

# $(R, S)$ -Modules and $(1, k)$ -Jointly Prime $(R, S)$ -Submodules

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*Abstract*—We introduced the notions of  $(1, k)$ -prime ideal and  $(1, k)$ -jointly prime  $(R, S)$ -submodule as a generalization of prime ideal and jointly prime  $(R, S)$ -submodule, respectively. We provide a relationship between  $(1, k)$ -prime ideal and  $(1, k)$ -jointly prime  $(R, S)$ -submodule. Characterizations of  $(1, k)$ -jointly prime  $(R, S)$ -submodules are also given.

*Keywords*— $(R, S)$ -module,  $(1, k)$ -prime ideal,  $(1, k)$ -jointly prime  $(R, S)$ -submodule.

## I. INTRODUCTION

THROUGHOUT this paper, let  $R$  and  $S$  be rings and  $M$  an abelian group.

*Definition 1.1:* [1] Let  $R$  and  $S$  be rings and  $M$  an abelian group under addition. We say that  $M$  is an  $(R, S)$ -**module** if there is a function  $\cdot : R \times M \times S \rightarrow M$  satisfying the following properties: for all  $r, r_1, r_2 \in R$ ,  $m, n \in M$  and  $s, s_1, s_2 \in S$ ,

- (i)  $r \cdot (m + n) \cdot s = r \cdot m \cdot s + r \cdot n \cdot s$
- (ii)  $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$
- (iii)  $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$
- (iv)  $r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2)$ .

We usually abbreviate  $r \cdot m \cdot s$  by  $rms$ . We may also say that  $M$  is an  $(R, S)$ -module under  $+$  and  $\cdot$ .

An  $(R, S)$ -**submodule** of an  $(R, S)$ -module  $M$  is a subgroup  $N$  of  $M$  such that  $rns \in N$  for all  $r \in R$ ,  $n \in N$  and  $s \in S$ .

*Definition 1.2:* [1] Let  $M$  be an  $(R, S)$ -module. A proper  $(R, S)$ -submodule  $P$  of  $M$  is called **jointly prime** if for each left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$INJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } N \subseteq P.$$

The structure of an  $(R, S)$ -module was created as a generalization of a module structure. The basic results of an  $(R, S)$ -module structure have been given by [1] and [2]. Almost all of those results was studied analogous to a module structure such as the primalities of  $(R, S)$ -submodules of  $(R, S)$ -modules and left multiplication  $(R, S)$ -modules; see [1] and [2].

In this paper, we introduce the notions of  $(1, 2)$ -prime ideal,  $(1, k)$ -prime ideal,  $(1, 2)$ -jointly prime  $(R, S)$ -submodule and  $(1, k)$ -jointly prime  $(R, S)$ -submodule and obtain equivalent conditions for an  $(R, S)$ -submodule to be  $(1, k)$ -jointly prime  $(R, S)$ -submodule.

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## II. $(1, 2)$ -JOINTLY PRIME $(R, S)$ -SUBMODULES

In this research, we modify the structure of a jointly prime  $(R, S)$ -submodules for more general. Now, we start this section by giving the definition of  $(1, 2)$ -jointly prime  $(R, S)$ -submodules.

*Definition 2.1:* A proper  $(R, S)$ -submodule  $P$  of  $M$  is called  $(1, 2)$ -**jointly prime** if for each left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$INJ^2 \subseteq P \text{ implies } IMJ^2 \subseteq P \text{ or } N \subseteq P.$$

By the dual of  $(1, 2)$ -jointly prime, we define  $(2, 1)$ -jointly prime as follow.

A proper  $(R, S)$ -submodule  $P$  of  $M$  is called  $(2, 1)$ -**jointly prime** if for each left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$I^2NJ \subseteq P \text{ implies } I^2MJ \subseteq P \text{ or } N \subseteq P.$$

It is clear that a jointly prime  $(R, S)$ -submodule is  $(1, 2)$ -jointly prime and  $(2, 1)$ -jointly prime. Next, we give a characterization of  $(1, 2)$ -jointly prime and  $(2, 1)$ -jointly prime  $(r\mathbb{Z}, s\mathbb{Z})$ -submodule of  $\mathbb{Z}$  where  $r, s \in \mathbb{Z}^+$ .

*Proposition 2.2:* Let  $r, s \in \mathbb{Z}^+$  and  $p \in \mathbb{Z}_0^+ \setminus \{1\}$ . Then

- (i)  $p\mathbb{Z}$  is an  $(1, 2)$ -jointly prime  $(r\mathbb{Z}, s\mathbb{Z})$ -module of  $\mathbb{Z}$  if and only if  $p = 0$ ,  $p$  is a prime integer or  $p | rs^2$ .
- (ii)  $p\mathbb{Z}$  is a  $(2, 1)$ -jointly prime  $(r\mathbb{Z}, s\mathbb{Z})$ -module of  $\mathbb{Z}$  if and only if  $p = 0$ ,  $p$  is a prime integer or  $p | r^2s$ .

*Proof:* (i)  $(\Rightarrow)$  Assume that  $p\mathbb{Z}$  is a  $(1, 2)$ -jointly prime  $(r\mathbb{Z}, s\mathbb{Z})$ -module of  $\mathbb{Z}$ . Suppose that  $p \neq 0$  and  $p$  is not a prime integer. Then  $p = mn$  for some integer  $m, n > 1$ . It implies that  $(rm\mathbb{Z})(n\mathbb{Z})(s^2\mathbb{Z}) = (rmns^2)\mathbb{Z} \subseteq p\mathbb{Z}$ . Since  $p\mathbb{Z}$  is  $(1, 2)$ -jointly prime and  $p \nmid n$ ,  $(rm\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$ . Note that  $(r\mathbb{Z})(m\mathbb{Z})(s^2\mathbb{Z}) = (rm\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$ . Since  $p\mathbb{Z}$  is  $(1, 2)$ -jointly prime and  $p \nmid m$ ,  $(r\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$ . Hence  $p | rs^2$ .

$(\Leftarrow)$  If  $p = 0$  or  $p$  is a prime integer or  $p | rs^2$ , then it is clear that  $p\mathbb{Z}$  is  $(1, 2)$ -jointly prime. ■

Now, we already have an example of  $(1, 2)$ -jointly prime but is not jointly prime.

*Example 2.3:* It is clear that  $\mathbb{Z}$  is a  $(2\mathbb{Z}, 3\mathbb{Z})$ -module. Then  $9\mathbb{Z}$  is a  $(1, 2)$ -jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$  but  $9\mathbb{Z}$  is not a jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$ .

The following is an example showing that  $(1, 2)$ -jointly prime and  $(2, 1)$ -jointly prime are exactly different.

*Example 2.4:* Recall that  $\mathbb{Z}$  is a  $(2\mathbb{Z}, 3\mathbb{Z})$ -module. Then  $4\mathbb{Z}$  is a  $(2, 1)$ -jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$  but  $4\mathbb{Z}$  is not a  $(1, 2)$ -jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$  and  $9\mathbb{Z}$

is a (1,2)-jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$  but  $9\mathbb{Z}$  is not a (2,1)-jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$ .

Moreover,  $p\mathbb{Z}$  is both a (1,2)-jointly prime and (2,1)-jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$  if and only if  $p\mathbb{Z}$  is a jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$ .

Note that (1,2)-jointly prime and (2,1)-jointly prime may be different even if ring  $R$  and  $S$  are commutative.

We have a question from Example 2.4 that for general, if  $P$  is (1,2)-jointly prime and (2,1)-jointly prime, then can  $P$  be a jointly prime  $(R, S)$ -submodule? The following is an answers.

*Example 2.5:* It easy to see that  $\mathbb{Z}$  is a  $(2\mathbb{Z}, 4\mathbb{Z})$ -module. Then  $16\mathbb{Z}$  is both a (1,2)-jointly prime and (2,1)-jointly prime  $(2\mathbb{Z}, 4\mathbb{Z})$ -submodule of  $\mathbb{Z}$  but  $16\mathbb{Z}$  is not a jointly prime  $(2\mathbb{Z}, 4\mathbb{Z})$ -submodule of  $\mathbb{Z}$ .

*Example 2.6:* Let  $\mathbb{Z}$  be a ring of integer and let

$$R = \left\{ \left[ \begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right] \mid x, y \in \mathbb{Z} \right\},$$

$$S = \left\{ \left[ \begin{array}{ccc} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{array} \right] \mid x, y \in \mathbb{Z} \right\} \text{ and}$$

$M$  is the set of all  $3 \times 3$  matrices on integer. Then  $M$  is an  $(R, S)$ -module. Since  $R^2 = 0 = S^2$ , all proper  $(R, S)$ -submodules of  $M$  are both (1,2)-jointly prime and (2,1)-jointly prime  $(R, S)$ -submodule of  $M$ . However,  $0$  is not a jointly prime  $(R, S)$ -submodule of  $M$ .

For each  $(R, S)$ -submodule  $P$  of  $M$  and  $k \in \mathbb{Z}^+$ , let

$$(P : M)_{R, S^k} = \{r \in R \mid rMS^k \subseteq P\}.$$

*Proposition 2.7:* Let  $P$  be an  $(R, S)$ -submodule of an  $(R, S)$ -module  $M$  and  $k \in \mathbb{Z}^+$ . The followings hold.

- (i)  $(P : M)_{R, S^k}$  is a subgroup of  $R$  under addition.
- (ii)  $(P : M)_{R, S^k} \subseteq (P : M)_{R, S^{k+1}}$ .
- (iii) If  $S^2 = S$ , then  $(P : M)_{R, S^k}$  is an ideal of  $R$ .

*Proof:* The proof is straightforward. ■

Next, we introduced a particular nonempty subset of  $R$  which play a role in this research.

Let  $R$  be a ring and  $T$  a proper ideal of  $R$ . Then  $T$  is said to be a (1,2)-**prime ideal** of  $R$  if for each ideal  $A$  and  $B$  of  $R$ , if  $AB^2 \subseteq T$ , then  $A \subseteq T$  or  $B^2 \subseteq T$ . A prime ideal of  $R$  is a (1,2)-prime ideal of  $R$  but the converse is not true. We show by observing the following example.

*Example 2.8:* Let  $p$  be an integer. If  $p = 0$  or  $p$  is a prime integer or  $p = q^2$  where  $q$  is a prime integer, then  $p\mathbb{Z}$  is a (1,2)-prime ideal of  $\mathbb{Z}$ .

It clear that  $4\mathbb{Z}$  is a (1,2)-prime ideal of  $\mathbb{Z}$  but is not a prime ideal of  $\mathbb{Z}$ .

*Proposition 2.9:* Let  $P$  be an  $(R, S)$ -submodule of an  $(R, S)$ -module  $M$  such that  $(P : M)_{R, S^2}$  is a proper ideal of  $R$ . If  $P$  is a (1,2)-jointly prime  $(R, S)$ -submodule of  $M$ , then  $(P : M)_{R, S^2}$  is a (1,2)-prime ideal of  $R$ .

*Proof:* Assume that  $P$  is a (1,2)-jointly prime  $(R, S)$ -submodule of  $M$ . Let  $A$  and  $B$  be ideals of  $R$  such that  $AB^2 \subseteq (P : M)_{R, S^2}$ . Hence  $(AB^2)MS^2S^2 \subseteq (AB^2)MS^2 \subseteq P$ . Thus  $A(B^2MS^2)S^2 \subseteq P$ . Since  $P$  is (1,2)-jointly prime,  $AMS^2 \subseteq P$  or  $B^2MS^2 \subseteq P$ . Therefore  $A \subseteq (P : M)_{R, S^2}$

or  $B^2 \subseteq (P : M)_{R, S^2}$ . This means that  $(P : M)_{R, S^2}$  is a (1,2)-prime ideal of  $R$ . ■

The converse of Proposition 2.9 is invalid. For example,  $6\mathbb{Z}$  is a  $(\mathbb{Z}, 2\mathbb{Z})$ -submodule of  $\mathbb{Z}$ . We see that  $(6\mathbb{Z} : \mathbb{Z})_{\mathbb{Z}, (2\mathbb{Z})^2} = 3\mathbb{Z}$  is a prime ideal of  $\mathbb{Z}$ , of course,  $3\mathbb{Z}$  is a (1,2)-prime ideal of  $\mathbb{Z}$  but  $6\mathbb{Z}$  is not a (1,2)-jointly prime  $(\mathbb{Z}, 2\mathbb{Z})$ -submodule of  $\mathbb{Z}$ .

### III. (1, k)-JOINTLY PRIME (R, S)-SUBMODULES

In this section, we extend the notion of (1,2)-jointly prime to (1,k)-jointly prime where  $k \in \mathbb{Z}^+$ . Similarly, we also extend the notion of (2,1)-jointly prime to (k,1)-jointly prime where  $k \in \mathbb{Z}^+$ .

*Definition 3.1:* Let  $k \in \mathbb{Z}^+$  and  $M$  be an  $(R, S)$ -module. A proper  $(R, S)$ -submodule  $P$  of  $M$  is called (1,k)-**jointly prime** if for each left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

Dually, a proper  $(R, S)$ -submodule  $P$  of  $M$  is called (k,1)-**jointly prime** if for each left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$I^kNJ \subseteq P \text{ implies } I^kMJ \subseteq P \text{ or } N \subseteq P.$$

Note here that jointly prime and (1,1)-jointly prime are identical.

*Proposition 3.2:* Let  $r, s, k \in \mathbb{Z}^+$  and  $p \in \mathbb{Z}_0^+ \setminus \{1\}$ . Then

- (i)  $p\mathbb{Z}$  is a (1,k)-jointly prime  $(r\mathbb{Z}, s\mathbb{Z})$ -module of  $\mathbb{Z}$  if and only if  $p = 0$ ,  $p$  is a prime integer or  $p \mid rs^k$ .
- (ii)  $p\mathbb{Z}$  is a (k,1)-jointly prime  $(r\mathbb{Z}, s\mathbb{Z})$ -module of  $\mathbb{Z}$  if and only if  $p = 0$ ,  $p$  is a prime integer or  $p \mid r^ks$ .

*Proof:* ( $\Rightarrow$ ) Assume that  $p\mathbb{Z}$  is a (1,k)-jointly prime  $(r\mathbb{Z}, s\mathbb{Z})$ -module of  $\mathbb{Z}$ . Suppose that  $p \neq 0$  and  $p$  is not a prime integer. Then  $p = mn$  for some integer  $m, n > 1$ . It implies that  $(rm\mathbb{Z})(n\mathbb{Z})(s^k\mathbb{Z}) = (rmns^k)\mathbb{Z} \subseteq p\mathbb{Z}$ . Since  $p\mathbb{Z}$  is (1,k)-jointly prime and  $p \nmid n$ ,  $(rm\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z}) \subseteq p\mathbb{Z}$ . Note that  $(r\mathbb{Z})(m\mathbb{Z})(s^k\mathbb{Z}) = (rm\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z}) \subseteq p\mathbb{Z}$ . Since  $p\mathbb{Z}$  is (1,k)-jointly prime and  $p \nmid m$ ,  $(r\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z}) \subseteq p\mathbb{Z}$ . Hence  $p \mid rs^k$ .

( $\Leftarrow$ ) If  $p = 0$  or  $p$  is a prime integer or  $p \mid rs^k$ , then it is clear that  $p\mathbb{Z}$  is (1,k)-jointly prime. ■

*Proposition 3.3:* Let  $k \in \mathbb{Z}^+$  and  $M$  be an  $(R, S)$ -module. Then

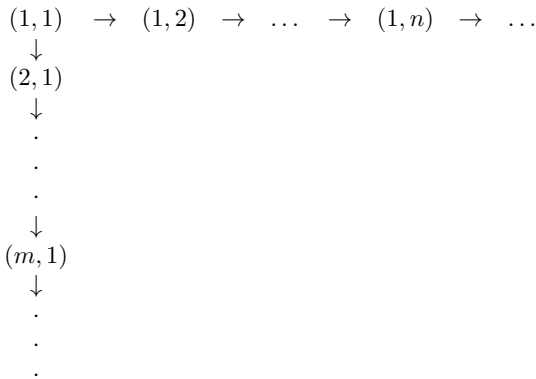
- (i) If  $P$  is (1,k)-jointly prime, then  $P$  is (1,k+1)-jointly prime.
- (ii) If  $P$  is (k,1)-jointly prime, then  $P$  is (k+1,1)-jointly prime.

*Proof:* Assume that  $P$  is a (1,k)-jointly prime  $(R, S)$ -submodule of  $M$ . Let  $I$  be a left ideal of  $R$ ,  $N$  an  $(R, S)$ -submodule of  $M$  and  $J$  be a right ideal of  $S$  such that  $INJ^{k+1} \subseteq P$ . Note that  $I(INJ)J^k \subseteq I^2NJ^{k+1} \subseteq INJ^{k+1} \subseteq P$ . Since  $P$  is (1,k)-jointly prime,  $IMJ^k \subseteq P$  or  $INJ \subseteq P$ . Note that  $J^{k+1} \subseteq J^k \subseteq J$ . If  $IMJ^k \subseteq P$ , then  $IMJ^{k+1} \subseteq P$ . Assume that  $INJ \subseteq P$ . Then  $INJ^k \subseteq P$ . Since  $P$  is (1,k)-jointly prime,  $IMJ^k \subseteq P$  or  $N \subseteq P$ . ■

The following example shows that the converse of Proposition 3.3 is false in general.

*Example 3.4:* Recall that  $\mathbb{Z}$  is a  $(2\mathbb{Z}, 3\mathbb{Z})$ -module. Then  $27\mathbb{Z}$  and  $54\mathbb{Z}$  are  $(1, 3)$ -jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$  but  $27\mathbb{Z}$  and  $54\mathbb{Z}$  are not  $(1, 2)$ -jointly prime  $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of  $\mathbb{Z}$ .

We obtain the following diagram from Proposition 3.3. Note that the order pair  $(m, n)$  means  $P$  is a  $(m, n)$ -jointly prime  $(R, S)$ -submodule of  $M$ .



In this point, we present a generalization of a  $(1, 2)$ -prime ideal of  $R$  which is called  $(1, k)$ -prime ideal of  $R$  where  $k \in \mathbb{Z}^+$ . Let  $R$  be a ring and  $T$  a proper ideal of  $R$  and  $k \in \mathbb{Z}^+$ . Then  $T$  is said to be a  $(1, k)$ -**prime ideal** of  $R$  if for each ideal  $A$  and  $B$  of  $R$ , if  $AB^k \subseteq T$ , then  $A \subseteq T$  or  $B^k \subseteq T$ .

*Proposition 3.5:* Let  $k \in \mathbb{Z}^+$  and  $P$  be an  $(R, S)$ -submodule of an  $(R, S)$ -module  $M$  such that  $(P : M)_{R;S^k}$  is a proper ideal of  $R$ . If  $P$  is a  $(1, k)$ -jointly prime  $(R, S)$ -submodule of  $M$ , then  $(P : M)_{R;S^k}$  is a  $(1, k)$ -prime ideal of  $R$ .

Note that  $(X)_l$  and  $(X)_r$  is the left ideal generated by  $X$  and the right ideal generated by  $X$ , respectively, for any subset  $X$  of a ring  $R$ .  $\langle Y \rangle$  is the  $(R, S)$ -submodule generated by  $Y$  for any  $(R, S)$ -submodule  $Y$  of an  $(R, S)$ -module  $M$  Next result needs the following lemma.

*Lemma 3.6:* Let  $M$  be an  $(R, S)$ -module and  $k \in \mathbb{Z}^+$ . The following statements hold:

- (i) For all left ideal  $I$  of  $R$ , left ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  and  $P$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } I^2N(J)_r^k \subseteq P.$$

- (ii) For all right ideal  $I$  of  $R$ , left ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  and  $P$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } (I)_lN(J^2)^k \subseteq P.$$

- (iii) For all right ideal  $I$  of  $R$ , left ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $P$  of  $M$ ,

$$(I)_lM(J^2)^k \subseteq P \text{ implies } (I)_l\langle IMJ^k \rangle J^k \subseteq P.$$

- (iv) For all right ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  and  $P$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } I^2N(J)_i^k \subseteq P.$$

- (v) For all right ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $P$  of  $M$ ,

$$I^2M(J)_i^k \subseteq P \text{ implies } I\langle IMJ^k \rangle (J)_i^k \subseteq P.$$

- (vi) For all left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  and  $P$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } (I)_rN(J^2)^k \subseteq P.$$

- (vii) For all left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $P$  of  $M$ ,

$$(I)_rM(J^2)^k \subseteq P \text{ implies } (I)_r\langle IMJ^k \rangle J^k \subseteq P.$$

*Proof:* (i) Let  $I$  be a left ideal of  $R$ ,  $J$  a left ideal of  $S$ ,  $N$  and  $P$  be  $(R, S)$ -submodules of  $M$ . Assume that  $INJ^k \subseteq P$ . Then

$$\begin{aligned}
 I^2N(J)_r^k &= I^2N(J + JS)^k \\
 &\subseteq I^2N(J^k + J^kS) \\
 &\subseteq INJ^k + I(INJ^k)S \\
 &\subseteq P.
 \end{aligned}$$

- (ii) Let  $I$  be a right ideal of  $R$ ,  $J$  a left ideal of  $S$ ,  $N$  and  $P$  be  $(R, S)$ -submodules of  $M$ . Assume that  $INJ^k \subseteq P$ . Then

$$\begin{aligned}
 (I)_lN(J^2)^k &= (I + RI)N(J^k)^2 \\
 &= (I + RI)N(J^k)J^k \\
 &\subseteq INJ^k J^k + RINJ^k J^k \\
 &\subseteq INJ^k + R(INJ^k)J^k \\
 &\subseteq P.
 \end{aligned}$$

- (iii) Let  $I$  be a right ideal of  $R$ ,  $J$  a left ideal of  $S$ ,  $N$  and  $P$  be  $(R, S)$ -submodules of  $M$ . Assume that  $(I)_lM(J^2)^k \subseteq P$ . Then

$$\begin{aligned}
 (I)_l\langle IMJ^k \rangle J^k &= (I)_l(\mathbb{Z}(IMJ^k) + R(IMJ^k)S)J^k \\
 &\subseteq (I)_l(\mathbb{Z}(IMJ^k))J^k + (I)_l(R(IMJ^k)S)J^k \\
 &\subseteq \mathbb{Z}(I)_lIM(J^k)^2 + (I)_lRIMJ^kS J^k \\
 &\subseteq \mathbb{Z}(I)_lM(J^2)^k + (I)_lM(J^2)^k \\
 &\subseteq P.
 \end{aligned}$$

- (iv) Let  $I$  be a right ideal of  $R$ ,  $J$  a right ideal of  $S$ ,  $N$  and  $P$  be  $(R, S)$ -submodules of  $M$ . Assume that  $INJ^k \subseteq P$ . Then

$$\begin{aligned}
 I^2N(J)_i^k &= I^2N(J + SJ)^k \\
 &\subseteq I^2N(J^k + SJ^k) \\
 &\subseteq INJ^k + I(INS)J^k \\
 &\subseteq INJ^k + INJ^k \\
 &\subseteq P + P \\
 &\subseteq P.
 \end{aligned}$$

- (v) Let  $I$  be a right ideal of  $R$ ,  $J$  a right ideal of  $S$ ,  $N$  and  $P$  be  $(R, S)$ -submodules of  $M$ . Assume that  $I^2M(J)_i^k \subseteq P$ . Then

$$\begin{aligned}
 I\langle IMJ^k \rangle (J)_i^k &= I(\mathbb{Z}(IMJ^k) + R(IMJ^k)S)(J)_i^k \\
 &\subseteq \mathbb{Z}(I^2M(J)_i^k) + IRIMJ^kS(J)_i^k \\
 &\subseteq \mathbb{Z}(I^2M(J)_i^k) + I^2M(J)_i^k \\
 &\subseteq P.
 \end{aligned}$$

(vi) Let  $I$  be a left ideal of  $R$ ,  $J$  a right ideal of  $S$ ,  $N$  and  $P$  be  $(R, S)$ -submodules of  $M$ . Assume that  $INJ^k \subseteq P$ . Then

$$\begin{aligned}(I)_r N(J^2)^k &= (I + IR)NJ^k J^k \\ &\subseteq INJ^k J^k + (IR)NJ^k J^k \\ &\subseteq INJ^k + I(RNJ^k)J^k \\ &\subseteq INJ^k \\ &\subseteq P.\end{aligned}$$

(vii) Let  $I$  be a left ideal of  $R$ ,  $J$  a right ideal of  $S$ ,  $N$  and  $P$  be  $(R, S)$ -submodules of  $M$ . Assume that  $(I)_r M(J^2)^k \subseteq P$ . Then

$$\begin{aligned}(I)_r \langle IMJ^k \rangle J^k &= (I)_r (\mathbb{Z}(IMJ^k) + R(IMJ^k)S)J^k \\ &\subseteq \mathbb{Z}((I)_r IMJ^k J^k) + (I)_r RIMJ^k S J^k \\ &\subseteq \mathbb{Z}(I)_r M(J^2)^k + (I)_r M(J^2)^k \\ &\subseteq P + P. \\ &\subseteq P.\end{aligned}$$

■

Next, we obtain equivalent conditions for an  $(R, S)$ -submodule to be  $(1, k)$ -jointly prime  $(R, S)$ -submodules.

*Theorem 3.7:* Let  $M$  be an  $(R, S)$ -module and  $P$  a proper  $(R, S)$ -submodule of  $M$  and  $k \in \mathbb{Z}^+$ . The following statements are equivalent:

- (i)  $P$  is an  $(1, k)$ -jointly prime  $(R, S)$ -submodule of  $M$ .
- (ii) For all left ideal  $I$  of  $R$ , left ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

- (iii) For all right ideal  $I$  of  $R$ , left ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

- (iv) For all right ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $(R, S)$ -submodule  $N$  of  $M$ ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

- (v) For all left ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $m \in M$ ,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

- (vi) For all left ideal  $I$  of  $R$ , left ideal  $J$  of  $S$  and  $m \in M$ ,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

- (vii) For all right ideal  $I$  of  $R$ , left ideal  $J$  of  $S$  and  $m \in M$ ,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

- (viii) For all right ideal  $I$  of  $R$ , right ideal  $J$  of  $S$  and  $m \in M$ ,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

*Proof:* This follows from Lemma 3.6. ■

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