International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:1, No:6, 2007

Quality Factor Variation with Transform Order in Fractional Fourier Domain

Sukrit Shankar, Chetana Shanta Patsa, K. Pardha Saradhi, and Jaydev Sharma

***Abstract*—**Fractional Fourier Transform is a powerful tool, which is a generalization of the classical Fourier Transform. This paper provides a mathematical relation relating the span in Fractional Fourier domain with the amplitude and phase functions of the signal, which is further used to study the variation of quality factor with different values of the transform order. It is seen that with the increase in the number of transients in the signal, the deviation of average Fractional Fourier span from the frequency bandwidth increases. Also, with the increase in the transient nature of the signal, the optimum value of transform order can be estimated based on the quality factor variation, and this value is found to be very close to that for which one can obtain the most compact representation. With the entire mathematical analysis and experimentation, we consolidate the fact that Fractional Fourier Transform gives more optimal representations for a number of transform orders than Fourier transform.

***Keywords*—**Fractional Fourier Transform, Quality Factor, Fractional Fourier span, transient signals.

I. Introduction

FRACTIONAL Fourier transform is a generalization of classical Fourier Transform. The traditional Fourier transform decomposes the signal in terms of sinusoids, which are perfectly localized in frequency, but are not at all localized in time [1]. FrFT expresses the signal in terms of an orthonormal basis formed by linear chirps. Linear chirps are the signals, whose instantaneous frequency varies linearly with time.

The Kernel for continuous Fractional Fourier Transform is given by [2]

$$K_\alpha(t,u) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}\, e^{j\frac{t^2+u^2}{2}\cot\alpha - jut\operatorname{cosec}\alpha}$$

Using this kernel of FrFT, the FRFT of signal *x(t)* with transform order (α) is computed as:

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\,dt$$

And x(t) can be recovered from the following equation,

$$x(t) = \int_{-\infty}^{\infty} X_\alpha(u) K_{-\alpha}(u,t)\,du$$

Manuscript received on 1st August, 2007.

Sukrit Shankar and K. Pardha Saradhi are with DSP Firmware team at Conexant Systems Pvt. Ltd., India.
Chetana Shanta Patsa is with IBM, India.
Jaydev Sharma is with EE Department, IIT Roorkee.

II. Mathematical Relation of Fractional Fourier Span

The normalized energies of a given signal can be thought of as signal's energy density function. To quantitatively characterize the signal's behaviour, we use the moment concepts from the probability theory. Here we use the first moment to compute mean for the signal in "*t*" domain and "*u*" domain[3].

For signal *x(t),*

$$\langle t \rangle = \frac{1}{E}\int_{-\infty}^{\infty} t\,|x(t)|^2\,dt$$

FrFT of *x(t)* is $X_\alpha(u)$. *So,*

$$\langle u \rangle = \frac{1}{E}\int_{-\infty}^{\infty} u\,|X_\alpha(u)|^2\,du$$

To expand the above equation, we take,

$$H_\alpha(u) = uX_\alpha(u)$$

So that,

$$\langle u \rangle = \frac{1}{E}\int_{-\infty}^{\infty} H_\alpha(u) X_\alpha^*(u)\,du = \frac{1}{E}\int_{-\infty}^{\infty} h(t) x^*(t)\,dt$$

(By Parseval's Relation)

It is simple to establish that

$$F_\alpha\left(\frac{d}{dt}x(t)\right) = \{-jt\cot\alpha + ju\operatorname{cosec}\alpha\} X_\alpha(u)$$

Using the above expression, it can be shown that,

$$\langle u \rangle = \frac{1}{E}\int_{-\infty}^{\infty} \sin\alpha\left[\frac{1}{j}\frac{d}{dt}x(t) + t\cot\alpha\, x(t)\right].x^*(t)\,dt$$

Putting $x(t) = A(t)\, e^{j\phi(t)}$,

Where *A(t)* and $\phi(t)$ are real. We have,

$$x^*(t) = A(t)\, e^{-j\phi(t)}$$

and

$$\frac{d}{dt}x(t)=A(t)\,j\phi'(t)e^{j\phi(t)}+A'(t)\,e^{j\phi(t)}$$

Solving further and with the fact that $<u>$ is always real, we get

$$<u>=\frac{\sin\alpha}{E}\int_{-\infty}^{\infty}[\phi'(t)+t\cot\alpha]|x(t)|^2\,dt$$

Now, the expressions for the span of the signal in time and Fractional Fourier domain about the respective mean values can be given as follows:

$$\Delta_t^2=\frac{1}{E}\int(t-<t>)^2|x(t)|^2\,dt$$

$$=\frac{1}{E}\int(t)^2|x(t)|^2\,dt-<t>^2$$

$$\Delta_u^2=\frac{1}{E}\int(u-<u>)^2|X(u)|^2\,du$$

$$=\frac{1}{E}\int(u)^2|X(u)|^2\,du-<u>^2$$

Now based on Parseval's theorem

$$\Delta_u^2=\frac{1}{E}\int(u-<u>)^2|X(u)|^2\,du$$

$$=\frac{1}{E}\int H(u)H^*(u)du$$

$$=\frac{1}{E}\int h(t)h^*(t)dt$$

Where,

$H(u)=(u-<u>)X(u)$, and

$h(t)=F_\alpha^{-1}[H(u)]$

So,

$$h(t)=F_\alpha^{-1}[(u-<u>)X(u)]$$

$$=\int_{-\infty}^{\infty}(u-<u>)X(u).K_{-\alpha}(t,u)du$$

$$=\int_{-\infty}^{\infty}u.X(u).K_{-\alpha}(t,u)du$$

$$-<u>\int_{-\infty}^{\infty}.X(u).K_{-\alpha}(t,u)du$$

$$=\int_{-\infty}^{\infty}uX(u).K_{-\alpha}(t,u)du-<u>x(t)$$

$$=F_\alpha^{-1}[u.X(u)]-<u>x(t)$$

From derivative property, it is easy to get that,

$$h(t)=\sin\alpha[-j\frac{d}{dt}x(t)+t\cot\alpha.x(t)]-<u>.x(t)=$$

$$e^{j\phi(t)}[A(t)\{\sin\alpha\phi'(t)+t\cos\alpha.-<u>\}-j\sin\alpha A'(t)]$$

Now,

$$h(t)h^*(t)=$$

$$e^{j\phi(t)}[A(t)\{\sin\alpha.\phi'(t)+t\cos\alpha-<u>\}-j\sin\alpha A'(t)]$$

$$*$$

$$e^{-j\phi(t)}[A(t)\{\sin\alpha.\phi'(t)+t\cos\alpha-<u>\}+j\sin\alpha A'(t)]$$

(Using the fact that $<u>^*=<u>$)

Expanding the above expression, we finally obtain,

$$\Delta_u^2=\frac{1}{E}\int h(t)h^*(t)dt$$

$$=\frac{1}{E}\int\{[A(t)\{\sin\alpha.\phi'(t)+t\cos\alpha-<u>\}]^2+[\sin\alpha A'(t)]^2\}dt$$

Hence,

$$\Delta_u^2=\frac{1}{E}\int\{\sin\alpha\phi'(t)+t\cos\alpha-<u>\}^2A^2(t)dt$$

$$+\frac{1}{E}\int\sin^2\alpha(A'(t))^2\,dt$$

III. Quality Factor Variation with the Transform Order

The quality factor for any domain is the ratio of the span of a signal in that domain and the mean. So, in Fractional Fourier domain, the quality factor Q is given by,

$$Q=\frac{<u>}{2\Delta_u}$$

We study the quality factor variation of the following signals with the transform order ranging from 0 to 2π.

$$x_1(t)=(\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}e^{j(20t)}$$

$$x_2(t)=(\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}e^{j(t^2+t+1)}$$

$$x_3(t)=(\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}e^{j(t^3+t^2+t+1)}$$

$$x_4(t)=(\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}e^{j(t^4+t^3+t^2+t+1)}$$

$$x_5(t)=(\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}e^{j(t^5+t^3+t^2+t+1)}$$

$$x_6(t)=(\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2}e^{j(t^6+t^3+t^2+t+1)}$$

$$x_7(t) = (\frac{1}{\pi})^{\frac{1}{4}}.e^{-\frac{1}{2}t^2} e^{j(t^7+t^3+t^2+t+1)}$$

The range of the transform order is helpful in showing the symmetry of the span and the quality factor about π. Also, the signals considered are all transient signals with the increasing complexity. The study of the quality factor variation with such signals is helpful for analysing the real time signals which are complex in nature and which at times may contain a huge number of transients.

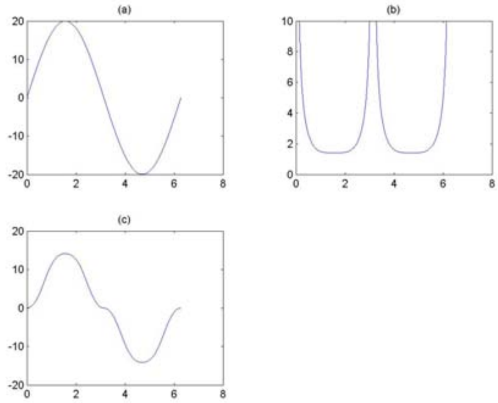

Fig. 1 Signal is $x_1(t)$. (a) <u> as a function of transform order. (b) Span as a function of transform order. (c) Quality Factor as a function of transform order

Fig. 1 shows that the variation if the quality factor with the transform order ranging from 0 to 2π, holds an anti-symmetry about π. Concentrating on the values between 0 to π, based on the quality factor curve, the optimum transform order for a normalized frequency modulated Gaussian signal can be found out.

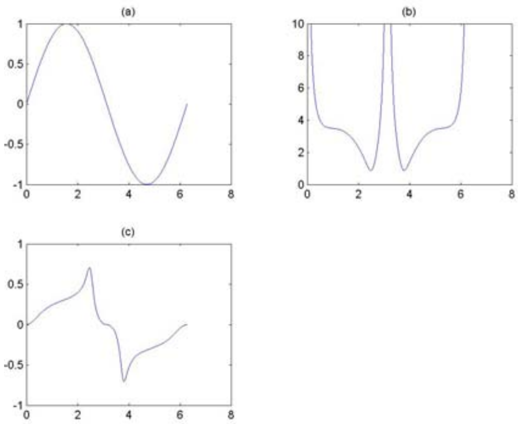

Fig. 2 Signal is $x_2(t)$. (a) <u> as a function of transform order. (b) Span as a function of transform order. (c) Quality Factor as a function of transform order

The figure above is the response for a linear chirp signal. The span starts showing a slight variation for the values for which the response was constant in the case for $x_1(t)$. Also, the transform order for the maximum quality factor changes.

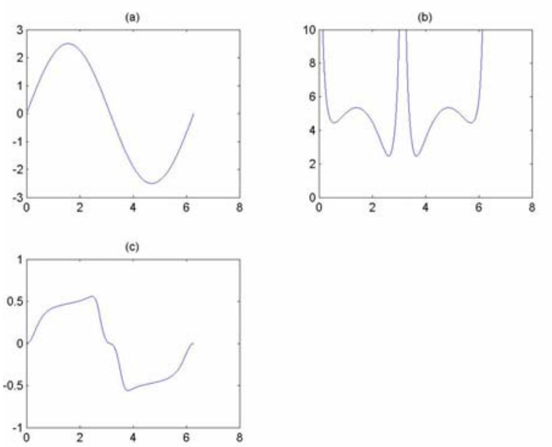

Fig. 3 Signal is $x_3(t)$. (a) <u> as a function of transform order. (b) Span as a function of transform order. (c) Quality Factor as a function of transform order

The variation of the span increases and it can now be seen clearly, that as the complexity in the form of transients is increasing, the span curve starts forming a convex hull with the turnaround point at $\pi/2$ and $3\pi/2$ respectively. The turnaround point remains same for all the subsequent cases, and it can be seen in Fig. 2,3,4,5,6,7 that only the transform order for the maximum quality factor becomes significantly different from $\pi/2$. This observation also consolidates the fact that for the increasing transient nature of the signals, the classical Fourier transform gives a decreasing optimal representation. Hence, it is logical to say here that Fractional Fourier Transform at an optimum transform order can be used to give a more compact representation as compared to the Fourier Transform.

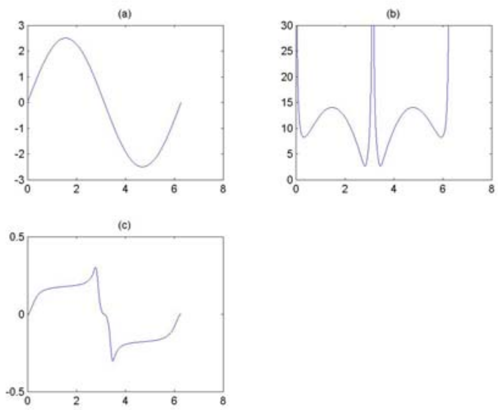

Fig. 4 Signal is $x_4(t)$. (a) <u> as a function of transform order. (b) Span as a function of transform order. (c) Quality Factor as a function of transform order

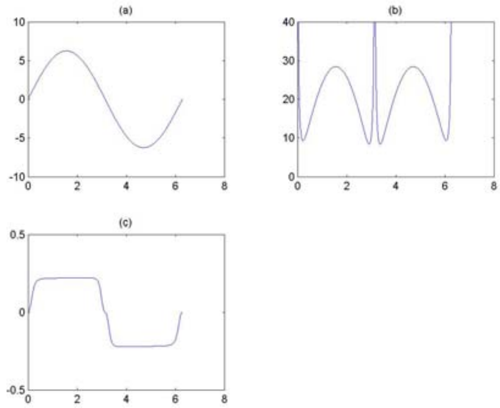

Fig. 5 Signal is $x_5(t)$. (a) <u> as a function of transform order. (b) Span as a function of transform order. (c) Quality Factor as a function of transform order

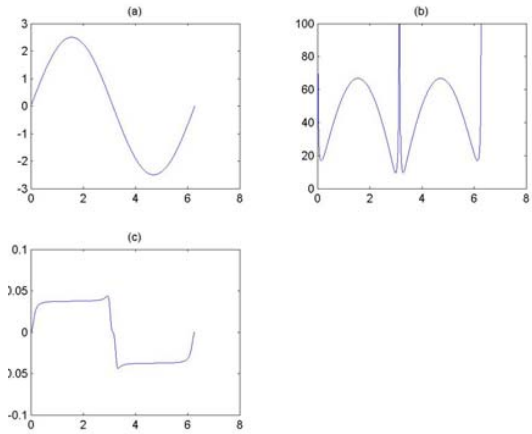

Fig. 6 Signal is $x_6(t)$. (a) <u> as a function of transform order. (b) Span as a function of transform order. (c) Quality Factor as a function of transform order

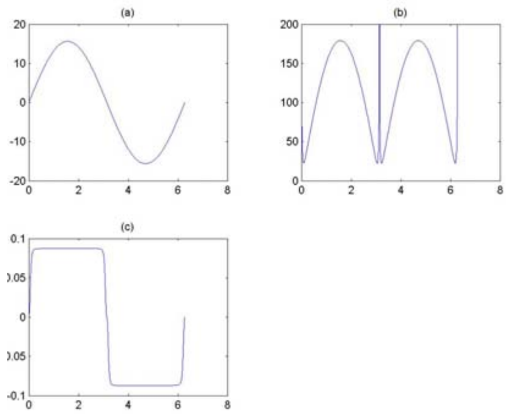

Fig. 7 Signal is $x_7(t)$. (a) <u> as a function of transform order. (b) Span as a function of transform order. (c) Quality Factor as a function of transform order

Fig. 5 shows that quality factor remains constant for many transform orders. So, in such a case the order for which the span is also minimum can be chosen as the optimal one.

It is noteworthy in all the figures above that the response at transform orders of π, 2π and 0 are almost the same and they indicate the maximum span at those points. This is synchronous with the fact that the Fractional Fourier transform gives the same signal, either erect or inverted at these transform orders.

IV. Conclusion

We have studied the variation of the quality factor with the transform order for signals with increasing transient complexity. The experiments which are based on the mathematical relation also consolidate the fact that with the increasing transient nature of the signals, the Fractional Fourier Transform can give more optimal representations with a lesser span and better quality factor for a number of transform orders than Fourier transform.

References

[1] S. C. Pei and M. H. Yeh, "Improved discrete fractional Fourier transform," Optics Letters, vol. 22, pp. 1047-1049, July 15 1997.

[2] Ahmed I. Zayed, "Relationship between the Fourier and Fractional Fourier Transforms", IEEE Signal Processing Letters, vol. 3, no. 12, December 1996.

[3] Tatiana Alieva and Martin J. Bastiaans, "On Fractional Fourier Transform Moments", IEEE Signal Processing Letters, vol. 7, no. 11, November 2000.