

Recovery of Missing Samples in Multi-channel Oversampling of Multi-banded Signals

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Abstract—We show that in a two-channel sampling series expansion of band-pass signals, any finitely many missing samples can always be recovered via oversampling in a larger band-pass region. We also obtain an analogous result for multi-channel oversampling of harmonic signals.

Keywords—oversampling, multi-channel sampling, recovery of missing samples, band-pass signal, harmonic signal

I. INTRODUCTION

FOR a bounded and closed band-region B , let PW_B be the Paley-Wiener space of finite energy (i.e. square integrable) signals of which frequencies are confined in B . That is,

$$PW_B := \{f(t) \in L^2(\mathbb{R}) : \text{supp } \hat{f}(\xi) \subset B\},$$

where $\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-it\xi} dt$ is the Fourier transform of $f(t)$ with inverse Fourier transform $f(t) = \mathcal{F}^{-1}(\hat{f})(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{it\xi} d\xi$.

If a signal $f(t)$ is single-banded with band-region $B = [-\pi\omega, \pi\omega]$ ($\omega > 0$), then $f(t)$ can be expanded as a Shannon sampling series:

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{\omega}\right) \frac{\sin \pi(t-n)}{\pi(t-n)},$$

in which all samples $\{f(\frac{n}{\omega}) : n \in \mathbb{Z}\}$ are independent. However, if we oversample $f(t)$ with higher rate than the optimal Nyquist rate ω , then the resulting samples are dependent. Using this observation, we may recover finitely many missing samples([2,3,5,8]). When we join oversampling and multi-channeling, we may or may not be able to recover finitely missing samples depending on the nature of the band-region B and pre-filters used in channeling ([6,10]). In this work, we show that in case of band-pass and harmonic signals, any finitely many missing samples can be always recovered through a multi-channel oversampling in a larger band-region of the same type.

II. OVERSAMPLING OF BAND-PASS SIGNALS

Consider a band-pass region $B = B_- \cup B_+$, where $w_0, w > 0$ and

$$B_- = [-\pi(\omega_0 + \omega), -\pi\omega_0] \text{ and } B_+ = [\pi\omega_0, \pi(\omega_0 + \omega)].$$

Then the optimal Nyquist rate for signals in PW_B is ω samples per second. For τ with $0 < \tau \leq w_0$, let $\tilde{B} = \tilde{B}_- \cup \tilde{B}_+$ be another band-pass region, where

$$\tilde{B}_- = [-\pi(\omega_0 + \omega + \tau), -\pi(\omega_0 - \tau)]$$

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and

$$\tilde{B}_+ = [\pi(\omega_0 - \tau), \pi(\omega_0 + \omega + \tau)].$$

We take τ so that $r := \frac{2\omega_0 + \omega}{2\tau + \omega}$ is a positive integer. Then $\tilde{B}_+ = \tilde{B}_- + r\pi(2\tau + \omega)$ so that \tilde{B} becomes a so-called selectively tiled band-region([4]) of length $2\pi\tilde{\omega}$ with $\tilde{\omega} = \omega + 2\tau$. Note that the smallest such τ is obtained when we take r to be the largest integer less than $1 + \frac{2\omega_0}{\omega}$. We now take two pre-filters of bounded measurable functions $A_j(\xi)$ ($j = 1, 2$) on \tilde{B} . We set

$$A(\xi) = \begin{bmatrix} A_1(\xi) & A_1(\xi + r\pi\tilde{\omega}) \\ A_2(\xi) & A_2(\xi + r\pi\tilde{\omega}) \end{bmatrix} \text{ on } \tilde{B}_-$$

and assume for some constant $\alpha > 0$, $|\det A(\xi)| \geq \alpha$ a.e. on \tilde{B}_- .

For any band-pass signal $f(t)$ in $PW_{\tilde{B}}$, let

$$c_j(f)(t) := \mathcal{F}^{-1}(A_j(\xi)\hat{f}(\xi))(t) = \frac{1}{\sqrt{2\pi}} \int_{\tilde{B}} A_j(\xi)\hat{f}(\xi)e^{it\xi} d\xi \quad (1)$$

be the channeled output signals of the input signal $f(t)$. Then ([4,7,8,9])

$$f(t) = \sum_{j=1}^2 \sum_n c_j(f)\left(\frac{2n}{\tilde{\omega}}\right) S_{j,n}(t), \quad (2)$$

which converges in $PW_{\tilde{B}}$ and also converges uniformly on \mathbb{R} . By taking Fourier transform on (2), we obtain

$$\hat{f}(\xi) = \sum_{j=1}^2 \sum_n c_j(f)\left(\frac{2n}{\tilde{\omega}}\right) \phi_{j,n}(\xi),$$

which converges in $L^2(B)$, where

$$\phi_{j,n}(\xi) = \frac{1}{\tilde{\omega}} \sqrt{\frac{2}{\pi}} U_j(\xi) e^{-i\frac{2n}{\tilde{\omega}}\xi}$$

and

$$A(\xi)^{-1} = \begin{bmatrix} U_1(\xi) & U_2(\xi) \\ U_1(\xi + r\pi\tilde{\omega}) & U_2(\xi + r\pi\tilde{\omega}) \end{bmatrix} \text{ on } \tilde{B}_-. \quad (3)$$

If $f(t)$ is in PW_B , i.e., $\text{supp } \hat{f} \subset B$, then

$$\hat{f}(\xi) = \sum_{j=1}^2 \sum_n c_j(f)\left(\frac{2n}{\tilde{\omega}}\right) \phi_{j,n}(\xi) \chi_B(\xi) \text{ in } L^2(B), \quad (4)$$

where $\chi_B(\xi)$ is the characteristic function of B . By taking inverse Fourier transform on (4), we have

$$f(t) = \sum_{j=1}^2 \sum_n c_j(f)\left(\frac{2n}{\tilde{\omega}}\right) T_{j,n}(t) \quad (5)$$

where $T_{j,n}(t) = \frac{1}{\sqrt{2\pi}} \int_B \phi_{j,n} e^{it\xi} d\xi$. We may call (5) a two-channel oversampling series expansion of $f(t)$ in PW_B .

III. RECOVERING MISSING SAMPLES

For a band-pass signal $f(t)$ in PW_B , consider its oversampled expansion (5).

Lemma 1. We have for any integer m

$$c_k(f)\left(\frac{2m}{\tilde{\omega}}\right) = \frac{1}{\pi\tilde{\omega}} \sum_n c_k(f)\left(\frac{2n}{\tilde{\omega}}\right) \int_{B_-} e^{i\frac{2}{\tilde{\omega}}(m-n)\xi} \quad (6)$$

for $k = 1, 2$.

Proof: By (1) and (4), we have

$$\begin{aligned} c_k(f)(t) &= \frac{1}{\sqrt{2\pi}} \int_{\tilde{B}} A_k(\xi) \hat{f}(\xi) e^{it\xi} d\xi \\ &= \frac{1}{\pi\tilde{\omega}} \sum_{j=1}^2 \sum_n c_j\left(\frac{2n}{\tilde{\omega}}\right) \int_{\tilde{B}} A_k(\xi) U_j(\xi) \chi_B(\xi) e^{i(t-\frac{2n}{\tilde{\omega}})\xi} d\xi. \end{aligned}$$

Hence for any integer m we have

$$\begin{aligned} c_k(f)\left(\frac{2m}{\tilde{\omega}}\right) &= \frac{1}{\pi\tilde{\omega}} \sum_{j=1}^2 \sum_n c_j\left(\frac{2n}{\tilde{\omega}}\right) \left[\int_{\tilde{B}_-} A_k(\xi) U_j(\xi) \chi_B(\xi) e^{i\frac{2}{\tilde{\omega}}(m-n)\xi} d\xi \right. \\ &\quad \left. + \int_{\tilde{B}_+} A_k(\xi) U_j(\xi) \chi_B(\xi) e^{i\frac{2}{\tilde{\omega}}(m-n)\xi} d\xi \right] \\ &= \frac{1}{\pi\tilde{\omega}} \sum_{j=1}^2 \sum_n c_j\left(\frac{2n}{\tilde{\omega}}\right) \int_{\tilde{B}_-} \left[A_k(\xi) U_j(\xi) \right. \\ &\quad \left. + A_k(\xi + r\pi\tilde{\omega}) U_j(\xi + r\pi\tilde{\omega}) \right] \chi_{B_-}(\xi) e^{i\frac{2}{\tilde{\omega}}(m-n)\xi} d\xi, \end{aligned}$$

from which (6) comes since $A_k(\xi) U_j(\xi) + A_k(\xi + \pi\tilde{\omega}) U_j(\xi + \pi\tilde{\omega}) = \delta_{jk}$ by (3). \square

Theorem 1. For any finite index sets of integers I_1 and I_2 , any finite missing samples $\{c_1(f)\left(\frac{2m}{\tilde{\omega}}\right) : m \in I_1\} \cup \{c_2(f)\left(\frac{2n}{\tilde{\omega}}\right) : n \in I_2\}$ can be uniquely recovered.

Proof: Set $I_1 = \{m_1, m_2, \dots, m_M\}$ if $I_1 \neq \emptyset$ and $I_2 = \{n_1, n_2, \dots, n_N\}$ if $I_2 \neq \emptyset$. Then we have from (6)

$$c_1(f)\left(\frac{2m_j}{\tilde{\omega}}\right) = \frac{1}{\pi\tilde{\omega}} \sum_{k=1}^M r(m_j, m_k) c_1(f)\left(\frac{2m_k}{\tilde{\omega}}\right) + g_{1j} \quad (7)$$

for $1 \leq j \leq M$ and

$$c_2(f)\left(\frac{2n_j}{\tilde{\omega}}\right) = \frac{1}{\pi\tilde{\omega}} \sum_{k=1}^N r(n_j, n_k) c_2(f)\left(\frac{2n_k}{\tilde{\omega}}\right) + g_{2j} \quad (8)$$

for $1 \leq j \leq N$ where g_{1j} 's and g_{2j} 's are known quantities and

$$r(m, n) := \int_{B_-} e^{i\frac{2}{\tilde{\omega}}(m-n)\xi} d\xi \text{ for } m, n \in \mathbb{Z}.$$

We may write (7-8) in a vector form as :

$$\begin{cases} (I - S_1)\mathbf{c}_1 = \mathbf{g}_1 \\ (I - S_2)\mathbf{c}_2 = \mathbf{g}_2 \end{cases} \quad (9)$$

where

$$\begin{aligned} \mathbf{c}_1 &:= \left(c_1(f)\left(\frac{2m_1}{\tilde{\omega}}\right), \dots, c_1(f)\left(\frac{2m_M}{\tilde{\omega}}\right) \right)^T, \\ \mathbf{c}_2 &:= \left(c_2(f)\left(\frac{2n_1}{\tilde{\omega}}\right), \dots, c_2(f)\left(\frac{2n_N}{\tilde{\omega}}\right) \right)^T, \\ \mathbf{g}_1 &:= (g_{11}, \dots, g_{1M})^T, \\ \mathbf{g}_2 &:= (g_{21}, \dots, g_{2N})^T, \end{aligned}$$

and

$$S_1 = \left[\frac{1}{\pi\tilde{\omega}} r(m_j, m_k) \right]_{j,k=1}^M, \quad S_2 = \left[\frac{1}{\pi\tilde{\omega}} r(n_j, n_k) \right]_{j,k=1}^N.$$

Note that S_1 and S_2 are self-adjoint. Now for any $u = (u_1, \dots, u_M) \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned} \langle S_1 u, u \rangle &= \frac{1}{\pi\tilde{\omega}} \sum_{j,k=1}^M r(m_j, m_k) u_k \bar{u}_j \\ &= \int_{\tilde{B}_-} \left| \sum_{j=1}^M \bar{u}_j \frac{1}{\sqrt{\pi\tilde{\omega}}} e^{i\frac{2}{\tilde{\omega}} m_j \xi} \right|^2 \chi_{B_-}(\xi) d\xi \\ &< \int_{\tilde{B}_-} \left| \sum_{j=1}^M \bar{u}_j \frac{1}{\sqrt{\pi\tilde{\omega}}} e^{i\frac{2}{\tilde{\omega}} m_j \xi} \right|^2 d\xi = \sum_{j=1}^M |u_j|^2 \end{aligned}$$

since $\left\{ \frac{1}{\sqrt{\pi\tilde{\omega}}} e^{i\frac{2}{\tilde{\omega}} m \xi} \right\}_{m \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\tilde{B}_-)$. Hence, 1 cannot be an eigenvalue of S_1 . Similarly, 1 cannot be an eigenvalue of S_2 . Therefore, both equations in (9) have unique solutions \mathbf{c}_1 and \mathbf{c}_2 . \square

Above process can be readily extended to multi-channel oversampling of harmonic signals (see [1] and Chapter 13 in [4]). Let $f(t)$ be a harmonic signal in PW_B , where

$$B := \bigcup_{i=1}^N [a_i, b_i]$$

is a harmonic band-region and

$$\begin{aligned} b_i - a_i &= \pi\omega \quad (1 \leq i \leq N) \\ a_{i+1} - b_i &= 2\pi\omega_0 \quad (1 \leq i < N) \text{ for } \omega, \omega_0 > 0. \end{aligned}$$

For $0 < \tau \leq \omega_0$, let $\tilde{B} := \bigcup_{i=1}^N \tilde{B}_i$ be another harmonic band-region, where

$$\tilde{B}_i = [a_i - \pi\tau, b_i + \pi\tau] \text{ for } 1 \leq i \leq N.$$

We take τ so that $r := \frac{2\omega_0 + \omega}{2\tau + \omega}$ is a positive integer. Then $\tilde{B}_j = \tilde{B}_i + (j-i)r\pi(2\tau + \omega)$ for $1 \leq i < j \leq N$ so that \tilde{B} becomes a so-called selectively tiled band-region of total length $N\pi\tilde{\omega}$, where $\tilde{\omega} = \omega + 2\tau$. We now take N pre-filters $A_j(\xi)$ ($j = 1, 2, \dots, N$) of bounded measurable functions on \tilde{B} . We set $A(\xi)$ be the $N \times N$ matrix whose (j, k) th component is given by

$$A_{jk}(\xi) = A_j(\xi + (k-1)r\pi\tilde{\omega})$$

and assume $|\det A(\xi)| \geq \alpha > 0$ a.e. on \tilde{B}_1 . Let

$$c_j(f)(t) := \mathcal{F}^{-1}(A_j(\xi) \hat{f}(\xi))(t) = \frac{1}{\sqrt{2\pi}} \int_{\tilde{B}} A_j(\xi) \hat{f}(\xi) e^{it\xi} d\xi$$

be the channeled output signals. Proceeding as in Section 2, we can obtain an oversampling formula for any harmonic signal $f(t)$ in PW_B (but viewed as a signal in $PW_{\bar{B}}$) as

$$f(t) = \sum_{j=1}^N \sum_n c_j(f) \left(\frac{2n}{\bar{\omega}} \right) T_{j,n}(t). \quad (10)$$

Then, we have the following multi-channel analog of Theorem 3.2.

Theorem 2. For any finite index sets of integers $I_i (i = 1, 2, \dots, N)$, any finite missing samples $\cup_{i=1}^N \{c_i(f) \left(\frac{2m}{\bar{\omega}} \right) : m \in I_i\}$ from the oversampling (10) can be uniquely recovered.

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