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Positive Solutions for Semipositone Discrete Eigenvalue Problems via Three Critical Points Theorem

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Abstract**—-** In this paper, multiple positive solutions for semipositone discrete eigenvalue problems are obtained by using a three critical points theorem for nondifferentiable functional.

Keywords**—-** Discrete eigenvalue problems, positive solutions, semipositone, three critical points theorem

I. Introduction

Let $\mathbf{Z}$ and $\mathbf{R}$ be the set of all integers and real numbers respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a \le b$.

In this paper, we study the positive solutions for semipositone discrete eigenvalue problems

$$\begin{cases} -\triangle^2 u(t-1) = \lambda f(u(t)), t \in \mathbf{Z}(1,T), \\ u(0) = 0, u(T+1) = 0, \end{cases} \tag{1}$$

where λ is a positive parameter, $T \ge 4$ is a positive integer, $\triangle u(t) = u(t+1) - u(t)$ is the forward difference operator, $\triangle^2 u(t) = \triangle(\triangle u(t))$. $f : [0, +\infty) \to \mathbf{R}$ is a continuous function, $f(0) < 0$. A sequence $\{u(t)\}_{t=0}^{T+1}$ is called a positive solution of (1) if $\{u(t)\}_{t=0}^{T+1}$ satisfies (1) and $u(t) > 0$ for $t \in \mathbf{Z}(1,T)$.

When $f(0) < 0$, such problems are usually referred in the literature as semipositone problems. Semipositone problems derive from [5], where Castro and Shivaji initially called them nonpositone problems, in contrast with the terminology positone problems, put forward by Cohn and Keller in [8], where the nonlinearity was positive and monotone. Semipositone problems arise in bulking of mechanical systems, design of suspension bridges, chemical reactions, astrophysics, combustion and management of natural resources.

In general, studying positive solutions for semipositone problems is more difficult than that for positone problems. The difficulty is due to the fact that in the semipostione case, solutions have to live in regions where the nonlinear term is negative as well as positive. However, many methods have been applied to deal with semipositone problems. The usual approaches are quadrature methods, fixed point theory, sub-super solutions and degree theory. The readers may refer to the survey paper [7] and the references therein.

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Very recently, critical point theory has been applied to study the positive solutions of semipositone problems. In [9], Costa, Tehrani and Yang investigated the positive solutions of semipositone Dirichlet problems by using nonsmooth mountain pass theorem, which has been developed by Chang [6]. Furthermore, in [12], Zhang and Liu considered the positive solutions of a class of semipositone discrete boundary value problems. In addition, three critical points theorem, which is a powerful tool in studying multiple solutions of differential equations, has been used to obtain the multiple solutions of nonlinear differential equation and difference equations. See [2, 10, 11]. However, to the author's best knowledge, it has not been applied to study positive solutions of semipositone discrete boundary value problems. For knowledge about difference equations, one can refer to [1].

Our main objective in this paper is to use a three critical points theorem for nondifferentaible functional to deal with the multiple positive solutions of semipositone problem (1). More precisely, we define the discontinuous nonlinear term

$$g(s) = \begin{cases} f(s) & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases}$$

Now we consider the slightly modified problem

$$\begin{cases} -\triangle^2 u(t-1) = \lambda g(u(t)), t \in \mathbf{Z}(1,T), \\ u(0) = 0, u(T+1) = 0. \end{cases} \tag{2}$$

We will prove in section III that the set of positive solutions of (1) and (2) do concide. Moreover, any nonzero solution of (2) is nonnegative.

II. Preliminaries

In this section, we recall some basic results on variational method for locally Lipschitz functional $I : X \to \mathbf{R}$ defined on a real Banach space X with norm $\|\cdot\|$, that is, for functional such that, for each $u \in X$, there is a neighbourhood $N = N(u)$ of u and a constant $M = M(u)$ for which

$$|I(x) - I(y)| \le L\|x - y\|, \forall\, x, y \in N.$$

This abstract theory has been developed by Chang [6].

Definition 1. For given $u, z \in X$, the generalized directional derivative of the functional I at u in the direction z is defined by

$$I^0(u; z) = \limsup_{k \to 0, t \to 0} \frac{1}{t}[I(u + k + tz) - I(u + k)].$$

Definition 2. The generalized gradient of I at u, denoted $\partial I(u)$, is defined to be the subdifferential of the convex function $I^0(u;z)$ at $z=0$, that is,

$$w \in \partial I(u) \subset X^* \Longleftrightarrow \langle w, z\rangle \leq I^0(u;z), \forall\, z \in X.$$

Definition 3. $u \in X$ is a critical point of the locally Lipschitz functional I if $0 \in \partial I(u)$.

Definition 4. I is said to satisfy nonsmooth Palais-Smale condition (nonsmooth (PS) condition for short), if any sequence $\{u_n\}$ such that

$$\begin{aligned}&I(u_n) \to c_0 \in \mathbf{R},\\ &I^0(u_n, v-u_n) \geq -\epsilon_n\|v-u_n\|, \text{ for all } v \in H,\\ &\text{where } \epsilon_n \to 0^+.\end{aligned}$$

has a strongly convergent subsequence.

Now we state the three critical points theorem for nondifferentiable functional, which plays an important role in proving the main results. It derives from [3], see also [4].

Lemma 1. ([3],[4]) Let X be a separable and reflexive real Banach space, and let $\Phi, J : X \to \mathbf{R}$ be two locally Lipschitz functionals. Assume that there exists $u_0 \in X$ such that $\Phi(u_0) = J(u_0) = 0$ and $\Phi(u) \geq 0$ for every $u \in X$, and that there exist $u_1 \in X$ and $r > 0$ such that

(i) $r < \Phi(u_1)$;

(ii) $\sup_{\Phi(u)<r} J(u) < r\frac{J(u_1)}{\Phi(u_1)}$.

Furthermore, assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous satisfying nonsmooth (PS) condition and

(iii) $\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda J(u)) = +\infty$ for every $\lambda \in [0, \bar{a}]$, where

$$\bar{a} = \frac{hr}{r\frac{J(u_1)}{\Phi(u_1)} - \sup_{\Phi(u)<r} J(u)} \text{ with } h > 1.$$

Then there exists an open interval $\Lambda \subseteq [0, \bar{a}]$ and a positive real number σ such that, for every $\lambda \in \Lambda$, the functional $\Phi - \lambda J$ admits at least three critical points whose norms are less than σ.

III. Proof of main results

Let E be the class of the functions $u : [0, T+1] \to \mathbf{R}$ such that $u(0) = u(T+1) = 0$. Equipped with the usual inner product and induced norm

$$(u,v) = \sum_{t=1}^{T}(u(t), v(t)), \|u\| = \left(\sum_{t=1}^{T} u^2(t)\right)^{1/2},$$

E is a T-dimensional Hilbert space.

Define the functional I on E as

$$\begin{aligned}I(u) &= \frac{1}{2}\sum_{t=1}^{T+1}[(\triangle u(t-1))^2 - 2G(u(t))]\\ &= \frac{1}{2}u^T Au - \sum_{t=1}^{T} G(u(t)) = \Phi(u) - J(u),\end{aligned}$$

where $u = \{u(1), u(2), \cdots, u(T)\}$, $G(x) = \int_0^x g(s)ds$, $\Phi(u) = \frac{1}{2}u^T Au$, $J(u) = \sum_{t=1}^{T} G(u(t))$ and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 2 & -1\\ 0 & 0 & 0 & \cdots & -1 & 2\end{pmatrix}_{T\times T}.$$

Clearly $J(u)$ is a locally Lipschitz functional and I defines a locally Lipschitz functional on E. Simple computation shows that

$$\frac{\partial}{\partial u(t)}\Phi(u) = 2u(t) - u(t+1) - u(t-1) = -\triangle^2 u(t-1).$$

By Theorem 2.1 of [6], $\partial J(u) \subset [\underline{g}(u(t)), \overline{g}(u(t))]$ can be obtained, where $\underline{g}(s) = \min[g(s+0), g(s-0)], \overline{g}(s) = \max[g(s+0), g(s-0)]$. Then the critical points of the functional I are solutions of the inclusion

$$-\triangle^2 u(t-1) \in [\underline{g}(u(t)), \overline{g}(u(t))], t \in \mathbf{Z}(1,T).$$

Remark 1. It is easy to verify that $\underline{g}(s) = \overline{g}(s) = f(s)$ for $s > 0$, $\underline{g}(s) = \overline{g}(s) = 0$ for $s < 0$. Then $\underline{g}(0) = f(0), \overline{g}(0) = 0$.

Remark 2. If $u > 0$, then the above inclusion becomes

$$-\triangle^2 u(t-1) = \lambda f(u(t)), t \in \mathbf{Z}(1,T).$$

It is clear that A is a positive definite matrix. Let $\lambda_T > 0, \lambda_1 > 0$ be the largest and smallest eigenvalues of matrix A respectively. Denote by $u^- = \max\{-u, 0\}$. Let $Y_1 = \{t \in \mathbf{Z}(1,T)|u(t) \leq 0\}, Y_2 = \{t \in \mathbf{Z}(1,T)|u(t) > 0\}$. Notice that $u^-(t) = 0$ for $t \in Y_2$ and $g(u(t)) = 0$ for $t \in Y_1$. Then

$$\sum_{t=1}^{T} g(u(t))u^-(t) = \sum_{t\in Y_1} g(u(t))u^-(t) + \sum_{t\in Y_2} g(u(t))u^-(t) = 0.$$

Lemma 2. If u is a solution of (2), then $u \geq 0$. Moreover, either $u > 0$, or $u = 0$.

Proof. It is not difficult to see that $(\triangle u^-(t) + \triangle u(t))\triangle u^-(t) \leq 0$ for $t \in \mathbf{Z}(0,T)$. In fact, no matter that $\triangle u(t) \geq 0$ or $\triangle u(t) < 0$, the former inequality holds. Hence $\triangle u^-(t)\cdot \triangle u(t) \leq -(\triangle u^-(t))^2$.

If u is a solution of (2), then we have

$$\begin{aligned}0 &= \sum_{t=1}^{T}[\triangle^2 u(t-1) + \lambda g(u(t))u^-(t)]\\ &= -\sum_{t=1}^{T+1}\triangle u(t-1)\triangle u^-(t-1) + \lambda\sum_{t=1}^{T} g(u(t))u^-(t)\\ &\geq \sum_{t=1}^{T+1}(\triangle u^-(t-1))^2 = (u^-)^T Au^- \geq \lambda_1|u^-\|^2.\end{aligned}$$

So $u^- = 0$. Hence $u \geq 0$. If $u(t) = 0$, then

$$u(t+1) + u(t-1) = \triangle^2 u(t-1) = -\lambda g(u(t)) = -\lambda g(0) = 0.$$

Therefore $u(t+1) = u(t-1) = 0$. It follows that $u = 0$ everywhere.

The following are the main results.

Theorem 1. Suppose that

(1) There exists a constant $\beta > 0$ such that if $u \in (0, \beta)$, then $g(u) < 0$, $g(\beta) = 0$; if $u \in (\beta, +\infty)$, then $g(u) > 0$;

(2) There are two constants $a, \gamma > 0$ which satisfy $\gamma < 2$ and

$$\tilde{G}(u) \le a(1 + |u|^{\gamma}),\ \forall u \in \mathbb{R};$$

(3) There exist two constants $c, d > 0$ satisfying $c < Nd$ with $\tilde{G}(d) > 0$ and

$$\frac{\sum_{t=1}^{N} \max_{|u(t)| \le c} \tilde{G}(u(t))}{c^2} < \frac{\mu_1}{\mu_N} \frac{\tilde{G}(d)}{Nd^2}.$$

Then for every $h > 1$, there exists an interval $\Lambda_1 \subseteq [0, \bar{a}]$, where

$$\bar{a} = \frac{\frac{\mu_1 hc^2}{2}}{\frac{\mu_1}{\mu_N}\frac{\tilde{G}(d)}{Nd^2} - \frac{\sum_{t=1}^{N} \max_{|u(t)| \le c} \tilde{G}(u(t))}{c^2}},$$

and positive real number σ_1 such that for any $\mu \in \Lambda_1$, (1) has at least two positive solutions on E, whose norms are less than σ_1.

Proof. Clearly $\Phi(u)$ is locally Lipschitz and weakly sequentially lower semicontinuous. Since E is finite-dimensional and f satisfies condition (ii), the assertion remains true regarding J too. Let $u_0(t) = 0, t \in \mathbb{Z}(1, N)$. Then $\Phi(u_0) = J(u_0) = 0$. Furthermore, let $u_1(t) = d, t \in \mathbb{Z}(1, N)$, then $\|u_1\|^2 = Nd^2$. Let $r = \frac{\mu_1}{2N} c^2$, by $c < Nd$,

$$\Phi(u_1) = \frac{1}{2} u_1^{\tau} A u_1 \ge \frac{1}{2}\mu_1 \|u_1\|^2 = \frac{1}{2}\mu_1 N d^2 > r,$$

condition (i) of Lemma 1 is satisfied.

On the other hand, $\Phi(u_1) = \frac{1}{2} u_1^{\tau} A u_1 \le \frac{1}{2}\mu_N \|u_1\|^2$ and

$$\frac{J(u_1)}{\Phi(u_1)} = \frac{N\tilde{G}(d)}{\frac{1}{2}u_1^{\tau} A u_1} \ge \frac{N\tilde{G}(d)}{\frac{1}{2}\mu_N \|u_1\|^2} = \frac{N\tilde{G}(d)}{\frac{1}{2}\mu_N N d^2} = \frac{\tilde{G}(d)}{\frac{1}{2}\mu_N d^2}.$$

Considering $u \in E$,

$$\max_{t \in \mathbf{Z}(1,N)} |u(t)| \le \sqrt{N} \|u\|.$$

By $\Phi(u) \le r$,

$$\frac{1}{2}\mu_1 \|u\|^2 \le \frac{1}{2} u^{\tau} A u = \Phi(u) \le r.$$

Therefore

$$\max_{t \in \mathbf{Z}(1,N)} |u(t)| \le \sqrt{\frac{2rN}{\mu_1}} = c.$$

Then by condition (3),

$$\sup_{\Phi(u) \le r} J(u) \le \sum_{t=1}^{N} \max_{|u(t)| \le \sqrt{\frac{2rN}{\mu_1}}} \tilde{G}(u(t)) < r \frac{J(u_1)}{\Phi(u_1)}.$$

Hence condition (ii) of Lemma 1 is satisfied.

By condition (2), for any $\mu \ge 0$,

$$\lim_{\|u\| \to \infty} (\Phi(u) - \mu J(u)) = +\infty, \tag{3}$$

Hence condition (iii) of Lemma 1 is satisfied.

For given $\mu > 0$, it is sure that $\Phi(u) - \mu J(u)$ satisfies nonsmooth (PS) condition. In fact, let $\{u_n\} \subset E$ be such that when $n \to \infty$,

$$\Phi(u_n) - \mu J(u_n) \to c_0 \in \mathbb{R},$$
$$(\Phi - \mu J)^0(u_n, u - u_n) \ge -\epsilon_n \|u - u_n\|, \text{for any } u \in E,$$
where $\epsilon_n \to 0^+$.

By (3), the boundedness $\{u_n\}$ is obtained. Since H is finite-dimensional, $\{u_n\}$ has a strongly convergent subsequence.

Therefore, noticing that

$$\bar{\mu} = \frac{\frac{\mu_1 hc^2}{2}}{\frac{\mu_1}{\mu_N}\frac{\tilde{G}(d)}{Nd^2} - \frac{\sum_{t=1}^{N} \max_{|u(t)| \le c} \tilde{G}(u(t))}{c^2}}, \text{ where } h > 1,$$

by Lemma 1, there exists an open interval $\Lambda_1 \subseteq [0, \bar{\mu}]$ and positive real number σ_1 such that for any $\mu \in \Lambda_1$, (2) has at least three solutions on E, whose norms are less than σ_1. By Lemma 2, (1) has at least two positive solutions.

Corollary 1. Suppose that (1), (2) hold and

(4) There are two constants $c, d > 0$ satisfying $c < Nd$ with $\tilde{G}(d) > 0$ and

$$\frac{\max_{|u| \le c} \tilde{G}(u)}{c^2} < \frac{\mu_1}{\mu_N} \frac{\tilde{G}(d)}{(Nd)^2}.$$

Then for every $h > 1$, there exists open interval $\Lambda_2 \subseteq [0, \tilde{\mu}]$, where

$$\tilde{\mu} = \frac{\frac{\mu_1 hc^2}{2N}}{\frac{\mu_1}{\mu_N}\frac{\tilde{G}(d)}{(Nd)^2} - \frac{\max_{|u| \le c} \tilde{G}(u)}{c^2}},$$

and positive real number σ_2 such that for any $\mu \in \Lambda_2$, (1) has at least two positive solutions on E, whose norms are less than σ_2.

Proof. Let u_0, u_1, r be the same as Theorem 1. To prove Corollary 1, it suffices to prove the condition (ii) of Lemma 1 is satisfied. However, by (4) the following result

$$\sup_{\Phi(u) \le r} J(u) \le N \max_{|u(t)| \le \sqrt{\frac{2rN}{\mu_1}}} \tilde{G}(u(t)) < r \frac{J(u_1)}{\Phi(u_1)}$$

holds. Therefore, there exist $\tilde{\mu}, \Lambda_2$ and σ_2 such that Corollary 1 holds.

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References

[1] R.P. Agarwal, *Difference equations and inequalities, in: Monographs and Textbooks in Pure and Applied Mathematics*, Vol. 228, Marcel Dekker Inc. New York, 2000.

[2] Gabriele Bonannoa, Pasquale Candito, "Nonlinear difference equations investigated via critical point methods", Nonlinear Anal., Vol. 70, pp. 3180-3186, May 2009.

[3] G. Bonanno, P. Candito, "On a class of nonlinear variational-hemivariational inequalities", Appl. Anal. Vol.83, pp. 1229-1244, Dec. 2004.

[4] G. Bonanno, N. Giovannelli, "An eigenvalue Dirichlet problem involving the p-Laplican with discontinuous nonlinearities", J. Math. Anal. Appl. Vol.308, pp. 596-604, Aug. 2005.

[5] A. Castro, R.Shivaji, "Nonnegative solutions for a class of nonpositone problems", Proc. Roy. Soc. Edin. Vol. 108A, pp. 291-302, 1988.

[6] K.C. Chang, "Variational methods for non-differential functional and their applications to PDE", J. Math. Anal. Appl. Vol. 80, pp. 102-129, Mar. 1981.

[7] A. Castro, C. Maya, R.Shivaji, “Nonlinear eigenvalue problems with semipositone structure”, Electronic J. Diff. Eqns. Conference 05, pp. 33-49, 2000.
[8] D.S. Cohn, H.B. Keller, “Some positone problems suggested by nonlinear heat generation”, J. Math. Mech. Vol. 16, pp. 1361-1376, June 1967.
[9] D.G. Costa, H. Tehrani, J. Yang, “On a variational approach to existence and multiplicity results for semipositone problems”, Electronic J. Diff. Eqns. vol. 2000, pp. 1-10, 2000.
[10] B. Ricceri, “On a three critical points theorem”, Arch. Math. (Basel) Vol. 75, pp. 220-226, 2000.
[11] Biagio Ricceri, “A three critical points theorem revisited”, Nonlinear Anal. Vol. 70, pp. 3084-3089, May 2009.
[12] G. Zhang, S. Liu, On a class of semipositone discrete boundary value problems, J. Math. Anal. Appl. Vol. 325, pp175-182, Jan. 2007.