig.1 are the rms error plots of the three location estimators for  $0.5 \leq \gamma \leq 6$  and  $M = 3$  and 10 to illustrate which is the most consistent for different tailed densities. The  $\gamma = 0.2$  values are left out because they are much larger and make it difficult to visualize the estimator performance for other values. The CLP Pitman estimator plots ( $\sigma_z^2 = 2$ ) have the least variation for different values of  $\gamma$  as illustrated by the more flat curves, followed by the sample median and lastly the sample mean. On the right-hand side of Fig.1 are the platykurtic rms errors ( $2 \leq \gamma \leq 6$ ) for the location estimators. There is almost no difference between the sample mean and the CLP Pitman estimator as predicted by *Theorem 1*. The sample median is least efficient for these densities. Here, the only advantage of the CLP Pitman estimator over the sample mean is the possibility of using larger values of  $\sigma_z^2$  as reflected by Table I.

#### IV. CONCLUSIONS

A new algorithm for noise suppression given several DUT measurements has been introduced. Prior to DUT excitation, the test signal is scaled by realizations of a Gaussian variable

of zero mean and pre-determined variance such that the DUT is not driven into non-linearity. The ratio of the captured measurements to the Gaussian sample is grouped into TCS ratio variables whose location parameters are shifted by the LTI convolution of the DUT impulse response and the test signal. Based on the CLT, the ratio sample for platykurtic noise densities tends to a Cauchy sample for which the CLP Pitman estimator is the minimum-variance location estimator. Therefore, the proposed algorithm performs as well as the sample mean and better than the sample median for platykurtic noise variables given the same FCR lower bound.

For leptokurtic densities, we used *Theorem 2* to illustrate how the weighting functions of the CLP Pitman estimator reduce the heavy-tailed densities to platykurtic densities, for which subsequent averaging (as the CLP Pitman estimator does) is more efficient than the sample median of the heavy-tailed densities. Consequently, the CLP Pitman estimator is more efficient than the sample mean for leptokurtic densities. Therefore by *Theorem 1* and 2, the CLP Pitman estimator is a more consistent and robust noise suppression algorithm than either the sample mean or median for different tailed noise density functions. Another major advantage of the proposed algorithm is that instead of noise suppression by increasing the number of measurements (as with the conventional methods), one can increase the variance of the scaling signal. If both the number of measurements and the variance of the scaling signal are increased, the CLP Pitman estimator is even more efficient than the sample mean and median as illustrated by

the simulation results in Section III. In conclusion, the results suggest that it is more reliable to use the CLP algorithm for noise suppression of unknown and first-order stationary noise sources given its lower uncertainties and better consistency over a wider range of different tailed density functions.

#### V. FUTURE WORK

It is clear that the scaling parameter of the Cauchy density function plays a major role on the selectivity of CLP Pitman estimator. In this paper, this value was estimated as half the IQR of the given samples. It is possible that for leptokurtic variables a certain range of values would yield better results and it would be better to use them instead of half the IQR. In future, an analysis of the effects of the scaling parameter with respect to other parameters will be investigated to help improve the convergence of the proposed algorithm.

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