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Pricing European Options under Jump Diffusion Models with Fast L-stable Padè Scheme

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Abstract—The goal of option pricing theory is to help the investors to manage their money, enhance returns and control their financial future by theoretically valuing their options. Modeling option pricing by Black-School models with jumps guarantees to consider the market movement. However, only numerical methods can solve this model. Furthermore, not all the numerical methods are efficient to solve these models because they have nonsmoothing payoffs or discontinuous derivatives at the exercise price. In this paper, the exponential time differencing (ETD) method is applied for solving partial integrodifferential equations arising in pricing European options under Merton's and Kou's jump-diffusion models. Fast Fourier Transform (FFT) algorithm is used as a matrix-vector multiplication solver, which reduces the complexity from $\mathcal{O}(M^2)$ into $\mathcal{O}(M \log M)$. A partial fraction form of Padè schemes is used to overcome the complexity of inverting polynomial of matrices. These two tools guarantee to get efficient and accurate numerical solutions. We construct a parallel and easy to implement a version of the numerical scheme. Numerical experiments are given to show how fast and accurate is our scheme.

Keywords—Integral differential equations, L-stable methods, pricing European options, Jump–diffusion model.

I. Introduction

BLACK-Scholes model is consider the cornerstone of option pricing theory. However, empirical studies revealed that the Black-Scholes model is inconsistent with market movements. Many studies have revealed to overcome these shortcomings, such as Lèvy models and jump-diffusion models. Two main Jump-diffusion models proposed by Merton [1] and Kou [2] are our concern in this paper. We intend to numerically solve a partial integrodifferential equation (PIDE) arising in jump-diffusion models. Unlike the Black-Scholes model, Jump diffusion models do not have closed forms. Therefore, extensive research has been conducted on this topic. Several numerical schemes were used such as Alternating Directions Implicit methods [3], [4], Multinomial trees method [5]–[7]. The latter method is restricted by the number of time steps. Operator-splitting approach was also considered in [8]–[10]. However, these approaches require to solve full dense matrices, which is computationally expensive. To avoid a full dense matrix inversion, Khaliq et al. [11] developed numerical schemes based on rational approximations of matrix exponential functions. The schemes were applied to the Black-Scholes model. We intend to extend their scheme for pricing European options under the jump-diffusion models. We overcome the difficulties of the dens matrix, arising from approximating the non-local integral term using the FFT algorithm. We also shall construct an accurate numerical scheme using partial fraction decomposition technique. Integrating these two techniques will considerably reduces the computational cost and as well as preserves the quadratic order of convergence in time as well as in space. This paper is organized as follows: the continuous problem is given in Section II. Discretization in space as well as in time are given in Sections III and IV respectively. Several numerical examples are given in Section V to show the efficiency and accuracy of our algorithm. Finally, short conclusion is given in Section VI.

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II. Mathematical Model

Before we go through the mathematical models and their formulas, we first need to give some important definitions and notations from option pricing theory for the sake of clarity.

Definition 1. An **asset** is a sale object that has a known value at present, but it can be changed in the future.

There are a few examples of assets, such as currencies, for example the value of one bitcoin in USD, shares in a company, value of gold or oil.

Definition 2. An **option** is a sale agreement between two parts, holder and writer, to purchase or to sell, but not the obligation, a particular asset for particular price at particular time in the future.

There are several types of options, but we are interested in some of them which will be divided into two types depending on the exercising type and exercising time. with respect to exercising type.

Definition 3. Call option is an option that gives the holder (buyer) the right to buy, but not the obligation, a particular asset for particular price at particular time in the future.

Definition 4. Put Option is an option that give the writer (seller) the right to sell, but not the obligation, a particular asset for particular price at particular time in the future.

Definition 5. European option is an option that can only be exercised (bought or sold) on the expiry date.

A. Jump Diffusion Model

Consider $v(x, \tau)$ is the value price of the asset x at time τ satisfying the following initial value problem

$$v_\tau - \frac{1}{2}\sigma^2 v_{xx} - (r - \frac{1}{2}\sigma^2 - \kappa\lambda)v_x + (r + \lambda)v$$

$$-\lambda \int_{-\infty}^{\infty} v(z,\tau)\phi(z-x)dz = 0 \tag{1}$$

with the appropriate boundary and initial conditions depending on the option of interest.

- Call option

$$\begin{cases} v(x,0) = max(Ee^x - E), \\ v(X_{min},\tau) = 0, \\ v(X_{max},\tau) = Ee^{x_{max}} - Ee^{-rt}, \end{cases}$$

- Put option

$$\begin{cases} v(x,0) = max(E - Ee^x), \\ v(X_{min},\tau) = Ee^{-rt} - Ee^{X_{min}}, \\ v(X_{max},\tau) = 0, \end{cases}$$

where E is the exercise price, σ is the volatility and r is the rate of interest, λ is the Poisson intensity and κ is the expectation of the the impulse function. For the boundary conditions, X_{min} and X_{max} are a boundaries of the truncated domain $\Omega \in (-\infty,\infty)$ and ϕ is the density function of the normal distribution function is given by

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(\xi-\mu)^2}{2\delta^2}}$$

where μ and δ^2 are the mean and the variance of the the normal distribution respectively. κ w.r.t the normal distribution function is given by

$$\kappa = e^{(\mu - \frac{\delta^2}{2})} - 1$$

III. Spatial Discretization

Most of the PDEs or PIDEs arising in finance are discretized by finite difference methods. We shall use the second order central finite difference method to approximate the truncated infinite space domain $\Omega = [X_{min}, X_{max}]$ [12], then we discretize the differentiation term of the PIDE (1) in space and approximate the integral term using the composite Trapezoidal Rule at each subinterval in space. Thus, the approximation of the differentiation part of of PIDE (1) is given by a tri-diagonal matrix

$$A = -\,tridiag\Big[\frac{\sigma^2}{2h^2} - \frac{2r - 2\lambda\kappa - \sigma^2}{4h}, -\frac{\sigma^2}{h^2} - r - \sigma^2 \\ , \frac{\sigma^2}{2h^2} + \frac{2r - 2\lambda\kappa - \sigma^2}{4h}\Big].$$

where h is the step size. The infinite integral is split into local and non-local integrals. The local integral (i.e over Ω) is approximated by composite trapezoidal rule which gives a Toeplitz matrix given by

$$[G]_{j,i} = h\phi(h(i-j)), \quad i,j = 1,2,...,M \tag{2}$$

where M is the number of space-steps. The non-local integrals are either vanished due to the boundary conditions or computed analytically depending on the distribution function and option type. Therefore, the PIDE(1) can be written as a semi-linear system of ODEs given by

$$v'(\tau) + Av = \lambda F(v,\tau) \tag{3}$$

where F is the approximated integral term.

IV. Time Stepping Schemes

Padé approximation is a ratio of two polynomials determined from the coefficients of the Taylor series expansion of a function given by [11]:

$$R_m^n(x) = P_m^n(x)/Q_m^n(x)$$

where $P_m^n(x)$ and $Q_m^n(x)$ are two polynomials of order n and m respectively. Our interest is to use the second order L-acceptable (0,2)-Padè approximation to be able to construct an L-stable method.

$$R_2^0(x) = 2(2 + 2x + x^2)^{-1}$$

For the semi-discretized system of the ODE given in (3), we set $k \geq 0$, $\tau_n = nk$, $0 \leq n \leq N$. Following [11], (3) has an exact solution using Duhamel principle given by

$$v_{n+1} = e^{-kA}v_n + k\int_0^1 e^{-kA(1-s)}F(v(\tau_n + ks), \tau_n + ks)ds \tag{4}$$

where $\nu - \tau = ks$ and $v_n = v(\tau_n)$.

To solve (4), several exponential time differencing Runge-Kutta schemes (ETDRK) were proposed (see [13], [14]). However, these methods require to invert matrix higher order polynomials which causes computational difficulties as well as instability due to the ill-conditioning. Follwoing [15] we overcame all these difficulties by using the partial fraction form of the Pade` approximation (see [16], [17]). Therefore, our scheme becomes

$$v_{n+1} = \alpha_n + k\phi_1(kA)\big[F(\alpha_n, \tau_{n+1}) - F(v_n, \tau_n)\big] \tag{5}$$

where

$$a_n = R_2^0(kA)v_n + k\phi_2(kA)F(v_n, \tau_n)$$

and

$$\phi_1(kA) = (I + kA)(2I + 2kA + k^2A^2)^{-1}$$
$$\phi_2(kA) = (kA)^{-1}(I - R_2^0(kA))$$

where

$$R_2^0(x) = 2\Re\left(\frac{w_1}{x - \rho_1}\right)$$

where $\rho_1 = i - 1$, is a shared pole of $R_2^0(x), \phi_1(x)$ and $\phi_2(x)$. Whereas, $w_1 = -i, w_2 = \frac{1}{2}$ and $w_3 = \frac{1}{2} - \frac{i}{2}$ are the weights of $R_0^2(x), \phi_1(x)$ and $\phi_2(x)$ corresponded to that pole respectively.

Definition 6. A method has an **absolutely stable region D** if $|R_m^n(x)| < 1$ for all $x \in D$.

Definition 7. A method is called **A-stable** if its absolutely stable region contains the right-half plan, i.e; $|R_m^n(x)| > 0$

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Definition 8. A method is called **L-stable** if it is an A-stable and satisfies

$$\lim_{x\to\infty} R^n_m(x) = 0$$

A. Stability Region

Consider the nonlinear ODE,

$$v_\tau = cv + F(v) \tag{6}$$

where $F(v)$ is the non-linear term. We assume that there exist a fixed point $v_0 = v(\tau_0)$, such that $cv_0 + F(v_0) = 0$. We linearize about the fixed point to lead to

$$v_t = cv + \lambda v. \tag{7}$$

where v becomes the perturbation to v_0, whereas, $\lambda = F'(v_0)$. Following [13], if $\Re(c + \lambda) < 0$, then the fixed point v_0 is stable.

To obtain the stability region of the numerical methods, we first denote $x = \lambda k$ and $y = ck$, where k is the time step-size, then we apply (5) to the ODE (6) leading to a recurrence relation involving v_n and v_{n+1}. The following amplification factor corresponding to the (0,2)-Padè scheme can be computed by any mathematical software.

$$r(x, y) = \frac{x^2y^2 - 3x^2y + 2(x - y)^2 + xy^2 + 4x - 4y + 4}{(y^2 - 2y + 2)^2} \tag{8}$$

Generally speaking, the parameters c and λ are complex so are x and y. Therefore, the the stability region of the (0,2)-Padè scheme is four dimensional, which makes it difficult to plot the stability region [13]. Hence different approaches have been used to overcome this issue such as in [13] who put both x and y are real, whereas, [18] assumed that x is complex and t y is fixed and real.

According to [18], for a better useful method, the stability regions grow as $|y|$ becomes larger. Therefore, we shall fix y with several negative real values , $y = 0$, $y = -5$, $y = -10$ and $y = -20$, in the complex x-plane.

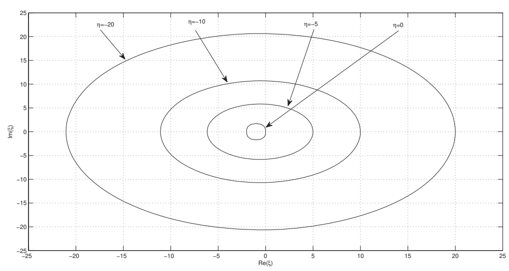

Fig. 1 Stability regions of (0,2)-Pad`e scheme in the complex ξ-plane

We can observe from Fig. 1 that the stability region tends to the second order Runge-Kutta scheme as $y \to 0$, and it grows as y decreases from -10 to -20. This result gives an indication of the stability of the (0,2)-Padè scheme.

B. The $ETD - Pad\grave{e}(0,2)$ Algorithm

$$\begin{cases} \textit{Step1:}\ \text{Solve for X} \\ \qquad (kA - \rho_1 I)X = w_1 v_n + w_3 k F(v_n, \tau_n) \\ \textit{Step2:}\ \text{Set} \\ \qquad \alpha_n = 2\,\Re(X) \\ \\ \textit{Step3:}\ \text{Solve for Y} \\ \quad (kA - \rho_1 I)Y = w_2 k \Big[F(\alpha_n, \tau_{n+1}) - F(\alpha_n, \tau_n) \Big] \\ \textit{Step4:}\ \text{Set} \\ \qquad v_{n+1} = \alpha_n + 2\,\Re(Y) \\ \text{End} \end{cases}$$

V. Numerical Experiments

In this section, we test performance of our algorithm by showing its efficiency and accuracy. Therefore, the first tow examples compare computational cost in different numerical methods. Whereas, the third numerical example shows the order of convergence independently of the maturity state (T). All the numerical experiments were computed using $Matlab^{©}$ with processor core i3 and RAM 4GB.

A. FFT Algorithm Efficiency

Thanks to the approximation of the integral part which leads to a Toeplitz matrix, we can reduce the cost of the vector-matrix multiplication from $O(M^2)$ to $O(M\,logM)$ by using using what is so-called Fast Fourier Transform (FFT) algorithm [8], [19].

We compare the FFT algorithm as a matrix-vector multiplication and the straightforward matrix-vector multiplication. We set the following inputs: $\Omega = [-6, 6]$, with the parameters from the literature [8] given by $E = 1$, $\sigma = 0.2$, $\rho = 0.5$, $r = 0$, $\lambda = 0.2$, $\alpha_1 = 3$,$\alpha_2 = 2$ and $T = 0.2$. We can observe from Table I that the FFT algorithm is taking much less time than the straightforward multiplication. As the size of the matrix getting larger and the number of iteration is higher, the FFT algorithm performs much better.

TABLE I
Comparison between Matrix-Vector Multiplication by the FFT Algorithm and the Straightforward Multiplication

		Straightforward Multiplication	FFT Algorithm
M	N	CPU(seconds)	CPU(seconds)
257	40	0.250	0.277
513	80	0.939	0.822
1025	160	4.777	4.169
2049	320	27.683	22.018
4097	640	176.423	130.603
8194	1280	1265.331	740.895

B. Padè Scheme Efficiency

In this experiment we consider the numerical solution for European call option under Merton's jump obtained by (0,2)-Padè scheme and exponential Time Integrator method (ETI) method used by [20], with the following parameters

$\Omega = [-2, 2]$, $E = 100$, $\sigma = 0.3$, $\delta = 0.5$, $r = 0$, $\lambda = 1$,$\mu = 0$ and $T = 0.5$

TABLE II
EUROPEAN CALL OPTION UNDER MERTON'S JDM OBTAINED BY (0,2)-PAD `E ETDRK2 SCHEME AND ETI METHOD

		(0,2)-Padè ETDRK2		ETI	
M	N	Error	CPU	Error	CPU
40	40	2.706e -02	0.03714	1.929e-02	0.27055
80	80	3.180 e-03	0.23670	1.216e-03	0.48838
160	160	5.778 e-05	0.39437	8.343 e-06	2.72692
320	320	1.365e-05	1.50878	1.260e-6	48.91743
640	640	3.918e-06	10.36289	8.074e-7	683.06578

We can easily observe from Table II that the accuracy of the solution is almost the same in the both methods.However, the (0,2)-Padè scheme is extremely faster that the ETI scheme. especially' when the time-steps is small as well as the size of the matrix A is large.

C. Padè Scheme Convergence

To test the convergence in time, we use our scheme in different maturities.

TABLE III
ORDER OF CONVERGENCE OF (0,2)-PAD `E SCHEME FOR EUROPEAN CALL OPTION UNDER MERTON'S JDM AT $S = E$

	T=0.5		T=1	
Time Steps	*Error*	*Order*	*Error*	*Order*
40	1.2283e-03	–	5.9796e-3	–
80	3.0948e-4	1.98877	1.5008e-3	1.99432
160	7.7761e-05	1.99273	3.7514e-4	2.00022
320	1.9492e-05	1.99615	9.3706e-5	2.00122
640	4.8372e-06	2.0106	2.3296e-05	2.00803

The convergence results are computed by multiplying the number the time-steps N by two starting by 40 nodes and uniformly refined to 640. Whereas, the space step size is fixed by $h = 0.001$. The error is calculated by the difference between the exact and approximated solution at the asset price $S = E$. The exact solution at $T = 0.5$ and $T = 1$ is 10.4219064 and 15.66668082 respectively

VI. CONCLUSION

We have developed an efficient and stable scheme for pricing European options under Morton's jump diffusion model. We integrated two useful methods to overcome computational cost of dense matrices operations. Our scheme considerably reduced the processing time compared with other schemes. Quadratic rate of convergence is also achieved.

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REFERENCES

[1] R. C. Merton, "Option pricing when underlying stock returns are discontinuous," *Journal of financial economics*, vol. 3, no. 1-2, pp. 125–144, 1976.
[2] S. G. Kou, "A jump-diffusion model for option pricing," *Management science*, vol. 48, no. 8, pp. 1086–1101, 2002.
[3] L. Andersen and J. Andreasen, "Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing," *Review of Derivatives Research*, vol. 4, no. 3, pp. 231–262, 2000.
[4] K. J. in't Hout and J. Toivanen, "Adi schemes for valuing european options under the bates model," *Applied Numerical Mathematics*, vol. 130, pp. 143–156, 2018.
[5] K. I. Amin, "Jump diffusion option valuation in discrete time," *The journal of finance*, vol. 48, no. 5, pp. 1833–1863, 1993.
[6] S. Zhang, "Radial basis functions method for valuing options: A multinomial tree approach," *Journal of Computational and Applied Mathematics*, vol. 319, pp. 97–107, 2017.
[7] N. Cantarutti and J. Guerra, "Multinomial method for option pricing under variance gamma," *International Journal of Computer Mathematics*, vol. 96, no. 6, pp. 1087–1106, 2019.
[8] A. Almendral and C. W. Oosterlee, "Numerical valuation of options with jumps in the underlying," *Applied Numerical Mathematics*, vol. 53, no. 1, pp. 1–18, 2005.
[9] A. Itkin, "Efficient solution of backward jump-diffusion partial integro-differential equations with splitting and matrix exponentials," *Journal of Computational Finance*, vol. 19, no. 3, 2016.
[10] L. Boen *et al.*, "Operator splitting schemes for the two-asset merton jump–diffusion model," *Journal of Computational and Applied Mathematics*, 2019.
[11] A. Khaliq, B. Wade, M. Yousuf, and J. Vigo-Aguiar, "High order smoothing schemes for inhomogeneous parabolic problems with applications in option pricing," *Numerical Methods for Partial Differential Equations*, vol. 23, no. 5, pp. 1249–1276, 2007.
[12] Y. dHalluin, P. A. Forsyth, and G. Labahn, "A penalty method for american options with jump diffusion processes," *Numerische Mathematik*, vol. 97, no. 2, pp. 321–352, 2004.
[13] S. Cox and P. Matthews, "Exponential time differencing for stiff systems," *Journal of Computational Physics*, vol. 176, no. 2, pp. 430–455, 2002.
[14] A.-K. Kassam and L. N. Trefethen, "Fourth-order time-stepping for stiff pdes," *SIAM Journal on Scientific Computing*, vol. 26, no. 4, pp. 1214–1233, 2005.
[15] A. Khaliq, J. Martin-Vaquero, B. Wade, and M. Yousuf, "Smoothing schemes for reaction-diffusion systems with nonsmooth data," *Journal of Computational and Applied Mathematics*, vol. 223, no. 1, pp. 374–386, 2009.
[16] A. Khaliq, E. Twizell, and D. Voss, "On parallel algorithms for semidiscretized parabolic partial differential equations based on subdiagonal padé approximations," *Numerical Methods for Partial Differential Equations*, vol. 9, no. 2, pp. 107–116, 1993.
[17] E. Gallopoulos and Y. Saad, "On the parallel solution of parabolic equations," in *Proceedings of the 3rd international conference on Supercomputing*, pp. 17–28, ACM, 1989.
[18] G. Beylkin, J. M. Keiser, and L. Vozovoi, "A new class of time discretization schemes for the solution of nonlinear pdes," *Journal of computational physics*, vol. 147, no. 2, pp. 362–387, 1998.
[19] C. R. Vogel, *Computational methods for inverse problems*, vol. 23. Siam, 2002.
[20] N. Rambeerich, D. Tangman, A. Gopaul, and M. Bhuruth, "Exponential time integration for fast finite element solutions of some financial engineering problems," *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 668–678, 2009.

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