

# Relative Injective Modules and Relative Flat Modules

Jianmin Xing, Rufeng Xing

**Abstract**—Let  $R$  be a ring,  $n$  a fixed nonnegative integer. The concepts of  $(n, 0)$ -FI-injective and  $(n, 0)$ -FI-flat modules, and then give some characterizations of these modules over left  $n$ -coherent rings are introduced. In addition, we investigate the left and right  $n$ - $\mathcal{FL}$ -resolutions of  $R$ -modules by left (right) derived functors  $\text{Ext}_n(-, -)$  ( $\text{Tor}^n(-, -)$ ) over a left  $n$ -coherent ring, where  $n$ - $\mathcal{FL}$  stands for the categories of all  $(n, 0)$ -injective left  $R$ -modules. These modules together with the left or right derived functors are used to study the  $(n, 0)$ -injective dimensions of modules and rings.

**Keywords**— $(n, 0)$ -injective module,  $(n, 0)$ -injective dimension,  $(n, 0)$ -FI-injective(flat) module, (Pre)cover, (Pre)envelope.

## I. INTRODUCTION

**T**HROUGHOUT this paper,  $n$  is a positive integer unless a special note.  $R$  denotes an associative ring with identity and all modules considered are unitary.  $M_R$  ( ${}_R M$ ) denotes a right(left)  $R$ -module. For an  $R$ -module  $M$ ,  $E(M)$  stands for the injective envelope of  $M$ , the character module  $\text{Hom}_Z(M, Q/Z)$  is denoted by  $M^+$ , and  $\text{id}(M)$  ( $\text{fd}(M)$ ) is the injective(flat) dimension of  $M$ .

B. Stenström [11] defined and studied FP-injective modules. FP-injective modules are also called absolutely pure modules[9], these modules have been studied by many authors. In the paper [11], right Noetherian rings, right coherent rings, right semihereditary rings and regular rings are characterized by FP-injective right  $R$ -modules. It has been recently proven that every left  $R$ -module has an FP-injective cover over a left coherent ring  $R$  in the paper [9]. On the other hand, every left  $R$ -module  $M$  has an FP-injective preenvelope over any ring in the paper [6]. In the paper [7], L.X.Mao and N.Q.Ding introduced the definitions of FI-injective and FI-flat modules and give some characterizations of these modules over left coherent rings. FI-injective and FI-flat modules together with the left derived functors of  $\text{Hom}$  are used to study the FP-injective dimensions of modules and rings.

As generalizations of the paper [7], we introduce the definitions of  $(n, 0)$ -FI-injective and  $(n, 0)$ -FI-flat modules and give some characterizations of these modules over left  $n$ -coherent rings. In addition, we investigate the left and right  $n$ - $\mathcal{FL}$ -resolutions of  $R$ -modules by left (right) derived functors  $\text{Ext}_n(-, -)$  ( $\text{Tor}^n(-, -)$ ) over a left  $n$ -coherent ring, where  $n$ - $\mathcal{FL}$  stands for the categories of all  $(n, 0)$ -injective left  $R$ -modules. These modules together with the left

or right derived functors are used to study the  $(n, 0)$ -injective dimensions of modules and rings.

We recall some known notions and facts needed in the sequel.

Let  $R$  be a ring and  $n$  be a non-negative integer. A left  $R$ -module  $M$  is called  $n$ -presented in case there is an exact sequence of left  $R$ -modules  $F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$  in which every  $F_i$  is a finitely generated free [3], equivalently projective left  $R$ -module. Let  $n, d$  be non-negative integers. According to [13], a left  $R$ -module  $M$  is called  $(n, d)$ -injective(respectively  $(n, d)$ -flat) if  $\text{Ext}^{d+1}(N, M) = 0$ (respectively  $\text{Tor}_{d+1}(N, M) = 0$ ) for all  $n$ -presented left (respectively right)  $R$ -modules  $N$ . The  $(n, 0)$ -injective( $(n, 0)$ -flat) dimension of  $M$ [14], denoted by  $(n, 0)\text{-id}(M)$  ( $(n, 0)\text{-fd}(M)$ ), is defined to be the smallest nonnegative integer  $m$  such that  $\text{Ext}^{m+1}(F, M) = 0$  ( $\text{Tor}_{m+1}(F, M) = 0$ ) for every  $n$ -presented left  $R$ -module  $F$  (if no such  $m$  exists, set  $(n, 0)\text{-id}(M)$  ( $(n, 0)\text{-fd}(M)$ ) =  $\infty$ ), and  $l.(n, 0)\text{-dim}(R)$  ( $l.(n, 0)\text{-wdim}(R)$ ) is defined as  $\sup\{(n, 0)\text{-id}(M)$  ( $(n, 0)\text{-fd}(M)$ ) :  $M$  is a left  $R$ -module}.

Let  $\mathcal{C}$  be a class of  $R$ -modules and  $M$  an  $R$ -module. Following [5], we say that a homomorphism  $\varphi : M \rightarrow C$  is a  $\mathcal{C}$ -preenvelope if  $C \in \mathcal{C}$  and the abelian group homomorphism  $\text{Hom}(\varphi, C') : \text{Hom}(C, C') \rightarrow \text{Hom}(M, C')$  is surjective for each  $C' \in \mathcal{C}$ . A  $\mathcal{C}$ -preenvelope  $\varphi : M \rightarrow C$  is said to be a  $\mathcal{C}$ -envelope if every endomorphism  $g : C \rightarrow C$  such that  $g\varphi = \varphi$  is an isomorphism. Dually we have the definitions of a  $\mathcal{C}$ -precover and a  $\mathcal{C}$ -cover.  $\mathcal{C}$ -envelopes ( $\mathcal{C}$ -covers) may not exist in general, but if they exist, they are unique up to isomorphism. A homomorphism  $\varphi : M \rightarrow C$  with  $C \in \mathcal{C}$  is said to be a  $\mathcal{C}$ -envelope with the unique mapping property [5] if for any homomorphism  $f : M \rightarrow C'$  with  $C' \in \mathcal{C}$ , there is a unique homomorphism  $g : C \rightarrow C'$  such that  $g\varphi = f$ . Dually we have the definition of a  $\mathcal{C}$ -cover with the unique mapping property.

In what follows, we write  ${}_R\mathcal{M}$  and  $n$ - $\mathcal{FL}$  for the categories of all left  $R$ -modules and all  $(n, 0)$ -injective left  $R$ -modules, respectively. According to Costa[7], a ring  $R$  is called a left  $n$ -coherent ring in case every  $n$ -presented left  $R$ -module is  $(n+1)$ -presented. It is easy to see that  $R$  is left 0-coherent(resp.1-coherent)if and only if it is left noetherian(resp. coherent), and every  $n$ -coherent ring is  $m$ -coherent for  $m \geq n$ .  $n$ -coherent rings have been investigated by many authors(see Chen and Ding[1,4],Costa[3]). For  $n \geq 1$ , it has been proven that every left  $R$ -module  $M$  has an  $(n, 0)$ -injective preenvelope over any ring in [8]. So  $M$  has a right  $n$ - $\mathcal{FL}$ -resolution, that is, there is a  $\text{Hom}(-, n\text{-}\mathcal{FL})$  exact complex  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow$

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... with each  $F^i(n, 0)$ -injective. Obviously, the complex is exact. Let

$$L^0 = M, L^1 = \text{coker}(M \rightarrow F_0),$$

$$L^i = \text{coker}(F^{i-2} \rightarrow F^{i-1}) \quad \text{for } i \geq 2$$

The  $n$ th cokernel  $L_n (n \geq 0)$  is called the  $n$ th  $n$ - $\mathcal{FL}$ -cosyzygy of  $M$ .

On the other hand, for  $n \geq 1$ , it has been proven that every left  $R$ -module has an  $(n, 0)$ -injective cover over a left  $n$ -coherent ring  $R$  [8]. So every left  $R$ -module  $M$  has a left  $n$ - $\mathcal{FL}$ -resolution, that is, there is a  $\text{Hom}(n\text{-}\mathcal{FL}, -)$  exact complex  $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$  (not necessarily exact) with each  $F_i(n, 0)$ -injective. Write

$$K_0 = M, K_1 = \ker(F_0 \rightarrow M),$$

$$K_i = \ker(F_{i-1} \rightarrow F_{i-2}) \quad \text{for } i \geq 2.$$

The  $n$ th kernel  $K_n (n \geq 0)$  is called the  $n$ th  $n$ - $\mathcal{FL}$ -syzygy of  $M$ .

Note that  $\text{Hom}(-, -)$  is left balanced on  ${}_R\mathcal{M} \times_R \mathcal{M}$  by  $n$ - $\mathcal{FL} \times n$ - $\mathcal{FL}$  for a left  $n$ -coherent ring  $R$  (see [6, Definition 8.2.13]). Thus the  $n$ th left derived functor of  $\text{Hom}(-, -)$ , which is denoted by  $\text{Ext}_n(-, -)$ , can be computed using a right  $n$ - $\mathcal{FL}$ -resolution of the first variable or a left  $n$ - $\mathcal{FL}$ -resolution of the second variable. Following [6, Definition 8.4.1], the left  $n$ - $\mathcal{FL}$ -dimension of a left  $R$ -module  $M$ , denoted by  $\text{left } n\text{-}\mathcal{FL}\text{-dim} M$ , is defined as  $\inf\{m : \text{there is a left } n\text{-}\mathcal{FL}\text{-resolution of the form } 0 \rightarrow F_m \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0 \text{ of } M\}$ . If there is no such  $m$ , set  $\text{left } n\text{-}\mathcal{FL}\text{-dim}(M) = \infty$ . The global left  $n$ - $\mathcal{FL}$ -dimension of  ${}_R\mathcal{M}$ , denoted by  $\text{gl left } n\text{-}\mathcal{FL}\text{-dim} \mathcal{M}$ , is defined to be  $\sup\{\text{left } n\text{-}\mathcal{FL}\text{-dim}(M) : M \in {}_R\mathcal{M}\}$  and is infinite otherwise. The right versions can be defined similarly.

Recall that a left  $R$ -module  $M$  is called reduced [6] if  $M$  has no nonzero injective submodules.

In Section II of this paper, we introduce the concepts of  $(n, 0)$ -FI-injective and  $(n, 0)$ -FI-flat modules. It is shown that a left  $R$ -module  $M$  is  $(n, 0)$ -FI-injective if and only if  $M$  is a kernel of an  $(n, 0)$ -injective precover  $A \rightarrow B$  with  $A$  injective. For a left  $n$ -coherent ring  $R$ , we prove that a left  $R$ -module  $M$  is  $(n, 0)$ -FI-injective if and only if  $M$  is a direct sum of an injective left  $R$ -module and a reduced  $(n, 0)$ -FI-injective left  $R$ -module; an  $n$ -presented right  $R$ -module  $M$  is  $(n, 0)$ -FI-flat if and only if  $M$  is a cokernel of an  $(n, 0)$ -flat preenvelope of a right  $R$ -module.

In Section III, we investigate the  $(n, 0)$ -injective dimensions of modules and rings in terms of  $(n, 0)$ -FI-injective and  $(n, 0)$ -FI-flat modules and the left derived functors  $\text{Ext}_n(-, -)$ . Let  $R$  be a left  $n$ -coherent ring. We first give some characterizations of left  $n$ -hereditary rings. It is proven that  $R$  is left  $n$ -hereditary (i.e.,  $\text{l.}(n, 0)\text{-dim}(R) \leq 1$ ) if and only if the canonical map  $\sigma : \text{Ext}_0(M, N) \rightarrow \text{Hom}(M, N)$  is a monomorphism for all left  $R$ -modules  $M$  and  $N$  if and only if every  $(n, 0)$ -FI-injective left  $R$ -module is injective if and only if every  $(n, 0)$ -FI-flat right  $R$ -module is flat. Then it is shown that  $\text{l.}(n, 0)\text{-dim}(R) \leq m (m \geq 2)$  if and only if  $\text{Ext}_{m+k}(M, N) = 0$  for all left  $R$ -modules  $M, N$  and all  $k \geq -1$ .

In Section IV, we first investigate that the  $-\otimes-$  on  $\mathcal{M}_R \times_R \mathcal{M}$  is right balanced by  $n\text{-}\mathcal{F} \times n\text{-}\mathcal{FL}$  in the  $n$ -coherent ring, where  $n\text{-}\mathcal{F}$  stands for the class of all  $(n, 0)$ -flat modules. Then we introduce the right derived functors  $\text{Tor}^n(-, -)$  and give some characteristic of right  $n\text{-}\mathcal{F}\text{-dim} M$  and  $n\text{-}\mathcal{FL}\text{-dim} M$  for any  $R$ -module  $M$  in the  $n$ -coherent ring  $R$ .

Let  $M$  and  $N$  be  $R$ -modules.  $\text{Hom}(M, N)$  (respectively  $\text{Ext}^n(M, N)$ ) means  $\text{Hom}_R(M, N)$  (respectively  $\text{Ext}_R^n(M, N)$ ), and similarly  $M \otimes N$  (respectively  $\text{Tor}_n(M, N)$ ) denotes  $M \otimes_R N$  (respectively  $\text{Tor}_n^R(M, N)$ ) for an integer  $n \geq 1$  throughout this paper. For unexplained concepts and notations, we refer the reader to [6, 10, 12].

## II. $(n, 0)$ -FI-INJECTIVE MODULES AND $(n, 0)$ -FI-FLAT MODULES

**Definition 1** A left  $R$ -module  $M$  is called  $(n, 0)$ -FI-injective if  $\text{Ext}^1(G, M) = 0$  for any  $(n, 0)$ -injective left  $R$ -module  $G$ .

A right  $R$ -module  $N$  is said to be  $(n, 0)$ -FI-flat if  $\text{Tor}_1(N, G) = 0$  for any  $(n, 0)$ -injective left  $R$ -module  $G$ .

**Remark 1** (1) A right  $R$ -module  $M$  is  $(n, 0)$ -FI-flat if and only if  $M^+$  is  $(n, 0)$ -FI-injective by the standard isomorphism:  $\text{Ext}^1(N, M^+) \simeq \text{Tor}_1(M, N)^+$  for any left  $R$ -module  $N$ .

(2) We note that by the above definitions that  $(1, 0)$ -FI-injective (flat) modules are FI-injective (flat) module in [7] and any FI-injective (flat) module is  $(n, 0)$ -FI-injective (flat) for any  $n \geq 1$ .

**Proposition 1** Let  $\{M_i\}_I$  be family of right  $R$ -module

(1)  $\bigoplus_I M_i$  is  $(n, 0)$ -FI-flat if and only if each  $M_i$  is  $(n, 0)$ -FI-flat;

(2)  $\prod_I M_i$  is  $(n, 0)$ -FI-injective if and only if each  $M_i$  is  $(n, 0)$ -FI-injective.

**Proof** (1) By  $\text{Tor}_1(G, \bigoplus_I M_i) \simeq \bigoplus_I \text{Tor}_1(G, M_i)$ ;

(2) By  $\text{Ext}^1(G, \prod_I M_i) \simeq \prod_I \text{Ext}^1(G, M_i)$ .

**Definition 2** A ring  $R$  is said to be  $(n, 0)$ -IP-ring if every  $(n, 0)$ -injective  $R$ -module is projective;  $R$  is said to be  $(n, 0)$ -IF-ring if every  $(n, 0)$ -injective  $R$ -module is flat. It is trivial to show that if  $n \geq n'$ , then every  $(n, 0)$ -IP(IF) ring is an  $(n', 0)$ -IP(IF) ring and every  $(0, 0)$ -IP-ring is a quasi-Frobenius ring and every  $(0, 0)$ -IF-ring is an IF ring.

Next, we shall see that the class of right  $(n, 0)$ -IP(IF)-rings contains several important known rings.

**Proposition 2** Let  $R$  be a ring.

(1)  $R$  is a right  $(n, 0)$ -IP-ring if and only if every right module is  $(n, 0)$ -FI-injective.

(2)  $R$  is a right  $(n, 0)$ -IF-ring if and only if every left module is  $(n, 0)$ -FI-flat.

(3) If  $R$  is a right  $(n, 0)$ -IP-ring, then  $R$  is a right  $(n, 0)$ -IF-ring.

**Proof** Directly by the definitions.

**Corollary 1** Let  $R$  be a ring.

(1)  $R$  is a right quasi-Frobenius if and only if every right module is FI-injective.

(2)  $R$  is a right IF-ring if and only if every left module is FI-flat.

(3) If  $R$  is a right quasi-Frobenius, then  $R$  is a right IF-ring.

**Proposition 3** The following hold for a left  $n$ -coherent ring  $R$ :

(1) A left  $R$ -module  $M$  is injective if and only if  $M$  is  $(n, 0)$ -FI-injective and  $(n, 0)$ -id( $M$ )  $\leq 1$ .

(2) A right  $R$ -module  $N$  is flat if and only if  $N$  is  $(n, 0)$ -FI-flat and  $(n, 0)$ -fd( $N$ )  $\leq 1$ .

**Proof** (1) "Only if" part is trivial.

"If" part. Let  $M$  be an  $(n, 0)$ -FI-injective left  $R$ -module and  $(n, 0)$ -id( $M$ )  $\leq 1$ . Then there is an exact sequence  $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$  with  $E$  injective. Note that  $L$  is  $(n, 0)$ -injective by [14, Theorem 2.12] since  $R$  is a left  $n$ -coherent ring. So the exact sequence is split, and hence  $M$  is injective.

(2) "Only if" part is trivial.

"If" part. For any  $(n, 0)$ -FI-flat right  $R$ -module  $N$  with  $(n, 0)$ -fd( $N$ )  $\leq 1$ , we have  $N^+$  is  $(n, 0)$ -FI-injective by Remark 2.2 Thus  $N^+$  is injective by (1) since  $(n, 0)$ -id( $N^+$ )  $\leq 1$  by [14, Theorem 2.15]. So  $N$  is flat.

**Proposition 4** The following are equivalent for a left  $R$ -module  $M$ :

(1)  $M$  is  $(n, 0)$ -FI-injective.

(2) For every exact sequence  $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ , where  $E$  is  $(n, 0)$ -injective,  $E \rightarrow L$  is an  $(n, 0)$ -injective precover of  $L$ .

(3)  $M$  is a kernel of an  $(n, 0)$ -injective precover  $f : A \rightarrow B$  with  $A$  injective.

(4)  $M$  is injective with respect to every exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , where  $C$  is  $(n, 0)$ -injective.

**Proof** (1)  $\Rightarrow$  (2) and (1)  $\Rightarrow$  (4) are clear by definitions.

(2)  $\Rightarrow$  (3) is obvious since there exists a short exact sequence  $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$ .

(3)  $\Rightarrow$  (1) Let  $M$  be a kernel of an  $(n, 0)$ -injective precover  $f : A \rightarrow B$  with  $A$  injective. Then we have an exact sequence  $0 \rightarrow M \rightarrow A \rightarrow A/M \rightarrow 0$ . So, for any  $(n, 0)$ -injective left  $R$ -module  $N$ , the sequence  $\text{Hom}(N, A) \rightarrow \text{Hom}(N, A/M) \rightarrow \text{Ext}^1(N, M) \rightarrow 0$  is exact. It is easy to verify that  $\text{Hom}(N, A) \rightarrow \text{Hom}(N, A/M) \rightarrow 0$  is exact by (3). Thus  $\text{Ext}^1(N, M) = 0$ , and so (1) follows.

(4)  $\Rightarrow$  (1). For each  $(n, 0)$ -injective left  $R$ -module  $N$ , there exists a short exact sequence  $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$  with  $P$  projective, which induces an exact sequence  $\text{Hom}(P, M) \rightarrow \text{Hom}(K, M) \rightarrow \text{Ext}^1(N, M) \rightarrow 0$ . Note that  $\text{Hom}(P, M) \rightarrow \text{Hom}(K, M) \rightarrow 0$  is exact by (4). Hence  $\text{Ext}^1(N, M) = 0$ , as desired.

**Proposition 5** Let  $R$  be a left  $n$ -coherent ring. Then the following are equivalent for a left  $R$ -module  $M$ :

(1)  $M$  is a reduced  $(n, 0)$ -FI-injective left  $R$ -module.

(2)  $M$  is a kernel of an  $(n, 0)$ -injective cover  $f : A \rightarrow B$  with  $A$  injective.

**Proof** (1)  $\Rightarrow$  (2) By Proposition 4, the natural map  $\pi : E(M) \rightarrow E(M)/M$  is an  $(n, 0)$ -injective precover. Note that  $E(M)/M$  has an  $(n, 0)$ -injective cover, and  $E(M)$  has no nonzero direct summand  $K$  contained in  $M$  since  $M$  is reduced. It follows that  $\pi : E(M) \rightarrow E(M)/M$  is an  $(n, 0)$ -injective cover by [12, Corollary 1.2.8], and hence (2) follows.

(2)  $\Rightarrow$  (1) Let  $M$  be a kernel of an  $(n, 0)$ -injective cover  $\alpha : A \rightarrow B$  with  $A$  injective. By Proposition 4,  $M$  is  $(n, 0)$ -FI-injective. Now let  $K$  be an injective submodule of  $M$ . Suppose  $A = K \oplus L, p : A \rightarrow L$  is the projection and  $i : L \rightarrow A$  is the inclusion. It is easy to see that  $\alpha(ip) = \alpha$  since  $\alpha(K) = 0$ . Therefore  $ip$  is an isomorphism since  $\alpha$  is a cover. Thus  $i$  is epic, and hence  $A = L, K = 0$ . So  $M$  is reduced.

**Theorem 1** Let  $R$  be a left  $n$ -coherent ring. Then a left  $R$ -module  $M$  is  $(n, 0)$ -FI-injective if and only if  $M$  is a direct sum of an injective left  $R$ -module and a reduced  $(n, 0)$ -FI-injective left  $R$ -module.

**Proof** "If" part is clear.

"Only if" part. Let  $M$  be an  $(n, 0)$ -FI-injective left  $R$ -module. Consider the exact sequence  $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$ . Note that  $E(M) \rightarrow E(M)/M$  is an  $(n, 0)$ -injective precover of  $E(M)/M$  by Proposition 2.8. But  $E(M)/M$  has an  $(n, 0)$ -injective cover  $L \rightarrow E(M)/M$ , so we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \xrightarrow{f} & L & \rightarrow & E(M)/M \rightarrow 0 \\ & & \downarrow \varphi & & \downarrow \gamma & & \parallel \\ 0 & \rightarrow & M & \xrightarrow{\alpha} & E(M) & \rightarrow & E(M)/M \rightarrow 0 \\ & & \downarrow \sigma & & \downarrow \beta & & \parallel \\ 0 & \rightarrow & K & \xrightarrow{f} & L & \rightarrow & E(M)/M \rightarrow 0 \end{array}$$

Note that  $\beta\gamma$  is an isomorphism, and so  $E(M) = \ker(\beta) \oplus \text{im}(\gamma)$ . Thus  $L$  and  $\ker(\beta)$  are injective (for  $\text{im}(\gamma) \simeq L$ ). Therefore  $K$  is a reduced  $(n, 0)$ -FI-injective module by Proposition 9. Since  $\sigma\varphi$  is an isomorphism by the Five Lemma, we have  $M = \ker(\sigma) \oplus \text{im}(\varphi)$ , where  $\text{im}(\varphi) \simeq K$ . In addition, we get the commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \ker(\sigma) & \rightarrow & \ker(\beta) & \rightarrow & 0 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & M & \xrightarrow{\alpha} & E(M) & \rightarrow & E(M)/M \rightarrow 0 \\ & & \downarrow \sigma & & \downarrow \beta & & \parallel \\ 0 & \rightarrow & K & \xrightarrow{f} & L & \rightarrow & E(M)/M \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Hence  $\ker(\sigma) \simeq \ker(\beta)$  by the  $3 \times 3$  Lemma [10, Exercise 6.16, p.175]. This completes the proof.

It is well known that if  $R$  is a left  $n$ -coherent ring, then every right  $R$ -module has a  $(n, 0)$ -flat preenvelope (see [13]). Here we have

**Proposition 6** Let  $R$  be a left  $n$ -coherent ring.

(1) If  $L$  is a cokernel of a  $(n, 0)$ -flat preenvelope  $f : K \rightarrow F$  of a right  $R$ -module  $K$ , where  $F$  is flat, then  $L$  is  $(n, 0)$ -FI-flat.

(2) If  $M$  is an  $n$ -presented  $(n, 0)$ -FI-flat right  $R$ -module, then  $M$  is a cokernel of an  $(n, 0)$ -flat preenvelope.

**Proof** (1) There is an exact sequence  $0 \rightarrow \text{im}(f) \xrightarrow{i} F \rightarrow L \rightarrow 0$ . It is clear that  $i : \text{im}(f) \rightarrow F$  is an  $(n, 0)$ -flat preenvelope. For any  $(n, 0)$ -injective left  $R$ -module  $N, N^+$  is  $(n, 0)$ -flat by [14, Theorem 2.15]. Thus we obtain an exact

sequence

$$\text{Hom}(F, N^+) \rightarrow \text{Hom}(\text{im}(f), N^+) \rightarrow 0,$$

which yields the exactness of  $(F \otimes N)^+ \rightarrow (\text{im}(f) \otimes N)^+ \rightarrow 0$ . So the sequence  $0 \rightarrow \text{im}(f) \otimes N \rightarrow F \otimes N$  is exact. But the flatness of  $F$  implies the exactness of  $0 = \text{Tor}_1(F, N) \rightarrow \text{Tor}_1(L, N) \rightarrow \text{im}(f) \otimes N \rightarrow F \otimes N$ , and hence  $\text{Tor}_1(L, N) = 0$ .

(2) Let  $M$  be an  $n$ -presented  $(n, 0)$ -FI-flat right  $R$ -module. There is an exact sequence  $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  finitely generated projective and  $K$  is  $(n - 1)$ -presented. We claim that  $K \rightarrow P$  is an  $(n, 0)$ -flat preenvelope. In fact, for any  $(n, 0)$ -flat right  $R$ -module  $F$ , we have  $\text{Tor}_1(M, F^+) = 0$ , and so we get the following commutative diagram with the first row exact:

$$\begin{array}{ccccc} 0 & \longrightarrow & K \otimes F^+ & \xrightarrow{\alpha} & P \otimes F^+ \\ & & \downarrow \tau_{K,F} & & \downarrow \tau_{P,F} \\ & & \text{Hom}(K, F)^+ & \xrightarrow{\theta} & \text{Hom}(P, F)^+ \end{array}$$

Note that  $\tau_{K,F}$  is an epimorphism and  $\tau_{P,F}$  is an isomorphism by [2, Lemma 2]. Thus  $\theta$  is a monomorphism, and hence  $\text{Hom}(P, F) \rightarrow \text{Hom}(K, F)$  is epic, as required.

We shall say that a right  $R$ -module  $M$  is strongly  $(n, 0)$ -FI-flat if  $\text{Tor}_i(M, G) = 0$  for all  $(n, 0)$ -injective left  $R$ -modules  $G$  and all  $i \geq 1$ . Similarly, a left  $R$ -module  $N$  will be called strongly  $(n, 0)$ -FI-injective if  $\text{Ext}^i(G, N) = 0$  for all  $(n, 0)$ -injective left  $R$ -modules  $G$  and all  $i \geq 1$ .

**Theorem 2** Let  $R$  be a left and right  $n$ -coherent ring. Consider the following conditions:

- (1)  $(n, 0)$ -id $({}_R R) \leq 1$ .
- (2) Every submodule of an  $(n, 0)$ -FI-flat right  $R$ -module, which factor module is  $n$ -presented, is  $(n, 0)$ -FI-flat.
- (3) Every  $n$ -presented  $(n, 0)$ -FI-flat right  $R$ -module is strongly  $(n, 0)$ -FI-flat.
- (4) Every  $(n, 0)$ -FI-injective left  $R$ -module is strongly  $(n, 0)$ -FI-injective.
- (5) Every quotient of an  $(n, 0)$ -FI-injective left  $R$ -module is  $(n, 0)$ -FI-injective.

Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Leftarrow$  (4)  $\Leftarrow$  (5).

**Proof** (1)  $\Rightarrow$  (2) Let  $A$  be a submodule of an  $(n, 0)$ -FI-flat right  $R$ -module  $B$  such that  $B/A$  is  $n$ -presented and  $M$  an  $(n, 0)$ -injective left  $R$ -module. Then one gets an exact sequence  $\text{Tor}_2(B/A, M) \rightarrow \text{Tor}_1(A, M) \rightarrow \text{Tor}_1(B, M) = 0$ . On the other hand, there is a pure exact sequence  $0 \rightarrow M \rightarrow \prod (R_R)^+$  since  $(R_R)^+$  is a cogenerator in  $R\text{-Mod}$ . Thus we get a split exact sequence  $(\prod (R_R)^+)^+ \rightarrow M^+ \rightarrow 0$ . Note that  $(n, 0)\text{-fd}((R_R)^+) = (n, 0)\text{-id}(R_R) \leq 1$  by [14, Theorem 2.15], and so  $(n, 0)\text{-fd}(\prod (R_R)^+) \leq 1$  since  $R$  is right  $n$ -coherent. It follows that  $(n, 0)\text{-id}((\prod (R_R)^+)^+) = (n, 0)\text{-fd}(\prod (R_R)^+) \leq 1$  by [14, Theorem 2.15]. Hence  $(n, 0)\text{-fd}(M) = (n, 0)\text{-id}(M^+) \leq 1$ . Thus  $\text{Tor}_2(B/A, M) = 0$  by the condition, and so  $\text{Tor}_1(A, M) = 0$ . Therefore,  $A$  is  $(n, 0)$ -FI-flat.

(2)  $\Rightarrow$  (3) Let  $M$  be an  $n$ -presented  $(n, 0)$ -FI-flat right  $R$ -module. Then there is an exact sequence  $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  projective. So  $K$  is  $(n, 0)$ -FI-flat by (2). Thus  $M$  is strongly  $(n, 0)$ -FI-flat by induction.

(5)  $\Rightarrow$  (4) Let  $M$  be an  $(n, 0)$ -FI-injective left  $R$ -module. Then there is an exact sequence  $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$  with  $E$  injective. So  $L$  is  $(n, 0)$ -FI-injective by (5). It is easy to check that  $M$  is strongly  $(n, 0)$ -FI-injective by induction.

(4)  $\Rightarrow$  (3) holds by Remark 2 and the standard isomorphism:  $\text{Ext}^n(N, M^+) \simeq \text{Tor}_n(M, N)^+$  for any right  $R$ -module  $M$ , any left  $R$ -module  $N$  and any  $n \geq 1$  (see [10, p.360]).

Recall that a short exact sequence of right  $R$ -modules  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is called  $n$ -pure if every  $n$ -presented right  $R$ -module is projective with respect to this sequence [14]. In this case,  $A$  is said to be an  $n$ -pure submodule of  $B$ . It is easy to see that the pure exact sequence is 1-pure exact in this definition, and the pure exact sequence must be  $n$ -pure. Let  $A$  be a pure submodule of the right  $R$ -module  $B$ ,  $A$  must be an  $n$ -pure submodule of  $B$ .

**Proposition 7** A left  $(n, 0)$ -FI-injective  $R$ -module  $N$  is  $(n, 0)$ -injective if and only if, for every  $n$ -presented left  $R$ -module  $M$ , every homomorphism  $f : M \rightarrow L$  factors through an injective left  $R$ -module, where  $L$  is a cokernel of injective envelope of  $N$ .

**Proof** "Only if" part. There is an exact sequence  $0 \rightarrow N \rightarrow E(N) \xrightarrow{\pi} L \rightarrow 0$  with  $E$  injective. Since the exact sequence is  $n$ -pure, there exists  $g : M \rightarrow E$  such that  $\pi g = f$ , as required.

"If" part. It is enough to show that the exact sequence  $0 \rightarrow N \xrightarrow{i} E(N) \xrightarrow{\pi} L \rightarrow 0$  is  $n$ -pure by [14, Theorem 2.2]. Let  $M$  be any  $n$ -presented right  $R$ -module. For any  $f : M \rightarrow L$ , there exist an injective left  $R$ -module  $Q$  and  $g : M \rightarrow Q$  and  $h : Q \rightarrow L$  such that  $f = hg$  by hypothesis. Note that  $E(N) \xrightarrow{\pi} L$  is a precover of  $L$ , since  $N$  is FI-injective by Proposition 4. Thus there exists  $\alpha : Q \rightarrow E(N)$  such that  $h = \pi\alpha$ , and so  $f = \pi\alpha g$ . Therefore we get an exact sequence  $\text{Hom}(M, E(N)) \rightarrow \text{Hom}(M, L) \rightarrow 0$ . So  $N$  is  $(n, 0)$ -injective.

### III. $(n, 0)$ -INJECTIVE DIMENSIONS AND THE LEFT DERIVED FUNCTORS OF HOM

As is mentioned in the introduction, if  $R$  is a left  $n$ -coherent ring, then  $\text{Hom}(-, -)$  is left balanced on  ${}_R \mathcal{M} \times_R \mathcal{M}$  by  $n\text{-}\mathcal{FI} \times n\text{-}\mathcal{FI}$ . Let  $\text{Ext}_n(-, -)$  denote the  $n$ th left derived functor of  $\text{Hom}(-, -)$  with respect to the pair  $n\text{-}\mathcal{FI} \times n\text{-}\mathcal{FI}$ . Then, for two left  $R$ -modules  $M$  and  $N$ ,  $\text{Ext}_n(M, N)$  can be computed using a right  $n\text{-}\mathcal{FI}$ -resolution of  $M$  or a left  $n\text{-}\mathcal{FI}$ -resolution of  $N$ .

Let  $0 \rightarrow M \xrightarrow{g} F^0 \xrightarrow{f} F^1 \rightarrow \dots$  be a right  $n\text{-}\mathcal{FI}$ -resolution of  $M$ . Applying  $\text{Hom}(-, N)$ , we obtain the deleted complex  $\dots \rightarrow \text{Hom}(F^1, N) \xrightarrow{f^*} \text{Hom}(F^0, N) \rightarrow 0$ . Then  $\text{Ext}_n(M, N)$  exactly the  $n$ th homology of the complex above. There is a canonical map  $\sigma :$

$$\text{Ext}_0(M, N) = \text{Hom}(F^0, N) / \text{im}(f^*) \rightarrow \text{Hom}(M, N)$$

defined by  $\sigma(\alpha + \text{im}(f^*)) = \alpha g$  for  $\alpha \in \text{Hom}(F^0, N)$ .

**Proposition 8** Let  $R$  be a left  $n$ -coherent ring. The following are equivalent for a left  $R$ -module  $M$ :

- (1)  $M$  is  $(n, 0)$ -injective.

(2) The canonical map  $\sigma : \text{Ext}_0(M, N) \rightarrow \text{Hom}(M, N)$  is an epimorphism for any left  $R$ - module  $N$ .

(3) The canonical map  $\sigma : \text{Ext}_0(M, M) \rightarrow \text{Hom}(M, M)$  is an epimorphism.

**Proof** (1)  $\Rightarrow$  (2) is obvious by letting  $F^0 = M$ .

(2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1). By (3), there exists  $\alpha \in \text{Hom}(F^0, M)$  such that  $\sigma(\alpha + \text{im}(f^*)) = \alpha g = 1_M$ . Thus  $M$  is isomorphic to a direct summand of  $F^0$ , and hence it is  $(n, 0)$ -injective.

**Corollary 2** The following are equivalent for a left  $n$ -coherent ring  $R$ .

(1)  ${}_R R$  is  $(n, 0)$ -injective.

(2) The canonical map  $\sigma : \text{Ext}_0({}_R R, N) \rightarrow \text{Hom}({}_R R, N)$  is an epimorphism for any left  $R$ - module  $N$ .

(3) The canonical map  $\sigma : \text{Ext}_0({}_R R, {}_R R) \rightarrow \text{Hom}({}_R R, {}_R R)$  is an epimorphism.

(4) Every  $(n$ -presented) left  $R$ -module has an epic  $(n, 0)$ -injective cover.

(5) Every  $(n$ -presented) right  $R$ -module has a monic  $(n, 0)$ -flat preenvelope.

(6) Every  $(n$ -presented) right  $R$ -module is a submodule of a  $(n, 0)$ -flat right  $R$ -module.

**Proof** (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) follow from Proposition 8.

(1)  $\Rightarrow$  (4). Let  $M$  be a left  $R$ -module, then  $M$  has an  $(n, 0)$ -injective cover  $g$ . On the other hand, there is an exact sequence  $F \rightarrow M \rightarrow 0$  with  $F$  free. Since  $F$  is  $(n, 0)$ -injective by (1),  $g$  is an epimorphism.

(4)  $\Rightarrow$  (1). Let  $f : N \rightarrow {}_R R$  be an epic  $(n, 0)$ -injective cover. Then  ${}_R R$  is isomorphic to a direct summand of  $N$ , and so  ${}_R R$  is  $(n, 0)$ -injective.

(1)  $\Leftrightarrow$  (5). by [13, Theorem 4.5]

(5)  $\Rightarrow$  (6) is obvious.

(6)  $\Rightarrow$  (5) follows since  $R$  is a left  $n$ -coherent ring and by [13, Proposition 4.1].

**Proposition 9** Let  $R$  be a left  $n$ -coherent ring. Then the following are equivalent for a left  $R$ - module  $M$ :

(1) right  $n$ - $\mathcal{FT}$ -dim  $M \leq 1$ .

(2) The canonical map  $\sigma : \text{Ext}_0(M, N) \rightarrow \text{Hom}(M, N)$  is a monomorphism for any left  $R$ - module  $N$ .

**Proof** (1)  $\Rightarrow$  (2). By (1),  $M$  has a right  $n$ - $\mathcal{FT}$ -resolution  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow 0$ . Thus we get an exact sequence  $0 \rightarrow \text{Hom}(F^1, N) \rightarrow \text{Hom}(F^0, N) \rightarrow \text{Hom}(M, N)$  for any left  $R$ -module  $N$ . Hence  $\sigma$  is a monomorphism.

(2)  $\Rightarrow$  (1). Consider the exact sequence  $0 \rightarrow M \rightarrow F^0 \rightarrow L^1 \rightarrow 0$ , where  $M \rightarrow F^0$  is an  $(n, 0)$ -injective preenvelope. We only need to show that  $L^1$  is  $(n, 0)$ -injective. By [6, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{array}{ccccc} \text{Ext}_0(L^1, L^1) & \longrightarrow & \text{Ext}_0(F^0, L^1) & & \\ \downarrow \sigma_1 & & \downarrow \sigma_2 & & \\ 0 \longrightarrow \text{Hom}(L^1, L^1) & \longrightarrow & \text{Hom}(F^0, L^1) & & \\ & & \longrightarrow \text{Ext}_0(M, L^1) & \longrightarrow & 0 \\ & & \downarrow \sigma_3 & & \\ & & \longrightarrow \text{Hom}(M, L^1) & & \end{array}$$

Note that  $\sigma_2$  is an epimorphism by Proposition 8 and  $\sigma_3$  is a monomorphism by (2). Hence  $\sigma_1$  is an epimorphism by the

Snake Lemma[10, Theorem 6.5]. Thus  $L^1$  is  $(n, 0)$ -injective by Proposition 8, and so (1) follows.

**Lemma 1** Let  $R$  be a left  $n$ -coherent ring. Then

(1) right  $n$ - $\mathcal{FT}$ - dim  $(M) = (n, 0)$ -id $(M)$  for any left  $R$ -module  $M$ ;

(2)  $(n, 0)$ -wdim $(R) = l.(n, 0)$ -dim $(R) = \text{gl right } n$ - $\mathcal{FT}$ -dim  $\mathcal{M}$ .

**Proof** (1) It is clear that  $(n, 0)$ -id $(M) \leq \text{right } n$ - $\mathcal{FT}$ -dim  $M$ . Conversely, we may assume that  $(n, 0)$ -id $(M) = m < \infty$ . Let  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots \rightarrow F^{m-1} \rightarrow L \rightarrow 0$  be a partial right  $n$ - $\mathcal{FT}$ -resolution of  $M$ . Then we get an exact sequence  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots \rightarrow F^{m-1} \rightarrow L \rightarrow 0$ . Therefore,  $L$  is  $(n, 0)$ -injective by [14, Theorem 2.12], and so right  $n$ - $\mathcal{FT}$ -dim  $M \leq m$ , as desired.

(2) follows from [14, Theorem 2.15 ] and (1).

**Lemma 2** ([7]) Let  $\mathcal{C}$  be a class of  $R$ -modules and  $M$  an  $R$ -module.

(1) If  $F \rightarrow M$  and  $G \rightarrow M$  are  $\mathcal{C}$ -precovers with kernels  $K$  and  $L$ , respectively, then  $K \oplus G \simeq L \oplus F$ .

(2) If  $M \rightarrow F$  and  $M \rightarrow G$  are  $\mathcal{C}$ -preenvelopes with cokernels  $K$  and  $L$ , respectively, then  $K \oplus G \simeq L \oplus F$ .

Recall that a left  $R$  is called left  $n$ -hereditary[14] if every  $(n - 1)$ - presented submodule of projective left  $R$ -module is projective.

Clearly, a ring  $R$  is left semihereditary if and only if it is right 1- hereditary. Left  $n$ -hereditary ring is left  $(n + 1)$ -hereditary.

**Lemma 3** ([14]) The following statements are equivalent for a ring  $R$ :

(1)  $R$  is left  $n$ -hereditary.

(2)  $R$  is left  $n$ -coherent and  $l.(n, 0)$ -dim $(R) \leq 1$ .

(3) Factor module of  $(n, 0)$ -injective left  $R$ -module is  $(n, 0)$ -injective.

(4) Factor module of injective left  $R$ -module is  $(n, 0)$ -injective.

(5)  $R$  is a right  $(n, 1)$ -ring.

**Theorem 3** The following are equivalent for a left  $n$ -coherent ring  $R$ :

(1)  $R$  is a left  $n$ -hereditary ring (i.e.  $l.(n, 0)$ -dim $(R) \leq 1$ ).

(2) The canonical map  $\sigma : \text{Ext}_0(M, N) \rightarrow \text{Hom}(M, N)$  is monic for all left  $R$ -modules  $M$  and  $N$ .

(3) Every left  $R$ -module has a monic  $(n, 0)$ -injective cover.

(4) Every  $(n, 0)$ -FI-injective left  $R$ -module is injective.

(5) Every  $(n, 0)$ -FI-injective left  $R$ -module is  $(n, 0)$ -injective.

(6) Every  $(n$ -presented)  $(n, 0)$ -FI-flat right  $R$ -module is flat.

(7) The kernel of any  $(n, 0)$ -injective (pre)cover of a left  $R$ -module is  $(n, 0)$ -injective.

(8) The cokernel of any  $(n, 0)$ -injective preenvelope of a left  $R$ -module is  $(n, 0)$ -injective.

(9) The kernel of any  $(n, 0)$ -flat (pre)cover of a right  $R$ -module is flat.

**Proof** (1)  $\Leftrightarrow$  (2) holds by Proposition 9 and Lemma 1.

(1)  $\Rightarrow$  (4) follows from Proposition 3 and Lemma 1.

(4)  $\Rightarrow$  (5) is trivial.

(5)  $\Rightarrow$  (6). Let  $M$  be an  $(n, 0)$ -FI-flat right  $R$ -module. Then  $M^+$  is  $(n, 0)$ -FI-injective by Remark 1, and hence  $M^+$  is

$(n, 0)$ -injective by (5). So  $M$  is  $(n, 0)$ -flat by [14, Theorem 2.15].

(1)  $\Rightarrow$  (3) follows from Lemma 3 and [13, Proposition 4.9].

(3)  $\Rightarrow$  (7). Let  $f : F \rightarrow M$  be an  $(n, 0)$ -injective precover of a left  $R$ -module  $M$  and  $K = \ker(f)$ . Since there exists a monic  $(n, 0)$ -injective cover  $g : G \rightarrow M$  by (3), we have  $K \oplus G \simeq F$  by Lemma 2(1). So  $K$  is  $(n, 0)$ -injective.

(7)  $\Rightarrow$  (1). It is enough to show that any quotient of an  $(n, 0)$ -injective left  $R$ -module is  $(n, 0)$ -injective. But it is clear by Lemma 2.

(1)  $\Leftrightarrow$  (8) follows from Lemma 1.

(1)  $\Leftrightarrow$  (9) is obvious.

**Theorem 4** Let  $R$  be a left  $n$ -coherent ring and an integer  $m \geq 2$ . The following are equivalent for a left  $R$ -module  $M$ :

(1) right  $n$ - $\mathcal{FL}$ -dim  $M \leq m$ .

(2)  $\text{Ext}_{m+k}(M, N) = 0$  for all left  $R$ -modules  $N$  and all  $k \geq -1$ .

(3)  $\text{Ext}_{m-1}(M, N) = 0$  for all left  $R$ -modules  $N$ .

**Proof** (1)  $\Rightarrow$  (2). Let  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots \rightarrow F^m \rightarrow 0$  be a right  $n$ - $\mathcal{FL}$ -resolution of  $M$ , which induces an exact sequence  $0 \rightarrow \text{Hom}(F^m, N) \rightarrow \text{Hom}(F^{m-1}, N) \rightarrow \text{Hom}(F^{m-2}, N)$  for any left  $R$ -module  $N$ . Hence  $\text{Ext}_m(M, N) = \text{Ext}_{m-1}(M, N) = 0$ . Note that it is clear that  $\text{Ext}_{m+k}(M, N) = 0$  for all  $k \geq 1$ . Then (2) holds.

(2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1). Let  $0 \rightarrow M \rightarrow F^0 \rightarrow \dots \rightarrow F^{m-2} \xrightarrow{f} F^{m-1} \xrightarrow{g} F^m \rightarrow \dots$  be a right  $n$ - $\mathcal{FL}$ -resolution of  $M$ , with  $L^m = \text{coker}(F_{m-2} \rightarrow F_{m-1})$ . We only need to show that  $L^m$  is  $(n, 0)$ -injective. In fact, we have the exact sequence  $F^{m-1} \xrightarrow{\pi} L^m \rightarrow 0$  and  $0 \rightarrow L^m \xrightarrow{\lambda} F^{m-1}$  such that  $g = \lambda\pi$  by (3),  $\text{Ext}_{m-1}(M, L^m) = 0$ . Thus the sequence  $0 \rightarrow \text{Hom}(F^m, L^m) \xrightarrow{g^*} \text{Hom}(F_{m-1}, L^m) \xrightarrow{f^*} \text{Hom}(F^{m-2}, L^m)$  is exact. Since  $f^*(\pi) = \pi f = 0, \pi \in \ker(f^*) = \text{im}(g^*)$ . Thus there exists  $h \in \text{Hom}(F^m, L^m)$  such that  $\pi = g^*(h) = hg = h\lambda\pi$ , and hence  $h\lambda = 1$  since  $\pi$  is epic. Therefore  $L^m$  is  $(n, 0)$ -injective.

**Corollary 3** The following are equivalent for a left  $n$ -coherent ring  $R$  and an integer  $m \geq 2$ :

(1)  $l.(n, 0)$ -dim  $(R) \leq m$ .

(2)  $\text{Ext}_{m+k}(M, N) = 0$  for all left  $R$ -modules  $M$  and  $N$ , and all  $k \geq -1$ .

(3)  $\text{Ext}_{m-1}(M, N) = 0$  for all left  $R$ -modules  $M$  and  $N$ .

**Proof** It follows from Lemma 1 and Theorem 4.

It has been proven that  $R$  is a left coherent ring and  $l.\text{FP-dim}(R) \leq 2$  if and only if every right  $R$ -module has an FP-injective cover with the unique mapping property. Now we have

**Theorem 5** The following are equivalent for a ring  $R$ :

(1)  $R$  is left  $n$ -coherent and  $l.(n, 0)$ -dim  $(R) \leq 2$ .

(2) Every left  $R$ -module has an  $(n, 0)$ -injective cover with the unique mapping property.

(3)  $R$  is left  $n$ -coherent and  $\text{Ext}_1(M, N) = 0$  for all left  $R$ -modules  $M$  and  $N$ .

(4)  $R$  is left  $n$ -coherent and  $\text{Ext}_k(M, N) = 0$  for all left  $R$ -modules  $M, N$  and all  $k \geq 1$ .

**Proof** (1)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) follow from Corollary 3.

(1)  $\Rightarrow$  (2). Let  $M$  be any left  $R$ -module. Then  $M$  has an  $(n, 0)$ -injective cover  $f : F \rightarrow M$ . It is enough to

show that, for any  $(n, 0)$ -injective left  $R$ -module  $G$  and any homomorphism  $g : G \rightarrow F$  such that  $fg = 0$ , we have  $g = 0$ . In fact, there exists  $\beta : F/\text{im}(g) \rightarrow M$  such that  $\beta\pi = f$  since  $\text{im}(g) \subseteq \ker(f)$ , where  $\pi : F \rightarrow F/\text{im}(g)$  is the natural map. Since  $l.(n, 0)$ -dim  $(R) \leq 2, F/\text{im}(g)$  is  $(n, 0)$ -injective. Thus there exists  $\alpha : F/\text{im}(g) \rightarrow F$  such that  $\beta = f\alpha$ , and so we get the commutative diagram with an exact row:

$$\begin{array}{ccccccc} G & \xrightarrow{g} & F & \xleftarrow{\pi} & F/\text{im}(g) & \longrightarrow & 0 \\ & \searrow^0 & \downarrow f & \swarrow \beta & & & \\ & & M & & & & \end{array}$$

Thus  $f\alpha\pi = f$ , and hence  $\alpha\pi$  is an isomorphism. Therefore,  $\pi$  is monic, and so  $g = 0$ .

(2)  $\Rightarrow$  (1). We first prove that  $R$  is a left  $n$ -coherent ring. Let  $\{C_i, \varphi_j^i\}$  be a direct system with each  $C_i$   $(n, 0)$ -injective. By hypothesis,  $\varinjlim C_i$  has an  $(n, 0)$ -injective cover  $\alpha : E \rightarrow \varinjlim C_i$  with the unique mapping property. Let  $\alpha_i : C_i \rightarrow \varinjlim C_i$  satisfy  $\alpha_i = \alpha_j \varphi_j^i$  whenever  $i \leq j$ . Then there exists  $f_i : C_i \rightarrow E$  such that  $\alpha_i = \alpha f_i$  for any  $i$ . It follows that  $\alpha f_i = \alpha f_j \varphi_j^i$ , and so  $f_i = f_j \varphi_j^i$  whenever  $i \leq j$ . Therefore, by the definition of direct limits, there exists  $\beta : \varinjlim C_i \rightarrow E$  such that  $f_i = \beta \alpha_i$  and  $f_j = \beta \alpha_j$ . Thus  $(\alpha\beta)\alpha_i = \alpha(\beta\alpha_i) = \alpha f_i = \alpha_i$  for any  $i$ . Therefore  $\alpha\beta = 1_{\varinjlim C_i}$ , by the definition of direct limits, and hence  $\varinjlim C_i$  is a direct summand of  $E$ . So  $\varinjlim C_i$  is  $(n, 0)$ -injective. Thus  $R$  is a left  $n$ -coherent ring by [1].

Next we prove that  $l.(n, 0)$ -dim  $(R) \leq 2$ . Let  $M$  be any left  $R$ -module. Then  $M$  has an  $(n, 0)$ -injective cover  $f : F \rightarrow M$  with the unique mapping property. So  $0 \rightarrow F \rightarrow M \rightarrow 0$  is a left  $n$ - $\mathcal{FL}$ -resolution. Thus  $gl$  left  $n$ - $\mathcal{FL}$ -dim  $_R \mathcal{M} = 0$ , and hence  $l.(n, 0)$ -dim  $(R) \leq 2$  by Corollary 3.

**Proposition 10** Let  $R$  be a left  $n$ -coherent ring. If  $M$  is an  $n$ -pure-injective left  $R$ -module, then  $(n, 0)$ -id  $(M) \leq m (m \geq 0)$  if and only if for the minimal left  $n$ - $\mathcal{FL}$ -resolution  $\dots \rightarrow F_m \rightarrow F_{m-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0$  of all  $n$ -pure-injective left  $R$ -module  $N, \text{Hom}(M, F_m) \rightarrow \text{Hom}(M, K_m)$  is an epimorphism.

**Proof** The proof is modeled on that of [6, Lemma 8.4.34].

We will proceed by induction on  $m$ . Let  $m = 0$ . If  $M$  is  $(n, 0)$ -injective, it is clear that  $\text{Hom}(M, F_0) \rightarrow \text{Hom}(M, K_0)$  is an epimorphism, since  $F_0 \rightarrow N$  is an  $(n, 0)$ -injective cover of  $N$ . Conversely, put  $N = M$ . Then  $\text{Hom}(M, F_0) \rightarrow \text{Hom}(M, M)$  is an epimorphism, and so  $M$  is  $(n, 0)$ -injective.

Let  $m \geq 1$ . There is an exact sequence  $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$  with  $E$  injective. Then we have the following exact commutative diagrams:

$$\begin{array}{ccccc} \text{Hom}(E, F_n) & \longrightarrow & \text{Hom}(E, K_n) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \text{Hom}(M, F_n) & \longrightarrow & \text{Hom}(M, K_n) & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Hom}(L, K_m) & \longrightarrow & \text{Hom}(L, F_{m-1}) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Hom}(E, K_m) & \longrightarrow & \text{Hom}(E, F_{m-1}) & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \text{Hom}(M, K_m) & \longrightarrow & \text{Hom}(M, F_{m-1}) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & \\
 & & & & & & \\
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & \longrightarrow & \text{Hom}(L, K_{m-1}) & & & & \\
 & & \downarrow & & & & \\
 & \longrightarrow & \text{Hom}(E, K_{m-1}) & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \\
 & \longrightarrow & \text{Hom}(M, K_{m-1}) & & & & 
 \end{array}$$

Thus  $(n, 0)\text{-id}(M) \leq m$  if and only if  $(n, 0)\text{-id}(L) \leq m - 1$  by [14, Theorem 2.12.], if and only if  $\text{Hom}(L, F_{m-1}) \rightarrow \text{Hom}(L, K_{m-1})$  is an epimorphism by induction if and only if  $\text{Hom}(E, K_m) \rightarrow \text{Hom}(M, K_m)$  is an epimorphism by the second diagram if and only if  $\text{Hom}(M, F_m) \rightarrow \text{Hom}(M, K_m)$  is an epimorphism by the first diagram.

IV.  $(n, 0)$ -INJECTIVE DIMENSIONS AND THE RIGHT DERIVED FUNCTORS OF TOR

In this section, we introduce the right derived functors of Tor. If  $R$  is  $n$ -coherent, the  $- \otimes -$  on  $\mathcal{M}_R \times_R \mathcal{M}$  is right balanced by  $n\text{-}\mathcal{F} \times n\text{-}\mathcal{FT}$ , where  $n\text{-}\mathcal{F}$  stands for the class of all  $(n, 0)$ -flat modules. In fact, we need to show that if  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  is a right  $n\text{-}\mathcal{F}$ -resolution, which exists by [13, Lemma 4.1], and  $G$  is an  $(n, 0)$ -injective left  $R$ -module, then  $0 \rightarrow M \otimes G \rightarrow F^0 \otimes G \rightarrow F^1 \otimes G \rightarrow \dots$  is exact. Applying the functor  $\text{Hom}_Z(-, Q/Z)$  and using a standard identity we see the sequence  $0 \leftarrow \text{Hom}(M, G^+) \leftarrow \text{Hom}(F^0, G^+) \leftarrow \text{Hom}(F^1, G^+) \leftarrow \dots$ . But  $G^+$  is  $(n, 0)$ -flat by [14, Theorem 2.15] and so this sequence is exact. This means the desired sequence is exact. Since right  $n\text{-}\mathcal{FT}$ -resolutions are exact, let  $0 \rightarrow N \rightarrow G^0 \rightarrow G^1 \rightarrow \dots$  of a left  $R$ -module  $N$ , then  $\dots \rightarrow G^{1+} \rightarrow G^{0+} \rightarrow N^+ \rightarrow 0$  is a left  $n\text{-}\mathcal{F}$ -resolution. So applying the functor  $\text{Hom}(F, -)$  to above sequence, we get the exact sequence  $\dots \rightarrow \text{Hom}(F, G^{1+}) \rightarrow \text{Hom}(F, G^{0+}) \rightarrow \text{Hom}(F, N^+) \rightarrow 0$  for  $F \in n\text{-}\mathcal{F}$ . Using a standard identity we get the exact sequence  $0 \rightarrow F \otimes N \rightarrow F \otimes G^0 \rightarrow F \otimes G^1 \rightarrow \dots$ .

Let  $\text{Tor}^n(-, -)$  denote the  $n$ th right derived functor of  $- \otimes -$  with respect to the pair  $n\text{-}\mathcal{F} \times n\text{-}\mathcal{FT}$ . Then, for two left  $R$ -modules  $M$  and  $N$ ,  $\text{Tor}^n(M, N)$  can be computed using a right  $n\text{-}\mathcal{F}$ -resolution of  $M$  or a right  $n\text{-}\mathcal{FT}$ -resolution of  $N$ .

**Lemma 4** If  $M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4$  is an exact sequence of left  $R$ -modules such that for every  $n$ -presented right  $R$ -module  $P$ ,  $P \otimes M_1 \rightarrow P \otimes M_2 \rightarrow P \otimes M_3 \rightarrow P \otimes M_4$  is exact, then  $K = \ker(M_3 \rightarrow M_4)$  is an  $n$ -pure submodule of  $M_3$ .

**Proof**  $P \otimes M_1 \rightarrow P \otimes M_2 \rightarrow P \otimes M_3 \rightarrow P \otimes M_4$  is exact and  $P \otimes K \rightarrow P \otimes M_3 \rightarrow P \otimes M_4$  is a complex. Thus exactness of the first sequence means  $0 \rightarrow P \otimes K \rightarrow P \otimes M_3$  is exact. This means  $K$  is an  $n$ -pure submodule of  $M_3$ .

**Theorem 6** Let  $R$  be a left  $n$ -coherent ring and an integer  $m \geq 2$ . The following are equivalent for a left  $R$ -module  $N$ :

- (1) right  $n\text{-}\mathcal{FT}\text{-dim } N \leq m$ .
- (2)  $\text{Tor}^{m+k}(M, N) = 0$  for all right  $R$ -modules  $M$  and all  $k \geq -1$ .
- (3)  $\text{Tor}^m(M, N) = \text{Tor}^{m-1}(M, N) = 0$  for all right  $R$ -modules  $M$ .
- (4)  $\text{Tor}^m(M, N) = \text{Tor}^{m-1}(M, N) = 0$  for all right  $n$ -presented  $R$ -modules  $M$ .

**Proof** (1)  $\Rightarrow$  (2) Let  $0 \rightarrow N \rightarrow A^0 \rightarrow \dots \rightarrow A^n \rightarrow 0$  be a right  $n\text{-}\mathcal{FT}$ -resolution of  $N$ . Then  $M \otimes A^{n-2} \rightarrow M \otimes A^{n-1} \rightarrow M \otimes A^n \rightarrow 0$  is exact and so  $\text{Tor}^{m-1}(M, N) = \text{Tor}^m(M, N) = 0$ . But clearly  $\text{Tor}^{m+k}(M, N) = 0$  for  $k \geq -1$ . Hence (2) holds.

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) is trivial.

(4)  $\Rightarrow$  (1). Let  $0 \rightarrow N \rightarrow A^0 \rightarrow A^1 \rightarrow \dots$  be a right  $n\text{-}\mathcal{FT}$ -resolution of  $N$ . Then for any  $n$ -presented  $R$ -module  $M$ ,  $M \otimes A^{n-2} \rightarrow M \otimes A^{n-1} \rightarrow M \otimes A^n \rightarrow M \otimes A^{n+1}$  is exact. So by Lemma 4,  $K = \ker(A^n \rightarrow A^{n+1})$  is  $n$ -pure in  $A^n$ . But an  $n$ -pure submodule of  $(n, 0)$ -injective module is  $(n, 0)$ -injective by [14, Proposition 2.2]. But then  $0 \rightarrow N \rightarrow A^0 \rightarrow A^1 \rightarrow \dots \rightarrow A^{n-1} \rightarrow K \rightarrow 0$  is a right  $n\text{-}\mathcal{FT}$ -resolution of  $N$  and (1) holds.

**Theorem 7** Let  $R$  be a left  $n$ -coherent ring and an integer  $m \geq 2$ . The following are equivalent for a left  $R$ -module  $N$ :

- (1) right  $n\text{-}\mathcal{F}\text{-dim } M \leq m$ .
- (2)  $\text{Tor}^{m+k}(M, N) = 0$  for all right  $R$ -modules  $N$  and all  $k \geq -1$ .
- (3)  $\text{Tor}^m(M, N) = \text{Tor}^{m-1}(M, N) = 0$  for all right  $R$ -modules  $N$ .

**Proof** (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) is trivial.

(3)  $\Rightarrow$  (1). Let  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  be a right  $n\text{-}\mathcal{F}$ -resolution of  $N$ . Then for any  $R$ -module  $N$ ,  $F^{n-2} \otimes N \rightarrow F^{n-1} \otimes N \rightarrow F^n \otimes N \rightarrow F^{n+1} \otimes N$  is exact. So by Lemma 4,  $K = \ker(F^n \rightarrow F^{n+1})$  is  $n$ -pure in  $F^n$  and so is  $(n, 0)$ -flat. But  $F^{n-2} \rightarrow F^{n-1} \rightarrow K \rightarrow 0$  is exact. Therefore,  $L = \ker(F^{n-2} \rightarrow F^{n-1})$  is  $n$ -pure in  $F^{n-2}$  and so is  $(n, 0)$ -flat by [14, Corollary 2.20]. But then  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots \rightarrow F^{n-3} \rightarrow L \rightarrow 0$  is a right  $n\text{-}\mathcal{F}$ -resolution of  $M$  and so (1) holds.

**Theorem 8** Let  $R$  be a left  $n$ -coherent ring and an integer  $m \geq 0$ . The following are equivalent

- (1) For every  $(n, 0)$ -flat left  $R$ -module  $F$ , there is an exact sequence  $0 \rightarrow F \rightarrow E^0 \rightarrow \dots \rightarrow E^m \rightarrow 0$  with each  $E^i$  is  $(n, 0)$ -injective.
- (2) If  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  is a right  $n\text{-}\mathcal{F}$ -resolution of  $M$ , then the sequence is exact at  $F^k$  for  $k \geq m - 1$ , where  $F^{-1} = M$ .
- (3) There is an exact sequence  $0 \rightarrow R \rightarrow E^0 \rightarrow \dots \rightarrow E^m \rightarrow 0$  of left  $R$ -module with each  $E^i$  is  $(n, 0)$ -injective.

**Proof** (1)  $\Rightarrow$  (3) is immediate.

(3)  $\Rightarrow$  (2) We recall that  $-\otimes-$  is right balanced on  $\mathcal{M}_R \times_R \mathcal{M}$  by  $n\text{-}\mathcal{F} \times n\text{-}\mathcal{FI}$  with right derived functors  $\text{Tor}^k(-, -)$ .

If  $m \geq 2$ , using the exact sequence  $0 \rightarrow R \rightarrow E^0 \rightarrow \dots \rightarrow E^m \rightarrow 0$ , we get  $\text{Tor}^k(M, R) = 0$  for  $k \geq m - 1$ . Computing using  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  as in (2), we see that  $\text{Tor}^k(M, R)$  is just the  $k$ th homology group of this complex, giving the desired result.

For  $m = 1$ ,  $0 \rightarrow R \rightarrow E^0 \rightarrow E^1 \rightarrow 0$  exact sequence gives  $\text{Tor}^1(M, R) = 0$  so that, as above,  $F^0 \rightarrow F^1 \rightarrow F^2$  is exact and  $M \otimes R \rightarrow \text{Tor}^0(M, R)$  is onto. computing the latter morphism using  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1$  is exact.

If  $m = 0$  then (3) means  $R$  is  $(n, 0)$ -injective as a left  $R$ -module. But the balance of  $-\otimes-$  then gives  $0 \rightarrow M \otimes R \rightarrow F^0 \otimes R \rightarrow F^1 \otimes R \rightarrow \dots$  is exact. That is  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  is exact.

(2)  $\Rightarrow$  (1). Assume (2) with  $m \geq 2$ . Let  $0 \rightarrow F \rightarrow E^0 \rightarrow \dots \rightarrow E^m \rightarrow 0$  with each  $E^i$  is  $(n, 0)$ -injective. Then by (2), we get  $\text{Tor}^k(M, F) = 0$  for  $k \geq m - 1$  since  $F$  is  $(n, 0)$ -flat. Computing using  $0 \rightarrow E^0 \rightarrow E^1 \rightarrow \dots$  and using the Lemma 4, we get  $K = \ker(E^m \rightarrow E^{m+1})$  is  $n$ -pure in  $A^m$  and so  $K$  is also  $(n, 0)$ -injective. Hence  $0 \rightarrow F \rightarrow E^0 \rightarrow \dots \rightarrow E^{m-1} \rightarrow K \rightarrow 0$  gives the desired exact sequence.

Now let  $m = 1$ . Then (2) says  $M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  is exact. So  $\text{Tor}^k(M, F) = 0$  for  $k = 0$  and  $M \otimes F \rightarrow \text{Tor}^0(M, F)$  is onto. Hence if  $0 \rightarrow F \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow 0$  is exact,  $M \otimes F \rightarrow M \otimes E^0 \rightarrow M \otimes E^1 \rightarrow M \otimes E^2$  is exact for all  $n$ -presented  $M$ . By Lemma 25, we again get the desired exact sequence  $0 \rightarrow F \rightarrow E^0 \rightarrow K \rightarrow 0$  with  $K = \ker(E^1 \rightarrow E^2)$ .

If  $m = 0$  then  $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$  exact means  $\text{Tor}^k(M, F) = 0$  for  $k > 0$  and  $M \otimes F \rightarrow \text{Tor}^0(M, F)$  is isomorphism. This gives that  $0 \rightarrow M \otimes F \rightarrow M \otimes E^0 \rightarrow M \otimes E^1$  is exact for all  $M$  which implies  $F$  is an  $n$ -pure submodule of  $E^0$  and so is  $(n, 0)$ -injective.

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