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# Reachable Set Bounding Estimation for Distributed Delay Systems with Disturbances

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*Abstract*—The reachable set bounding estimation for distributed delay systems with disturbances is a new problem. In this paper,we consider this problem subject to not only time varying delay and polytopic uncertainties but also distributed delay systems which is not studied fully untill now. we can obtain improved non-ellipsoidal reachable set estimation for neural networks with time-varying delay by the maximal Lyapunov-Krasovskii fuctional which is constructed as the pointwise maximum of a family of Lyapunov-Krasovskii fuctionals corresponds to vertexes of uncertain polytope.On the other hand,matrix inequalities containing only one scalar and Matlabs LMI Toolbox is utilized to give a non-ellipsoidal description of the reachable set.finally,numerical examples are given to illustrate the existing results.

*Keywords*—Reachable set, Distributed delay,Lyapunov-Krasovskii function, Polytopic uncertainties.

## I. Introduction

REACHABLE set is a set that bounds all the states starting from the origin by inputs with peak values for a dynamic systems with distributed delays and bounded disturbance inputs. Recently ,the problem of finding a smallest bound of reachable set has received considerable attention.It is well known that time delay may result in instability , sustained oscillations,bifurcation or chaos of neural networks which degrades system performance[1]-[4].So we consider about the reachable set with time delays. It played a important role in peak-to-peak minimization in control theory extensively investigated for time-delay systems in recent years . For instance , Improved ellipsoidal bound of reachable set for time-delayde linear systems with disturbances was presented by Kim[5].A delay-dependent result expressed in the form of matrix inequalities containing only one non-convex scalar was geted by modified Lyapunov-Krasovskii type function . A non-ellipsoidal reachable set estimation for uncertain neural networks with time-varying delay was derived by Zuo[6]-[8].The maximal Lyapunov functional,combined with the Razumikhin methodology and S-procedure is applied to get better results.Reachable set estimation for distributed delay systems with bounded disturbances which inputs are regarded as unit-energy bounded or unit-peak bounded is proprosed by Zhang [9] .DelayCdependent conditions for estimating the reachable set of the system with distributed delays is derived in the terms of Lyapunov-Krasovskii functional approach and the delay-partitioning technique and so on. We consided the reachable set bound estimation of the time-delay systems with distributed delays in this paper.This problem was investigated by many researchers[10]-[12] . However,it has not been well addressed in many works [13]-[15],which motivates our study. Our main results are different from the works in follow several aspects.First, we select Lyapunov matrices which vertices are more than the vertices of uncertain polytope,which can reduce conservatism and get a more accurate description of the reachable set bound. Second , we consided the systems with not only state-delayed but also distributed delays which is distinguish from other existing works.Third,we introduce linear matrix inequality techniques as well as convex-hull properties to get a tighter reachable set estimation by spliting integral and bring conservatism with lower computational complexity to some extent.Finally,numerical examples as well as simulation results are obtained to illustrate the advantages of our treatments.

*Notation*: Thtoughout the whole paper.$\Re^n$ denotes the $n$ dimensional Euclidean space and $\Re^{m\times n}$ is the set of all $\Re^{m\times n}$ real matrices . The superscript $'T'$ denotes matrix transposition,and $I$ and $0$ denote the identity and zero matrix with appropriate dimension.The notation $P > 0 (P \geq 0)$ means that $P$ is symmetric and positive definite ( positive semi-definite ) . Co$\{\bullet\}$denotes a convex hull.The symmetric terms in a symmetric matrix are denoted by *. Matrix ,if not explicitly stated,are assumed to have compatible dimensions.

## II. Problem Statement and Preliminaries

Consider the following state-delayed systems with distributed delays and bounded disturbance inputs.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t)(t-h(t)) \\ \qquad +C\displaystyle\int_{t-d(t)}^{t} x(s)ds + D\omega(t) \end{cases} \tag{1}$$

Where $x(t) \in \Re^n$ is the state vector , $A \in \Re^{n\times n}, B \in \Re^{n\times n}$ . C, D are known constant matrices with appropriate dimensions belong to a given polytope;$\omega(t) \in \Re^q$ is the disturbance input,the discrete delay $h(t)$ is a continuous-time differentiable function and the disturbance is a bounded function .We denote $\Omega = [A, B, C, D]$ and $\Omega = \sum_{i=1}^{N} \theta_i \Omega_i$,$\theta_i \geq 0$,$\sum_{i=1}^{N} \theta_i = 1$. Here the $N$ vertices of the polytope are described by $\Omega_i =$

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$[A_i, B_i, C_i, D_i]$,$i \in [1, N]$. We assume that

$$\begin{aligned} &0 \le h_1 \le h(t) \le h_2, |\dot{h}(t)| \le h, \\ &0 \le d_1 \le d(t) \le d_2, |\dot{d}(t)| \le d, \omega^T(t)\omega \le \omega_m \end{aligned} \tag{2}$$

Where $h_1, h_2, d_1, d_2, h_1, h, d$ are constants.We denote the set of reachable states with $\omega(t)$ that satisfies Eq.(2) by

$$R_x \triangleq x(t) \in R_n | x(t), \omega(t) satisfies Eq.(1) and (2) \tag{3}$$

We construct a functional with maximum of a family of Lyapunov-Krasovskii functions corresponds to a vertex of the polytope .We denote $\varepsilon(P,1) = \{x \in R^n : x^T P x \le 1\}$with the matrix $P > 0$ and denote the 1-level set as $L_1 = \{x \in R^n : V(x) \le 1\}$ ,the pointwise maximum quadratic function $V_{1,max}$ is denoted as $V_{1,max} = max\{x^T P_j x\}$, here $P_j > 0$ ,For any $x_0 \in R^n$ ,without loss of generality,we assume that there exists an integer $m(1 < m < M)$ such that $V_{1,max}(x_0) = \{x_0^T P_j x_0\}$, for $j \in [1, m]$and $V_{1,max}(x_0) > \{x_0^T P_j x_0\}$, for $j > m$ . We can get that (1):for a vector $\zeta \in R^n$ ,the directional derivativre of $V_{1,max}$ at $x_0$ along $\zeta$ is

$$\begin{aligned} \bigtriangledown_\zeta V_{1,max}(x_0) &= \lim_{t \to 0^+} \frac{V_{1,max}(x_0 + t\zeta) - V_{1,max}(x_0)}{t} \\ &= \max_{\xi \in V_{1,max}(x_0)} \{\xi^T \zeta\}; \end{aligned} \tag{4}$$

(2) $\partial V_{1,max}(x_0) = Co\{2P_j x_0, j = 1, 2, \cdots, m\}$,where $\partial f(x_0)$ is the subdiffentential of the function $f(x_0)$ at $x_0$ Before proceeding further ,we will state lemmas which is used in following text.

**Lemma 1**.For any matrix $N > 0$ ,the following inequality holds:
$(\int_{t-h}^{t} f(s)ds)^T N (\int_{t-h}^{t} f(s)ds) \le h \int_{t-h}^{t} f^T(s) N f(s) ds$

**Lemma 2**.Let scalar functions $f_1, f_2, f_2, \cdots, f_N : R^n \to R$, be positive in an open subset F of $R^n$ . Then the reciprocally convex combination of $f_i$ over F has the property:

$$\sum_{i=1} \frac{1}{\lambda_i} f_i(t) \ge \sum_{i=1} f_i(t) + \sum_{i \ne j} g_{i,j}(t)$$

Subject to

$$\{g_{i,j} : R^n \to R, g_{i,j}(t) = g_{j,i}(t), \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t) \end{bmatrix} \ge 0\}$$

Where the real numbers $\lambda_i$ satisfy $\lambda_i > 0$, and

$$\sum_i \lambda_i = 1.$$

**Lemma 3**[1].Assume $V$ is a well-defined Lyapunov function for system(1),For some positive number $\alpha$ ,$V(x) \le 1$ ,if

$$\frac{dV(x)}{dx} + \alpha V(x) - \frac{\alpha}{\omega_m} \omega^T(t)\omega(t) \le 0$$

## III. Main Results

Our objection is to get an non-ellipsoid set as small as possible to bonud the renchable set defined in (3) Based on the system with distributed delay and disturbances.

**Theorem 1**. Consider the time-delay system(1) with distributed delay and disturbances based on (2),if there exist real symmetric matrices $S, Q, R, E > 0$,and $T_1, T_2, \cdots, T_6, T_7, \beta_{jk} > 0$,for all $j, k = \{1, 2, \cdots, M\}$,and a scalar $\alpha > 0$ satisfying the following matrix inequalities:

$$\Omega_i = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & A_i^T T_4 & A_i^T T_5 & \Omega_{16} & \Omega_{17} \\ * & \Omega_{22} & \Omega_{23} & -T_4 & -T_5 & \Omega_{26} & \Omega_{27} \\ * & * & \Omega_{33} & \Omega_{34} & \Omega_{35} & \Omega_{36} & \Omega_{37} \\ * & * & * & \Omega_{44} & \Omega_{45} & T_4^T C_i & T_4^T D_i \\ * & * & * & * & \Omega_{55} & T_5^T C_i & T_5^T D_i \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} \\ * & * & * & * & * & * & \Omega_{77} \end{bmatrix} < 0 \tag{5}$$

$$\begin{bmatrix} F_1 & H_1 \\ * & Q_1 \end{bmatrix} \ge 0 \tag{6}$$

$$\begin{aligned} \Omega_{11} &= S + Q + R + \alpha P_j + d_2 F_1 - \frac{e^{-\alpha d_2}}{d_2} Q_1 + T_1^T A_i \\ &\quad + A_i^T T_1 + \Sigma_{k=1}^{M} \beta_{jk} (P_j - P_k) \\ \Omega_{12} &= P_j + d_2 H_1 - T_1^T + A_i^T T_2 \\ \Omega_{13} &= \frac{e^{-\alpha d_2}}{d_2} Q_1 + T_1^T B_i + A_i^T T_3 \\ \Omega_{16} &= -\frac{e^{-\alpha d_2}}{d_2} H_1 + T_1^T C_i + A_i^T T_6 \\ \Omega_{17} &= T_1^T D_i + A_i^T T_7 \\ \Omega_{22} &= (h_2 - h_1)^2 E + d_2 Q_1 - T_2^T - T_2 \\ \Omega_{23} &= T_2^T B_i - T_3 \\ \Omega_{26} &= T_2^T C_i - T_6 \\ \Omega_{27} &= T_2^T D_i - T_7 \\ \Omega_{33} &= -\frac{e^{-\alpha d_2}}{d_2} Q_1 - (1-h) e^{-\alpha h_2} S + e^{-\alpha h_2} (-2E + G \\ &\quad + G^T) + T_3^T B_i + B_i^T T_3 \\ \Omega_{34} &= e^{-\alpha h_2} (E^T - G) + B_i^T T_4 \\ \Omega_{35} &= e^{-\alpha h_2} (E^T + G^T) + B_i^T T_5 \\ \Omega_{36} &= \frac{e^{-\alpha d_2}}{d_2} H_1 + T_3^T C_i + B_i^T T_6 \\ \Omega_{37} &= T_3^T D_i - B_i^T T_7 \\ \Omega_{44} &= -e^{-\alpha h_1} Q + e^{-\alpha h_2} E \\ \Omega_{45} &= e^{-\alpha h_2} G^T \\ \Omega_{55} &= -e^{-\alpha h_2} R - e^{-\alpha h_2} E \\ \Omega_{66} &= -\frac{e^{-\alpha d_2}}{d_2} F_1 + T_6^T C_i \\ \Omega_{67} &= T_6^T D_i + C_i^T T_7 \\ \Omega_{77} &= T_7^T D_i - \frac{\alpha}{\omega_m} I \end{aligned}$$

The reachable set of system(1) is bounded by the intersection of ellipsoids $\varepsilon(P_j,1)$ .

**Proof** : Denoting $T=[T_1,T_2,T_3,T_4,T_5,T_6,T_7]$ .

$$\xi^T(t)=[x^T(t),\dot{x}^T(t),x^T(t-h(t)),x^T(t-h_1(t)),x^T(t-h_2(t)),\int_{t-d(t)}^{t}x^T(s)ds,\omega^T(t)]$$

So we can select the maximal Lyapunov-Krasovskii functional candidate as follows:

$$\begin{aligned}
V(x)&=V_{1,max}(x)+V_2(x)+V_3(x)+V_4(x)\\
V_{1,j}(x)&=x^T(t)P_jx(t),V_{1,max}(x)=\max x^T(t)P_jx(t)\\
V_2(x)&=\int_{t-h(t)}^{t}e^{\alpha(s-t)}x^T(s)Sx(s)ds+\int_{t-h_1)}^{t}e^{\alpha(s-t)}\\
&\quad x^T(s)Qx(s)ds+\int_{t-h_2}^{t}e^{\alpha(s-t)}x^T(s)Rx(s)ds\\
V_3(x)&=(h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}e^{\alpha(s-t)}\dot{x}^T(s)E\dot{x}(s)ds\\
V_4(x)&=\int_{-d_t}^{0}\int_{t+\theta}^{t}e^{\alpha(s-t)}\begin{bmatrix}x(s)\\\dot{x}(s)\end{bmatrix}^T\begin{bmatrix}P_1&H_1*&Q_1\end{bmatrix}\\
&\quad\times\begin{bmatrix}x(s)\\\dot{x}(s)\end{bmatrix}dsd\theta
\end{aligned}$$

To imply the process of proof,we can denote the set :

$$M_{max}(x):=\{j\in\{1,2,\cdots,M\}:V_{1,j}(x)=V_{1,max}(x)\}.$$
$$\gamma_j=\{x\in R^n:V_j(x)\geq V_k(x),\forall k\neq j\}$$

So we can get the inequality $V_{1,j}(x)<V_{1,max}(x)$, if$j\notin M_{max}(x)$ as well as$M_{max}(x)\in[1,m]$ for some integer $m\leq M$ .where $m$ is the number of ellipsoids $\varepsilon(P_j,1)'s$ intersected at $x$ .So we can get that:

$$V_{1,j}(x)=V_{1,max}(x),forj\leq m$$
$$V_{1,j}(x)<V_{1,max}(x),forj>m$$

So we get that :

$x^T(P_j-P_k)x\geq 0$ , $\forall j\in\{1,2,3,\cdots,m\}$ ,$k\in[1,M]$ . (7)

Because $x$ is not differentiable everywhere ,we should consider two conditions for $x$ .

(1) If $x$ is differentiable

$$x\in\gamma_j\backslash\bigcup_{k\neq j}\gamma_k$$
$$V_{1,max}=\max\{x^TP_jx\}$$
$$V_{1,max}=\{2x^T(t)P_j\dot{x}(t)\}$$

(2) If $x$ is not differentiable

$$x\in\bigcap_{j=i_1}^{i_r}\gamma_j\backslash\bigcup_{k\neq i_1,i_2,\cdots,i_k}\gamma_k,r\geq 2$$
$$\bigtriangledown_{\dot{x}}V_{1,max}(x)\leq\max_{\xi\in\partial V_{1,max}(x)}\{\xi^T\dot{x}\}$$
$$=\max_{\xi\in Co\{2P_jx,j=1,\cdots,m)}\{\xi^T\dot{x}\}$$
$$\sum_{i=1}^{r}\alpha_r=1,\alpha_r>0,P_{il}>0,l=(1,2,3,\cdots,r)$$

$$\begin{aligned}
\dot{V}_2(x)&=x^T(t)Sx(t)-(1-\dot{h}(t))e^{-\alpha h(t)}x^T(t-h(t))S\\
&\quad\times x(t-h(t))+x^T(t)Qx(t)-e^{-\alpha h_1}x^T(t-h_1)Q\\
&\quad\times x(t-h_1)+x^T(t)Rx(t)-e^{-\alpha h_2}x^T(t-h_2)Rx(t-h_2)\\
&\quad-\alpha V_2\\
&\leq x^T(t)(S+Q+R)x(t)-(1-h)e^{-\alpha h_2}x^T(t-h(t))\\
&\quad\times Sx(t-h(t))-e^{-\alpha h_1}x^T(t-h_1)Qx(t-h_1)\\
&\quad-x^T(t)Rx(t)-e^{-\alpha h_2}x^T(t-h_2)Rx(t-h_2)-\alpha V_2\\
\dot{V}_3(x)&=(h_2-h_1)\int_{-h_2}^{-h_1}\dot{x}^T(t)E\dot{x}(t)d\theta-(h_2-h_1)\\
&\quad\times\int_{t-h_2}^{t-h_1}e^{\alpha(s-t)}\dot{x}^T(t)E\dot{x}(t)ds-\alpha V_3\\
&\leq(h_2-h_1)^2\dot{x}^T(t)E\dot{x}(t)-e^{-\alpha h_2}(h_2-h_1)\\
&\quad\times(\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)E\dot{x}(s)ds+\int_{t-h(t)}^{t-h_1}\dot{x}^T(s)E\dot{x}(s)ds)\\
&\quad-\alpha V_3\\
&\leq(h_2-h_1)^2\dot{x}^T(t)E\dot{x}(t)-e^{-\alpha h_2}(\frac{h_2-h_1}{h_2-h(t)}\\
&\quad\times\int_{t-h_2}^{t-h(t)}\dot{x}^T(s)dsE\int_{t-h_2}^{t-h(t)}\dot{x}(s)ds+\frac{h_2-h_1}{h_t-h(1)}\\
&\quad\times\int_{t-h_t}^{t-h(1)}\dot{x}^T(s)dsE\int_{t-h_t}^{t-h(1)}\dot{x}(s)ds)-\alpha V_3\\
&\leq(h_2-h_1)^2\dot{x}^T(t)E\dot{x}(t)-e^{-\alpha h_2}\\
&\quad\times\begin{bmatrix}x(t-h(t))-x(t-h_2)\\x(t-h_1)-x(t-h(t)\end{bmatrix}^T\begin{bmatrix}E&G*&E\end{bmatrix}\\
&\quad\times\begin{bmatrix}x(t-h(t))-x(t-h_2)\\x(t-h_1)-x(t-h(t)\end{bmatrix}-\alpha V_3\\
&\leq(h_2-h_1)^2\dot{x}^T(t)E\dot{x}(t)+e^{-\alpha h_2}\\
&\quad\times\begin{bmatrix}x(t-h_1)\\x(t-h_2)\\x(t-h(t))\end{bmatrix}^T\begin{bmatrix}-E&G^T&E-G^T*&-E&E+G*&*&-2E+G+G^T\end{bmatrix}\\
&\quad\times\begin{bmatrix}x(t-h_1)\\x(t-h_2)\\x(t-h(t))\end{bmatrix}-\alpha V_3\\
\dot{V}_4(x)&=d(t)\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}^T\begin{bmatrix}P_1&H_1*&Q_1\end{bmatrix}\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}\\
&\quad-\int_{t-d(t)}^{t}e^{\alpha(s-t)}\begin{bmatrix}x(s)\\\dot{x}(s)\end{bmatrix}^T\begin{bmatrix}P_1&H_1*&Q_1\end{bmatrix}\\
&\quad\times\begin{bmatrix}x(s)\\\dot{x}(s)\end{bmatrix}ds-\alpha V_4\\
&\leq d_2\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}^T\begin{bmatrix}P_1&H_1*&Q_1\end{bmatrix}\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}-\frac{e^{-\alpha d(t)}}{d(t)}\\
&\quad\times\int_{t-d(t)}^{t}\begin{bmatrix}x(s)\\\dot{x}(s)\end{bmatrix}^Tds\begin{bmatrix}P_1&H_1*&Q_1\end{bmatrix}\\
&\quad\times\int_{t-d(t)}^{t}\begin{bmatrix}x(s)\\\dot{x}(s)\end{bmatrix}ds-\alpha V_4\\
&\leq d_2\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}^T\begin{bmatrix}P_1&H_1*&Q_1\end{bmatrix}\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}-\frac{e^{-\alpha d_2}}{d_2}
\end{aligned}$$

$$\times \begin{bmatrix} \int_{t-d(t)}^{t} x(s)ds \\ x(t)-x(t-h(t)) \end{bmatrix}^T \begin{bmatrix} P_1 & H_1 \\ * & Q_1 \end{bmatrix}$$
$$\times \begin{bmatrix} \int_{t-d(t)}^{t} x(s)ds \\ x(t)-x(t-h(t)) \end{bmatrix} - \alpha V_4$$

we can get the following equation holds by the system(1):

$$2\xi^T(t)T^T[-\dot{x}(t)+Ax(t)+Bx(t-h(t))+C\int_{t-d(t)}^{t} x(s)ds + D\omega(t)] = 0$$
$$2(x^T(t)T_1^T+\dot{x}^T(t)T_2^T+x^T(t-h(t))T_3^T+x^T(t-h_1)T_4^T + x^T(t-h_2)T_5^T+\int_{t-d(t)}^{t} x^T(s)dsT_6^T+\omega(t)^T(t)T_7^T)$$
$$\times[-\dot{x}(t)+Ax(t)+Bx(t-h(t))+C\int_{t-d(t)}^{t} x(s)ds + D\omega(t)] = 0$$

(1)If $x$ is differentiable

$$x \in \gamma_j \backslash \bigcup_{k\neq j} \gamma_k$$

We can get following inequality by using the convex property of the polytope and S procedure .

$$\Pi \triangleq \frac{dV(x)}{dx}+\alpha V(x)-\frac{\alpha}{\omega_m}\omega^T(t)\omega(t)$$
$$\leq \xi^T(t)\Omega_1\xi(t)=\xi^T(t)\Sigma_{i=1}^{N}\theta_i\Omega_{i1}\xi(t)-\Sigma_{i=1}^{M}\beta_{jk}x^T(t)$$
$$(P_j-P_k)x(t)\leq 0$$

Where

$$\Omega_1=\begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \tilde{\Omega}_{13} & A^TT_4 & A^TT_5 & \tilde{\Omega}_{16} & \tilde{\Omega}_{17} \\ * & \Omega_{22} & \tilde{\Omega}_{23} & -T_4 & -T_5 & \tilde{\Omega}_{26} & \tilde{\Omega}_{27} \\ * & * & \tilde{\Omega}_{33} & \tilde{\Omega}_{34} & \tilde{\Omega}_{35} & \tilde{\Omega}_{36} & \tilde{\Omega}_{37} \\ * & * & * & \Omega_{44} & \Omega_{45} & T_4^TC & T_4^TD \\ * & * & * & * & \Omega_{55} & T_5^TC & T_5^TD \\ * & * & * & * & * & \tilde{\Omega}_{66} & \tilde{\Omega}_{67} \\ * & * & * & * & * & * & \tilde{\Omega}_{77} \end{bmatrix} < 0$$

$$\tilde{\Omega}_{11}=S+Q+R+\alpha P_j+d_2F_1-\frac{e^{-\alpha d_2}}{d_2}Q_1+T_1^TA + A^TT_1+\Sigma_{k=1}^{M}\beta_{jk}(P_j-P_k)$$
$$\tilde{\Omega}_{12}=P_j+d_2H_1-T_1^T+A^TT_2$$
$$\tilde{\Omega}_{13}=\frac{e^{-\alpha d_2}}{d_2}Q_1+T_1^TB+A^TT_3$$
$$\tilde{\Omega}_{16}=-\frac{e^{-\alpha d_2}}{d_2}H_1+T_1^TC+A^TT_6$$
$$\tilde{\Omega}_{17}=T_1^TD+A^TT_7$$
$$\tilde{\Omega}_{23}=T_2^TB-T_3$$
$$\tilde{\Omega}_{26}=T_2^TC-T_6$$
$$\tilde{\Omega}_{27}=T_2^TD-T_7$$
$$\tilde{\Omega}_{33}=-\frac{e^{-\alpha d_2}}{d_2}Q_1-(1-h)e^{-\alpha h_2}S+e^{-\alpha h_2}(-2E+G + G^T)+T_3^TB+B^TT_3$$
$$\tilde{\Omega}_{34}=e^{-\alpha h_2}(E^T-G)+B^TT_4$$
$$\tilde{\Omega}_{35}=e^{-\alpha h_2}(E^T+G^T)+B^TT_5$$
$$\tilde{\Omega}_{36}=\frac{e^{-\alpha d_2}}{d_2}H_1+T_3^TC+B^TT_6$$
$$\tilde{\Omega}_{37}=T_3^TD-B^TT_7$$
$$\tilde{\Omega}_{66}=-\frac{e^{-\alpha d_2}}{d_2}F_1+T_6^TC$$
$$\tilde{\Omega}_{67}=T_6^TD+C^TT_7$$
$$\tilde{\Omega}_{77}=T_7^TD-\frac{\alpha}{\omega_m}I$$

We can get the results:

$$\Pi-\Sigma_{i=1}^{M}\beta_{jk}x^T(t)(P_j-P_k)x(t)\leq 0$$
$$\Rightarrow \Pi\leq\Sigma_{i=1}^{M}\beta_{jk}x^T(t)(P_j-P_k)x(t)$$

This implies that $V_{max}\leq V(x)\leq 1$ by Lemma 3.
(2)If $x$ is not differentiable,

$$x\in\bigcap_{j=i_1}^{i_r}\gamma_j\backslash\bigcup_{k\neq i_1,i_2,\cdots,i_k}\gamma_k, r\geq 2$$

We can get following inequality by u sing the definition of the subdifferential and S procedure .

$$\Pi\triangleq\frac{dV(x)}{dx}+\alpha V(x)-\frac{\alpha}{\omega_m}\omega^T(t)\omega(t)$$
$$\leq 2x^TP_j\dot{x}+\alpha x^T(t)P_jx(t)+\dot{V}_2(t)+\dot{V}_3(t)+\dot{V}_4(t)$$
$$-\frac{\alpha}{\omega_m}\omega^T(t)\omega(t)$$
$$\leq\Sigma_{i=1}^{M}\beta_{jk}x^T(t)(P_j-P_k)x(t)\leq 0$$

Therefore , we can get $V_{max}\leq V(x)\leq 1$ by Lemma 3.This completes the proof of Theorem 1.

**Remark 1.**In this paper,we consided the reachable set bounding estimation for delay systems with disturbances.However, we consided just the systems before: $\dot{x}(t)=Ax(t)+Bx(t-h(t))+D\omega(t)$or the systems : $\dot{x}(t)=Ax(t)+C\int_{t-d(t)}^{t}x(s)ds+D\omega(t)$.What is more, $-(h_2-h_1)\int_{t-h_2}^{t-h_1}e^{\alpha(s-t)}\dot{x}^T(s)E\dot{x}(s)ds$ is enlarged by $-(h_2-h_1)\int_{t-h_2}^{t-h(t)}e^{\alpha(s-t)}\dot{x}^T(s)E\dot{x}(s)ds$,where an other term $-(h_2-h_1)\int_{t-h(t)}^{t-h_1}e^{\alpha(s-t)}\dot{x}^T(s)E\dot{x}(s)ds$ is ignored[16]-[18]. The useful term is explored in this paper. So we can reduce the conservativeness for our results.

**Remark 2.**in [3],we can pay attaition that the Lemma 1 of[3],Zuo ignored the term $-\frac{d(t)}{h-d(t)}(x^T(t-d(t))-x^T(t-h))R(x(t-d(t))-x(t-h))$ and the term $-\frac{h-d(t)}{h}(x^T(t)-x^T(t-d(t)))R(x^T(t)-x^T(t-d(t)))$ .In this paper,we concided the two terms above and can get

TABLE I
TABLE 1:WITH $h_2$=0.7 AND $\omega_m$=1.

| $d$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.9 |
|---|---|---|---|---|---|
| [4] | 1.89 | 2.00 | 2.19 | 2.60 | 3.51 |
| M=2 | 0.2301 | 0.2579 | 0.2998 | 0.3712 | 0.7319 |
| M=3 | 0.2279 | 0.2547 | 0.2911 | 0.3603 | 0.6217 |

TABLE II
TABLE 1:WITH $h_2$=0.7 AND $\omega_m$=1.

| Methods | [14] | ours |
|---|---|---|
| $d(t)$ is differentiable | 1.095 | 0.3763 |
| $d(t)$ is non-differentiable | 1.095 | 0.8725 |

the matrix $\begin{bmatrix} E & G \\ * & E \end{bmatrix}$ better than the matrix $\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$ . by using inequality . So it can be theoretically proven that it has less conservativeness and can be used more further.

**Remark 3.**we select free-weighting matrices $T = [T_1, T_2, T_3, T_4, T_5, T_6, T_7]$and can decoup between the system matrices and the Lyapnov matrices by using (11)

We can make a treating which is simple optimization problem formulation to look for the accurate bound for the reachable set of system(1).maximize $\sigma$ subjects to $\sigma I \leq P_j$ . It can be described as the following optimization problem:

Min :

$$\tilde{\sigma}(\tilde{\sigma} = \frac{1}{\sigma})$$

$$s.t. \begin{cases} (1) \begin{bmatrix} \tilde{\sigma} I & I \\ I & P_j \end{bmatrix} \geq 0 \\ (2) Inequality(4) holds. \end{cases} \quad (7)$$

## IV. NUMERICAL EXAMPLES

In this section, we provide the simulation of examples to illustrate the effectiveness of our method.We can select following paprameters:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix}, D_1 = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} -1 & 0 \\ -1 & -0.9 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -1.1 \end{bmatrix}, D_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1.1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -1 & -1 \\ 0 & 0.9 \end{bmatrix}, C_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

The resulting $\bar{\delta}'s$are listed in Table I for different values of $d$,by solving the optimization problem(7).Form the Table II ,we can get a tighter bounds than the approach derived in [4].

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