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Proposal of Additional Fuzzy Membership Functions in Smoothing Transition Autoregressive Models

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Abstract**—**In this paper we present, propose and examine additional membership functions for the Smoothing Transition Autoregressive (STAR) models. More specifically, we present the tangent hyperbolic, Gaussian and Generalized bell functions. Because Smoothing Transition Autoregressive (STAR) models follow fuzzy logic approach, more fuzzy membership functions should be tested. Furthermore, fuzzy rules can be incorporated or other training or computational methods can be applied as the error backpropagation or genetic algorithm instead to nonlinear squares. We examine two macroeconomic variables of US economy, the inflation rate and the 6-monthly treasury bills interest rates.

Keywords**—**Forecast , Fuzzy membership functions, Smoothing transition, Time-series

I. Introduction

EMPIRICAL analysis in macroeconomics as well as in financial economics is largely based on times series. This approach allows the model builder to use statistical inference in constructing and testing equations that characterize relationships between economic variables. There are two kinds of econometric modelling in time-series analysis. The first one contains the linear models like Autoregressive (AR), Moving Average (MA) and Autoregressive Moving Average (ARMA) models among other. The second is consisted b non-linear models, as the Threshold Autoregressive (TAR) models, Smoothing Transition Autoregressive (STAR) Models and Markov Switching Regime Autoregressive (MS-AR) model.

One criticism in STAR modeling is that the estimation procedure can be incomplete. To be specific the linear part is exactly like a linear Autoregressive (AR) process. But the non-linear part is actually a fuzzy database, where the values are the fuzzy membership grades of the inputs. So a first notice is that no rules and no linguistic terms are introduced and for inputs more than one the AND-OR operators are not considered. The studies employing STAR models do not examine and explore this feature and do not explain why the nonlinear part should be of this form. One of the very few studies which approximated that is the study of Aznarte *et al.* [1]. The second criticism is that the nonlinear squares with Levenberg-Marquardt algorithm might not be appropriate. Therefore taking the fuzzy rules and other optimization procedures like linear programming or neuro-fuzzy approach with error backpropagation algorithm can be more efficient optimization techniques.

The purpose of this paper it to propose some additional fuzzy membership functions and specifically the Gaussian and the Generalized Bell function, as well as the tangent hyperbolic function, which is used in neural networks with some appropriate modifications. A further proposal is to introduce fuzzy rules, linguistic terms and fuzzy operators in the estimation procedure. We examine linearity tests in order to find the lag order where we reject linear process, but we do not present the process of the test choosing either exponential or logistic smoothing functions. We are interesting in forecasting, because a good model is judged on its forecasting performance, so hence its estimations would be more reliable. Furthermore, the test of choosing exponential versus logistic STAR model, do not guarantee that the specific smoothing function gives necessary the best forecasts. In section II we present the methodology of STAR models and the membership functions used in this study, while in section III and IV we present the data and the empirical results.

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II. METHODOLOGY

A. Smoothing Transition Autoregressive Models

The smoothing transition auto-regressive (STAR) model was introduced and developed by Chan and Tong [2] and is defined as:

$$y_t = \pi_{10} + \pi_1' w_t + (\pi_{20} + \pi_2' w_t) F(y_{t-d}; \gamma, c) + u_t \quad (1)$$

,where $u_t \sim (0,\sigma^2)$, π_{10} and π_{20} are the intercepts in the middle (linear) and outer (nonlinear) regime respectively, $w_t = (y_{t-1} \ldots y_{t-j})$ *is* the vector of the explanatory variables consisting of the dependent variable with $j=1 \ldots p$ lags, y_{t-d} is the transition variable, parameter c is the threshold giving the location of the transition function and parameter γ is the slope of the transition function. The STAR model estimation is consisted by three steps according to Teräsvirta [3].

a) The specification of the autoregressive *(AR)* process of *j=1,... p*. One approach is to estimate *AR* models of different order and the maximum value of *j* can be chosen based on the *AIC* information criterion Besides this approach, *j* value can be selected by estimating the auxiliary regression (2) for various values of *j=1,...p,* and choose that value for which the *P-value* is the minimum, which is the process we follow.

b) The second step is testing linearity for different values of delay parameter *d*. We estimate the following auxiliary regression:

$$y_t = \beta_o + \beta_1 w_t + ...+ \sum_{j=1}^{p} \beta_{2j} y_{t-j} y_{t-d} + \sum_{j=1}^{p} \beta_{3j} y_{t-j} y_{t-d}^{\ 2} + \sum_{j=1}^{p} \beta_{4j} y_{t-j} y_{t-d}^{\ 3} + \varepsilon_t \qquad (2)$$

The null hypothesis of linearity is H_0: $\beta_{2j} = \beta_{3j} = \beta_{4j} = 0$. In order to specify the parameter *d* the estimation of (2) is carried out for a wide range of values $1 \le d \le D$ and we choose *d=1,...,5* In the cases where linearity is rejected for more than one values of *d*, then *d* is chosen by the minimum value of *p(d),* where *p(d)* is the *P-value* of the linearity test. We examine for *j=1,..., 5* and we choose those values of *j* and *d* , where the *P-value* is minimized.

c) The third and last step is the specification of STAR model. Because in this study we propose three additional membership or smoothing functions we do not examine the specific test as it is referred only in exponential and logistic STAR models. The fuzzy membership functions we examine in the non-linear part are the logistic, exponential, tangent hyperbolic, generalized bell function and Gaussian defined by (3)-(7) respectively.

$$\mu(x_i;\gamma,\mathrm{c}) = \frac{1}{1+e^{-\gamma(x_i-c)}} \qquad (3)$$

$$\mu(x_i;\gamma,\mathrm{c}) = 1 - e^{-\gamma(x_i-c)^2} \qquad (4)$$

$$\mu(x_i;\gamma,\mathrm{c}) = \frac{2}{(1+e^{-2\gamma(x_i-c)})-1} \qquad (5)$$

$$\mu(x_i;a,b,c) = \exp\left(-\left(\frac{x_i-c}{\gamma}\right)^{2b}\right) \qquad (6)$$

$$\mu(x_i;c,\sigma) = \exp\left(-\frac{(x_i-c)^2}{2\gamma^2}\right) \qquad (7)$$

,where *i* is the lag order. We apply a grid search procedure for equation (1) with non linear squares and Levenberg-Marquardt algorithm. For parameter *c* the grid search takes place in the range of the input data with increment 0.1, while we use the interval [1 10] with increment 0.5 for parameter *γ*. Finally for parameter *b* of Generalized Bell function we apply a grid search in the interval [0.5 2] with increment 0.1. The initial values for parameters *c*, *γ* and *b* are the mean value of the data, 1 and 0.5 respectively.

B. Unit Root and Stationary Tests

It is possible that the variables are not stationary in the levels, but probably are in the first or second differences. To be specific we confirm this assumption by applying Augmented Dickey-Fuller-ADF [4] and KPSS stationary test [5]. The ADF test is defined from the following relation:

$$\Delta y_t = \mu + \gamma y_{t-1} + \phi_1 \Delta y_{t-1} + ... + \phi_p \Delta y_{t-p} + \beta t + \varepsilon_t \quad (8)$$

, where y_t is the variable we examine each time. In the right hand of (8) the lags of the dependent variable are added with order of lags equal with *p*. Additionally, regression (8) includes the constant or drift *μ* and trend parameter *β*. The disturbance term is defined as ε_t. In the next step we test the hypotheses:

H_0: φ=1, β=0 => $y_t \sim I(0)$ with drift

against the alternative

H_1: |φ|<1 => $y_t \sim I(1)$ with deterministic time trend

The KPSS statistic is then defined as:

$$KPSS = T^{-2} \sum_{t=1}^{T} s^2{}_t \,/\, \hat{\sigma}^2(p) \,. \qquad (9)$$

, where *T* is the number of sample and $\hat{\sigma}^2(p)$ is the long-run variance of ε_t and can be constructed from the residuals ε_t as:

$$\hat{\sigma}^2(p) = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2 + \frac{2}{T}\sum_{t=1}^{p} w_j(p) + \sum_{t=j+1}^{T}\varepsilon_t\varepsilon_{t-j} \,, \qquad (10)$$

, where *p* is the truncation lag, $w_j(p)$ is an optional weighting function that corresponds to the choice of a special window [6]. Under the null hypothesis of level stationary,

$$KPSS \rightarrow \int_0^1 V_1(r)^2\, dx \qquad (11)$$

, where $V_1(x)$ is a standard Brownian bridge: $V_I(r) = B(r) - rB(1)$ and $B(r)$ is a Brownian motion (Wiener process) on $r \in [0, 1]$. Because relation (11) is refereed in testing only on the intercept and not in the trend and as we are testing with both intercept and trend we have the second-level Brownian bridge $V_2(x)$ and it is:

$$KPSS \rightarrow \int_0^1 V_2(r)^2\,dx \tag{12}$$

, where $V_2(x)$ is given by:

$$V_2(r) = W(r) + (2r - 3r^2)W_1 + (-6r + 6r^2)\int_0^1 W_s(s)ds \tag{13}$$

The forecasting performance of STAR models in both in-sample and out-of- sample periods is counted based on the Mean Absolute Error (MAE) and Root Mean Squared Error (RMSE) described respectively by (14) and (15).

$$MAE = \frac{1}{n}\sum_{i=1}^{n} | y_i - \hat{y}_i | \tag{14}$$

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (y_i - \hat{y}_i)^2} \tag{15}$$

III. DATA

We examine two macroeconomic variables of US economy, the inflation rate and the six monthly treasury bills interest rates. The data we use in our analysis are in monthly frequency. We examine the period 1950-2009. The period 1950 to 2005 is used for the in-sample period and the period 2006-2009, which is a period 48 observations, is left for the out-of-sample forecasting period. It should be noticed that the forecasting step is one period ahead.

IV. EMPIRICAL RESULTS

In Table I we present the results of ADF and KPSS tests, The results are mixed. For gross domestic product we reject unit root in α=0.05 and 0.10 based on ADF test, while we accept stationarity only in α=0.01 based on KPSS test. For example for unemployment and inflation rates we reject unit root based on ADF statistic in all statistical significance levels, but we reject stationarity hypothesis based on KPSS test. We accept that treasury bills interest rates are stationary in first differences, *I(1)*, based on both ADF and KPSS tests.

In Table II we report the linearity tests for the two macroeconomic variables we examine. The value of lag order *p* is chosen based on the minimum *p-value* and in the cases where there are more than one zero *p-values* lag order *p* is chosen based on the highest *F-statistic.* In all cases we found an autoregressive process AR(1), *p=1,* we choose the lag order of delay 1 and 2 respectively for inflation, and interest rates.

TABLE I
ADF AND KPSS TESTS

Indices	ADF-statistic	KPSS-statistic
Inflation Rate Levels	-5.040	0.3759
Inflation Rate First differences		0.1102
Treasury Bills Levels	-2.089	0.5547
Treasury Bills First differences	-8.461	0.0335
Critical values for ADF[1]	-4.086	α=0.01
	-3.471	α=0.01
	-3.162	α=0.10
Critical values for KPSS[2]	0.216	α=0.01
	0.146	α=0.01
	0.119	α=0.10

1 MacKinnon [7], 2 Kwiatkowski *et al.,*[5]

TABLE II
LINEARITY TESTS FOR INFLATION AND INTEREST RATES

Indices	Inflation Rate	Treasury Bills
p	1	1
d=1	24.820	7.503
	(0.000)	(0.0001)
d=2	13.458	17.551
	(0.000)	(0.000)
d=3	3.470	5.941
	(0.0160)	(0.0001)
d=4	6.278	3.069
	(0.0003)	(0.0274)
d=5	16.085	5.228
	(0.000)	(0.0014)

*p-values in parenthese

In tables III and IV the estimated results for inflation and interest rates respectively are reported. We observe, in the case of the inflation rate, that the fuzzy membership function parameters, *c* and *γ* are statistically significant, except from Generalized Bell function, where parameters *c* and *b* are statistically insignificant. On the other hand for Treasury bill interest rates, parameter *γ* is statistically insignificant in the case of logistic function, while parameter *c* is insignificant for tangent hyperbolic and Generalized Bell functions. In the most cases the estimated parameters in the linear and non-linear part are significant.

In Table V the Root Mean Squared Error (RMSE) and Mean Absolute Error (MAE) are reported. The results are mixed, as in the case of inflation rate and the in-sample period, Generalized Bell STAR function presents the highest performance, followed by tangent hyperbolic, logistic and exponential, while the lowest forecasting performance is reported for Gaussian. In the out-of-sample period for inflation rate exponential STAR model presents the highest forecasting performance, followed by Generalized Bell function, tangent hyperbolic, logistic and finally, Gaussian presents again the highest RMSE and MAE values.

TABLE III
ESTIMATIONS FOR INFLATION RATE

	Linear Part		***Non-Linear Fuzzy Part***	
Exponential STAR	π_{10}	π_{11}	π_{20}	π_{21}
	0.0509	0.5971	-0.1005	-2.0167
	(1.018)	(3.639)*	(-0.378)	(-4.506)*
	γ	c		
	2.9327	0.1163		
	(1.997)**	(1.671)***		
Logistic STAR	π_{10}	π_{11}	π_{20}	π_{21}
	0.2611	-0.2701	-0.3606	0.0317
	(2.173)**	(-0.639)	(-2.220)**	(0.067)
	γ	c		
	4.875	-0.0696		
	(2.034)**	(-2.723)*		
Tangent STAR	π_{10}	π_{11}	π_{20}	π_{21}
	0.5809	-0.2542	-0.6804	0.0159
	(2.075)**	(-1.161)	(-2.219)**	(0.0676)
	γ	c		
	4.876	-0.0696		
	(2.035)**	(-2.722)**		
Generalized Bell STAR	π_{10}	π_{11}	π_{20}	π_{21}
	-0.1854	0.6010	0.2143	-1.3164
	(-0.667)	(1.767)***	(0.757)	(-1.713)***
	γ	b	c	
	1.6512	0.3844	0.1743	
	(2.531)**	(0..0617)	(0.030)	
Gaussian STAR	π_{10}	π_{11}	π_{20}	π_{21}
	-0.5123	-0.9766	0.8722	1.5900
	(-2.023)**	(2.396)**	(3.260)*	(3.752)*
	γ	c		
	1.5073	0.8140		
	(2.654)**	(2911)**		

*,**,*** denotes significance in 0.01, 0.05 and 0.10 respectively, t-statistics in parentheses

TABLE IV
ESTIMATIONS FOR TREASURY BILLS INTEREST RATES

	Linear Part		***Non-Linear Fuzzy Part***	
Exponential STAR	π_{10}	π_{11}	π_{20}	π_{21}
	0.0424	0.7752	-0.0506	-1.1159
	(0.602)	(1.691)***	(-0.658)	(-2.111)**
	γ	c		
	3.335	0.601		
	(1.761)***	(15.916)*		
Logistic STAR	π_{10}	π_{11}	π_{20}	π_{21}
	0.0140	0.0022	-0.0489	-0.5597
	(0.683)	(0.017)	(-1.226)	(-2.548)**
	γ	c		
	4.206	0.638		
	(0.022)	(3.682)*		
Tangent STAR	π_{10}	π_{11}	π_{20}	π_{21}
	0.0013	0.4424	-0.0106	0.2480
	(0.007)	(4.034)*	(-0.620)	(2.260)
	γ	c		
	1.782	0.6001		
	(1.735)***	(0.008)		
Generalized Bell STAR	π_{10}	π_{11}	π_{20}	π_{21}
	-0.0079	0.6629	-0.0704	-2.1910
	(-0.461)	(7.639)*	(-0.179)	(-0.106)
	γ	b	c	
	1.167	1.749	1.1051	
	(5.412)*	(3.369)*	(0.060)	
Gaussian STAR	π_{10}	π_{11}	π_{20}	π_{21}
	-0.0082	-0.3406	0.0506	1.1159
	(-0.419)	(-2.541)*	(0.658)	(-2.115)**
	γ	c		
	2.601	-0.387		
	(15.916)*	(-2.540)**		

*,**,*** denotes significance in 0.01, 0.05 and 0.10 respectively, t-statistics in parentheses

On the other hand for six-monthly treasury bills, Generalized bell function presents again the highest forecasting performance followed by tangent hyperbolic, logistic and exponential STAR models, whose RMSE and MAE values are very close among them. Gaussian function presents again the highest RMSE and MAE vales and therefore the lowest forecasting performance. In the out-of-sample period logistic STAR model presents the highest performance, followed by Gaussian function.

The results indicate that we can examine additional membership functions in STAR modelling for further applications in more cases, as the gross domestic product, the stock returns and the exchange rates among others. To be specific we found that Gaussian function outperforms exponential STA model in the out-of-sample period, for interest rates. Furthermore, in the case of inflation rate and the out-of-sample period tangent hyperbolic and Generalized Bell functions outperform logistic STAR model, so additional membership functions for STAR modelling are proposed for further research applications.

TABLE V
RMSE AMD MAE VALUES

	In sample period			
	Inflation		Interest rates	
	RMSE	MAE	RMSE	MAE
Exponential	0.2719	0.2057	0.1641	0.1153
Logistic	0.2696	0.1981	0.1643	0.1119
Tangent hyperbolic	0.2684	0.1974	0.1643	0.1117
Generalized Bell	0.2674	0.1971	0.1494	0.0952
Gaussian	0.2760	0.2088	0.1661	0.1190
	Out-of-sample period			
	Inflation		Interest rates	
	RMSE	MAE	RMSE	MAE
Exponential	0.2791	0.2240	0.3249	0.2741
Logistic	0.2857	0.2363	0.2929	0.2333
Tangent hyperbolic	0.2819	0.2288	0.3420	0.2678
Generalized Bell	0.2816	0.2317	0.3259	0.2752
Gaussian	0.2898	0.2387	0.3032	0.2533

V. CONCLUSIONS

In this paper we proposed three additional fuzzy membership functions for the non-linear part of Smoothing Transition Autoregressive (STAR) models. Furthermore the STAR models are incomplete. To be specific the nonlinear part accounts for the membership grades of inputs but no rules are included. Additionally, STAR models assume only one or even no linguistic term and it should be for example for exchange rates or stock returns a fuzzy set of linguistic terms like {negative returns, zero returns, positive returns}, or even assigning more or different linguistic terms like {very negative returns, negative returns, zero returns, positive returns, very positive returns} and therefore taking the appropriate operator, max, min or product. More over additional membership functions can be proposed; besides those we have examined on this study, as the triangular, trapezoidal, the S-shaped or Z-shaped among others. Finally, other optimization methods can be applied in order to find the fuzzy parameters as the error backpropagation or genetic algorithms instead to nonlinear squares and Levenberg-Marquardt algorithm.

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