Restrictedly-Regular Map Representation of n-Dimensional Abstract Polytopes

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Abstract—Regularity has often been present in the form of regular polyhedra or tessellations; classical examples are the nine regular polyhedra consisting of the five Platonic solids (regular convex polyhedra) and the four Kleper-Poinsot polyhedra. These polytopes can be seen as regular maps. Maps are cellular embeddings of graphs (with possibly multiple edges, loops or dangling edges) on compact connected (closed) surfaces with or without boundary. The n-dimensional abstract polytopes, particularly the regular ones, have gained popularity over recent years. The main focus of research has been their symmetries and regularity. Planification of polyhedra helps its spatial construction, yet it destroys its symmetries. To our knowledge there is no "planification" for n-dimensional polytopes. However we show that it is possible to make a "surfacification" of the n-dimensional polytope, that is, it is possible to construct a restrictedly-marked map representation of the abstract polytope on some surface that describes its combinatorial structures as well as all of its symmetries. We also show that there are infinitely many ways to do this; yet there is one that is more natural that describes reflections on the sides ($(n-1)$-faces) of n-simplices with reflections on the sides of n-polygons. We illustrate this construction with the 4-tetrahedron (a regular 4-polytope with automorphism group of size 120) and the 4-cube (a regular 4-polytope with automorphism group of size 384).

Keywords—Maps, representation, polytopes.

I. INTRODUCTION

AN abstract n-polytope $\mathcal{P}$ is a partially ordered set (poset) of faces with a strictly monotone rank function of range $\{-1, 0, \ldots, n\}$, represented by a Hasse diagram with $n+1$ layers, where the poset obey the diamond condition and flags are strongly flag-connected. *Flags* are maximal chains of faces, that is, vectors consisting of $n+2$ faces of rank -1, 0, 1, ..., n respectively. There is a unique least face, the (-1)-face F_{-1} and a unique greatest face the n-face F_n. Faces of rank 0, 1 and $n-1$ are called *vertices*, *edges* and *facets*. Two flags are adjacent if they differ only by one face (entry). Flags are *strongly flag-connected* means that any two flags Ψ, Φ are connected by a sequence of flags $\Gamma_0 = \Psi$, Γ_1, $\ldots \Gamma_m = \Phi$ such that two successive flags Γ_i, Γ_{i+1} are adjacent and for any i, j, $\Gamma_i \cap \Gamma_j = \Psi \cap \Phi$. The *diamond condition* says that whenever F_{i-1} and F_{i+1} are faces of ranks $i-1$ and $i+1$ for some i, with $F_{i-1} < F_{i+1}$, then there are exactly two faces F_i of rank i containing F_{i-1} and contained in F_{i+1}, that is, $F_{i-1} < F_i < F_{i+1}$. That is, the poset of the section $F_{i+1}/F_{i-1} = \{F \in \mathcal{P} | F_{i-1} < F < F_{i+1}\}$ is like a diamond.

An abstract 2-polytope is just a polygon while a 3-polytope is a non-degenerate map (cellular embedding of a simple graph on some compact connected (i.e. closed) surface), with the property that every edge is incident with exactly two faces, and every vertex on a face is incident with two edges of that face. All polytopes and maps are finite in this paper. For a deeper reading see [3], [7].

II. Θ-MARKED MAPS

A map $\mathcal{M} = (\Omega; r_0, r_1, r_2)$ is determined by a set Ω of triangular pieces of surface called *flags*, and 3 involutory permutations r_0, r_1, r_2 on Ω satisfying $(r_0 r_1)^2 = 1$ and generating a transitive group on Ω called the *monodromy group* of the map. Lynne James in [6] introduced maps representations and associate it to a non-commutative multiplication operation between map type objects. Although restrictedly map representations [4] lie in a different category, they represent the same topological objects with a different perspective and semantic.

Consider the "right triangle" group $\Gamma = \langle R_0, R_2 \rangle * \langle R_1 \rangle \cong (C_2 \times C_2) * C_2$ generated by the three reflections R_0, R_1, R_2 in the sides of a hyperbolic right triangle with two zero internal angles. Every finite index subgroup $M < \Gamma$ determines a finite map $\mathcal{M} = (\Gamma/_{\!r} M; M^* R_0, M^* R_1, M^* R_2)$, where M^* is the core of M in Γ and each $M^* R_i$ acts as a permutation on the right cosets $\Gamma/_{\!r} M$ of M in Γ by right multiplication. M is called the *fundamental map subgroup* of $\mathcal{M}$ (or just "map subgroup"). Let Θ be a normal subgroup of Γ with finite index n. A map is Θ-*conservative* if M is a subgroup of Θ. In this case the flags of $\mathcal{M}$ are n coloured under the action of Θ, each colour determined by an orbit (the Θ-orbit) under the action of Θ. By the Kurosh's Subgroup Theorem [5, Proposition 3.6], Θ freely decomposes into a free product $C_2 * \ldots * C_2 * D_2 * \ldots * D_2 * C_\infty * \ldots * C_\infty = \langle Z_1, \ldots, Z_m \rangle$ for some finite number (possibly zero) of factors C_2, $D_2 = C_2 \times C_2$ and C_∞. This decomposition is unique up to a permutation of the factors. A Θ-conservative map can then be represented by a Θ-*marked map* $\mathcal{Q} = (\Omega; z_1, \ldots, z_m)$, where Ω is the set of right cosets $\Theta/_{\!r} M$ of M in Θ, and each $z_i = M_\Theta Z_i \in \Theta / M_\Theta$ (where M_Θ is the core of M in Θ). The geometric construction described in [1], which can be adapted to Γ [4], uses Θ-slices, polygonal regions determined by a Schreier transversal for Θ in Γ. Θ-slices represent the elements of Ω. For example, a Γ-slice is a "flag" and a Γ^+-slice is a "dart", where Γ^+ is the normal subgroup of index 2 in Γ consisting of the words of even length on R_0, R_1, R_2. The group generated by $z_1 \ldots, z_n$, called the *monodromy group* of $\mathcal{Q}$, or the Θ-monodromy group of $\mathcal{M}$, acts transitively on the set of the Θ-slices Ω. A morphism (or covering ψ from a Θ-marked map $\mathcal{Q}_1 = (\Omega_1; z_1, \ldots, z_m)$ to another Θ-marked map $\mathcal{Q}_2 = (\Omega_2; z'_1, \ldots, z'_m)$ is a function $\psi : \Omega_1 \longrightarrow \Omega_2$ that commutes the diagram.

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$$\begin{tikzcd}
\Omega_1 \times Mon(\mathcal{Q}_1) \arrow[r] \arrow[d, "\psi"', "\iota\colon z_i \mapsto z_i'"] & \Omega_1 \arrow[d, "\psi"] \\
\Omega_2 \times Mon(\mathcal{Q}_2) \arrow[r] & \Omega_2
\end{tikzcd}$$

An automorphism of $\mathcal{Q}$ is just a bijective morphism from $\mathcal{Q}$ to $\mathcal{Q}$. A Θ-marked map $\mathcal{Q}$ is *regular*, or the Γ-marked map $\mathcal{M}$ is Θ-regular, if M is a normal subgroup of Θ. In this case the automorphism group of $\mathcal{Q}$, which is the automorphism group of $\mathcal{M}$ preserving each Θ-orbit, coincides with the monodromy group $Mon(\mathcal{Q})$, but with different action on Ω. For a more detailed exposition see [1].

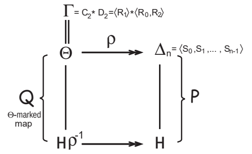
A *restrictedly-regular* (or resctrictly-regular) map is a Θ-regular for some (finite index) normal subgroup $\Theta \lhd \Gamma$. Any group G is the monodromy group (and hence the automorphism group) of a restrictedly-regular map ([2, Lemma 2.2] easily adapted to Γ).

III. REGULAR REPRESENTATION OF n-POLYTOPES BY RESTRICTEDLY-MARKED MAPS

A group with presentation $\langle s_0, s_1, \ldots, s_{n-1} \mid s_i^2 = (s_is_j)^{p_{ij}} = 1\rangle$ where $p_{ij} \geq 2$ is a positive integer possibly ∞, is a *Coxeter group*. If $p_{ij} = \infty$ then the relation $(s_is_j)^{p_{ij}}$ is not considered in the above presentation. Let $\Omega_{\mathcal{P}}$ denote the set of flags of a polytope $\mathcal{P}$. As a consequence of the diamond condition, for any $\Phi \in \Omega_{\mathcal{P}}$ and for any $0 \leq i \leq n-1$, the set $\{\Phi' \in \Omega_{\mathcal{P}} \mid \Phi'_j = \Phi_j, \forall j \neq i\}$ contains exactly two elements, one of them being Φ. Denote by $\Phi r_i = \Phi'$ the other flag of this set. We have n permutations $r_i = \prod_{\Phi \in \Omega_{\mathcal{P}}}(\Phi, \Phi r_i)$ for $i \in \{0, 1, \ldots, n-1\}$. They give rise to a flag transitive permutation group $G(\mathcal{P}) = \langle r_0, r_1, \ldots, r_{n-1}\rangle$, called the *connection group* of $\mathcal{P}$, that describes the polytope $\mathcal{P}$: each i-face F_i for $i \in \{0, 1, \ldots, n-1\}$, corresponds to an orbit of $\langle r_0, \ldots, \hat{r_i}, \ldots, r_{n-1}\rangle$ on $\Omega_{\mathcal{P}}$, where $\hat{r_i}$ means r_i is absent.

A polytope $\mathcal{P}$ can be identified with the $n+2$ tuple $(\Omega_{\mathcal{P}}; r_0, r_1, \ldots, r_{n-1})$. Denote by Δ_{n-1} the Coxeter group $\langle S_0, S_1, \ldots, S_{n-1} \mid S_i^2 = 1\rangle$. Then we have a natural epimorphism $\pi : \Delta_{n-1} \to G(\mathcal{P})$, mapping each S_i to r_i, inducing an action $\Phi d := \Phi d\pi$ of Δ_{n-1} on $\Omega(\mathcal{P})$. Similarly to [1, & 1.2], fixing a flag $\Phi \in \Omega_{\mathcal{P}}$ and letting P be the stabiliser of Φ in Δ_{n-1}, then Δ_{n-1} acts on $\Delta_{n-1}/_{\!r}P$ by right multiplication inducing a bijective function $\pi_\Phi : \Delta_{n-1}/_{\!r}P \to \Omega_{\mathcal{P}}$, $Pd \mapsto \Phi d\pi$. The kernel of π is the core P^* of P in Δ_{n-1} and the group Δ_{n-1}/P^* acts transitively on $\Delta_{n-1}/_{\!r}P$ by right multiplication in a similar way as $G(\mathcal{P})$ acts on $\Omega_{\mathcal{P}}$. Hence the polytope $(\Omega_{\mathcal{P}}; r_0, r_1, \ldots, r_{n-1})$ is isomorphic to $(\Delta_{n-1}/_{\!r}P; P^*S_0, P^*S_1, \ldots, P^*S_{n-1})$. Every polytope $\mathcal{P}$ is described by such $(n+2)$-tuples; the converse is false. The set of all such $(n+2)$-tuples will be called the set of $(n-1)$-*hypermaps.* So both n-polytopes are $(n-1)$-hypermaps, the converse is false. The subgroup P will be called a *fundamental subgroup* pf $\mathcal{P}$. This is unique up to a conjugacy in Δ_{n-1}.

Following Lynne's ideas [6], and more specifically [4], a regular representation of $(n-1)$-hypermaps by restrictedly-marked maps is a m tuple $(\Theta; X_0, X_1, \ldots, X_m)$, consisting of a normal subgroup Θ of Γ freely generated by $X_0, X_1, \ldots, X_m$ for some $m \geq n$, together with an epimorphism ρ from Θ to Δ_{n-1}. Such representation gives rise to a bijection between the set of $(n-1)$-hypermaps $\mathcal{P}$ with fundamental subgroup H to the set of regular Θ-marked maps with fundamental subgroup $H\rho^{-1}$, henceforth a representation of n-polytopes.



There are actually infinitely many regular restrictely-marked representations of $(n-1)$-hypermaps, and so of n-polytopes.

Theorem 1: There is a regular restrictedly-marked representation of n-polytopes such that

1 flags (n-tetrahedra for n-polytopes) correspond to n-polygons,
2 local reflections about facets ($(n-1)$-dimensional sides) of a n-tetrahedron corresponds to local reflections on the sides of a n-polygon,
3 the (full) automorphism group of the n-polytope is the (full) automorphism group of the restrictedly marked map.

Proof: Lynne James's first example [6], essentially given by an alternative construction, gives an answer to this question for $n = 4$. The proof resumes to find a normal subgroup Θ of Γ which is freely generated by reflections. Unfortunately there are only four subgroups that are freely generated only by reflections, namely $\Gamma_{2.1} = \langle R_0, R_1, R_2R_1R_2\rangle = C_2 * C_2 * C_2$, $\Gamma_{2.4} = \langle R_1, R_2, R_0R_1R_0\rangle = C_2 * C_2 * C_2$, $\Gamma_{2.5} = \langle R_1, R_2R_0, R_0R_1R_0\rangle = C_2 * C_2 * C_2$ and $\Gamma_{4.2} = \langle R_1, R_0R_1R_0, R_2R_1R_2, R_0R_2R_1R_2R_0\rangle = C_2 * C_2 * C_2 * C_2$. These solve the problem for $n = 3$ and 4. Denote by $\prod_k(R_i, R_j) =$ the product $R_iR_jR_iR_jR_i\ldots$ of R_i and R_j in alternate form, starting from R_i and counting k total factors. If $k = 0$ then let $\prod_0(R_i, R_j) = 1$. As a general construction we take the normal subgroup $\Gamma_n = \langle R_0, R_0^{R_1}, R_0^{R_1R_2}, \ldots, R_0^{\prod_{n-1}(R_1,R_2)}, (R_1R_2)^n\rangle$[1] of index $2n$ in Γ (Γ/Γ_n is a dihedral group of order $2n$). By the Kurosh's Subgroup Theorem, these generators decompose Γ_n as a free product $C_2 * C_2 * \ldots * C_2 * C_\infty$. We take the epimorphism $\rho : \Gamma_n \to \Delta_{n-1}$ by mapping each $R_0^{\prod_k(R_1,R_2)}$ to S_k, for $k = 0, 1, \ldots, n-1$, and $(R_1R_2)^n$ to 1. Then the regular map with dihedral automorphism group of size $2n$ corresponding to the quotient Γ/Γ_n, is a star graph cellularly embedded in the disk, thus a boundary map with one vertex and n edges. We need to cut open this disk to create a Γ_n-slice (see [4] for the constructing example of such a Γ_n-slice) for the restricted Γ_n-marked map, however we need to join the cut back to accomplish with $(R_1R_2)^n = 1$ imposed by the epimorphism ρ to create a Γ_n-slice for this representation ρ. Each $(n-1)$-hypermap $\mathcal{P}$, and hence each n-polyotpe, with

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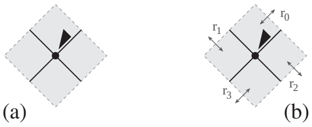
[1] There is another subgroup generated by reflections and one rotation that also decomposes as a free product $C_2 * C_2 * C_2 * \ldots * C_\infty$, it is the dual resulting from swapping R_0 with R_2. Another subgroup actually appears also with a free product decomposition $C_2 * C_2 * C_2 * \ldots * C_\infty$, yet one of the C_2 is generated by the rotation R_0R_2.

fundamental subgroup P, is isomorphic to a Γ_n-marked map $\mathcal{Q}$ with fundamental subgroup the inverse image $Q = P\rho^{-1}$. The rooted Γ_n-slice (Fig. 1a) for $\mathcal{Q}$ is the above n-polygon with a distinguished flag (in black): The monodromy group of the $(n-1)$-hypermap (which corresponds to the connection group of the n-polytope) is generated by the reflections on the sides of this n-polygon. The isomorphism $\bar{\rho}$ between the restricted Γ_n-marked map $\mathcal{Q}$ and $\mathcal{P}$ establishes the third statement. ■



(a) (b)

Fig. 1 A rooted Γ_4-slice (a) and sides identification (b)

IV. EXAMPLE: THE HYPERTETRAHEDRON AND THE HYPERCUBE

We take the hypertetrahedron, an orientable and regular 4-polytope with 120 flags, and the hypercube, an orientable and regular 4-polytope with 384 flags, for an illustration of the above theorem. The rooted Γ_4-slice of the restricted Γ_4-marked map representation is actually illustrated in the picture 1(a). To construct the regular restricted Γ_4-map $\mathcal{Q}$, that represents the hypertetrahedron (or the hypercube), we need to join the 120 (or the 384) rooted Γ_4-slices through their four sides according to the rule dictated by $r_0 = R_0$, $r_1 = R_0^{R_1}$, $r_2 = R_0^{R_1R_2}$ and $r_3 = R_0^{R_1R_2R_1}$. The automorphism group G of the hypertetrahedron and of the hypercube, is a Coxeter group of type $[3,3,3]$, and $[4,3,3]$ respectively. They have presentations respectively

$$\langle r_0, r_1, r_2, r_3 | r_0^2, r_1^2, r_2^2, r_3^2, (r_0r_2)^2, (r_0r_3)^2, (r_1r_3)^2, (r_0r_1)^3, (r_1r_2)^3, (r_2r_3)^3 \rangle$$

and

$$\langle r_0, r_1, r_2, r_3 | r_0^2, r_1^2, r_2^2, r_3^2, (r_0r_2)^2, (r_0r_3)^2, (r_1r_3)^2, (r_0r_1)^4, (r_1r_2)^3, (r_2r_3)^3 \rangle.$$

Since they are regular, their connection groups coincide with their automorphism groups (only their action on the flags are different), and their size is the number of flags (the action is regular on the set of flags). So the set of flags may be replaced by the automorphism group, in which case the action of the automorphism group is done by left multiplication while the action of the connection group is done by right multiplication. For the constructing we use this group as a connection group and label its elements 1, 2, 3, ... We start by labelling the first elements as follows: 1 for the identity element, 2 for $r_0 = R_0$, 3 for $r_1 = R_0^{R_1}$, 4 for r_0r_1, 5 for $r_0r_1r_0$, etc, until all the elements of the dihedral subgroup $\langle r_0, r_1 \rangle$ are labelled (this gives a central 12-gon with 6 sectors in the case of the hypertetrahedron (and a 16-gon with 8 sectors in the case of the hypercube). We only need to label all the elements of one sector; the remaining ones come by symmetry. In the figure below we show a constructed labelling of a sector of the hypertetrahedron. Bold numbers label the sides of this sector to be identified elsewhere; in bold red are those that will find an identification label inside the same sector while the others will be matched outside.

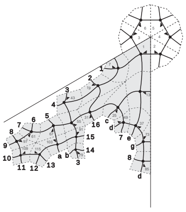


Fig. 2 The first sector of the hypertetrahedron

As we can see from the complete picture of the hypertetahedron, not all the sides were labelled; this is not necessary since by taking reflections and rotations about the central polygonal region we get all the remaining labels. For example, the bottom right side is not labelled; label it 17, horizontally reflect this to label d, see where the other d appears and then take the same reflection to see where the second d goes to and label that side 17. Moreover, there is no arrow indicating how the same labelled sides are identified. This is not necessary either: make the identification so to resemble the matched interior sides or just follow the word R_0, $R_0^{R_1}$, $R_0^{R_1R_2}$, $R_0^{R_1R_2R_1}$ that corresponds to the side (Fig. 1b); it will takes a root flag to a root flag.

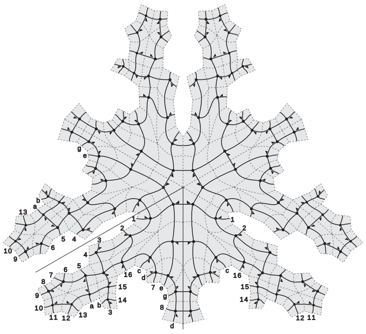


Fig. 3 The hypertetrahedron

The hypercube is done similarly.

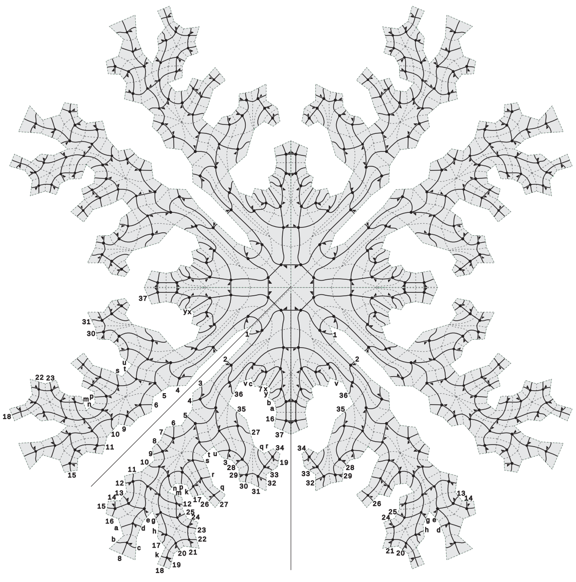


Fig. 4 The hypercube

REFERENCES

[1] A. Breda d'Azevedo, *A theory of restricted regularity of hypermaps*, J. Korean Math. Soc. 43 (2006), No.5, 991–1018.
[2] A. Breda d'Azevedo, *Riemann surfaces and restrictively-marked hypermaps*, Ars Mathematica Contemporanea 3 (2010), 87-98.
[3] A. Breda d'Azevedo, *Planification of n-dimensional abstract polytopes and hypertopes*, submitted.
[4] A. Breda d'Azevedo, D. Catalano, *Classification of the thin regular map-representation of hypermaps*, Submitted.
[5] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Springer, Berlin, 1977.
[6] Lynne D. James *Representation of Maps*, Dissertacciones Del Seminario de Matemáticas Fundamentales, Universidad Nacinal de Educación a Distancia, Lecture notes 27 March 1990.
[7] P. McMullen, E. Schulte, *Abstract Regular Polytopes*, Encyclopedia of Mathematics and Its Applications, Vol. 92. Cambridge Univ. Press, Cambridge-New York, 2002.