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Positive Solutions for Discrete Third-order Three-point Boundary Value Problem

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***Abstract*—**In this paper, the existence of multiple positive solutions for a class of third-order three-point discrete boundary value problem is studied by applying algebraic topology method.

***Keywords*—**Positive solutions, Discrete boundary value problem, Third-order, Three-point, Algebraic topology

I. INTRODUCTION

RECENTLY, positive solution for discrete second-order multi-point boundary value problems was widely investigated, see [5,7,8,10,12,13] and references therein. Usually, these results were obtained by using different fixed point theorems. However, to the author's best knowledge, there are few papers on positive solutions for discrete higher order multi-point boundary value problems, see [1-4,6,9,11]. In this paper, multiple positive solutions for discrete third-order three-point boundary value problem will be studied. Our result is based on algebraic topology method[10]. For convenience, we introduce our idea. To find the positive solutions of the discrete third-order three-point boundary value problem, we may turn this problem to find the solutions of a difference mapping. The difference mapping has the following property: whether there exists original of an element under a mapping is equivalent to whether the mapping image set contains the element and is equivalent to whether there exist the solution for the mapping equation whose mapping image is the element. If there are two mappings, one's image set is larger than the other's image set. Under the proper condition we can ascertain that the image set of the difference mapping of these two mappings contains a set, original of the set under the difference mapping is not empty, there exist solutions for the difference mapping equation such that the element of the set is mapping image. Theorem 1.1 is the result of such an idea which is proved via algebraic topology method.

In this paper, Z, R denote the set of all integers and real numbers. For convenience, for any integers a, b, we will define $Z[a,b] = \{a, a+1, \cdots, b\}$ when $a \le b$.

Consider the following discrete boundary value problem

$$\begin{cases} \Delta^3 x(t) = f(t, x(t+1)), t \in Z[t_1, t_3 - 1], \\ x(t_1) = 0, \alpha x(t_2) - \beta \Delta x(t_2) = 0, \gamma x(t_3) - \delta \Delta^2 x(t_3) = 0, \end{cases} \tag{1.1}$$

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where Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$. A sequence $\{u(t)\}_{t=t_1}^{t_3+2}$ is called a positive solution of (1.1) if $\{u(t)\}_{t=t_1}^{t_3+2}$ satisfies (1.1) and $u(t) > 0$ for $t \in Z[t_1+1, t_3-1]$. Firstly we assume that

(H1) $\alpha, \gamma \ge 0, \beta, \delta > 0$;

(H2) $k = \alpha\gamma(t_1 - t_2)(t_2 - t_3)(t_1 - t_3)$
$+\beta\gamma(t_3 - t_1)(t_1 + t_3 - 2t_2 - 1)$
$+2\delta[\alpha(t_1 - t_2) + \beta] > 0$;

(H3) $t_1 < t_2 < t_3$ are distinct integers with $t_2 - t_1 - 1 > t_3 - t_2$;

(H4) $f : Z[t_1, t_3 - 1] \times R \to R$ is continuous with respect to x and $f(t, x) \ge 0$ for $x \in R^+$, where R^+ denotes the set of nonnegative real numbers.

In the following, Theorem 1.1 plays an important role in proving our result, which is derived from [6].

Theorem 1.1. Suppose that Ω_1 is a contractible set of R^n, Ω_2 is a convex set of R^l, Ω_3, Ω_4 are sets of R^m, R^k respectively, Ω is a set of Ω_1, the sets of (positive, non-negative) continuous mapping from Ω_1 to Ω_3 and Ω_4 respectively are denoted by G_1, G_2; $a \in G_1, A, B \subseteq G_2$, $F_1(x, y, z_\alpha(|\alpha| \le n))$ and $F_2(x, y, z_\alpha(|\alpha| \le n))$ are continuous maps from

$$\Omega_1 \times \Omega \times \prod_{|\alpha| \le n} \Omega_2$$

to Ω_3, $H_1(x, y_\alpha(|\alpha| \le n), z)$, $H_2(x, y_\alpha(|\alpha| \le n), z)$ are mappings from

$$\Omega_1 \times \prod_{|\alpha| \le n} \Omega_2 \times \Omega_3$$

to Ω_4, $H_1(x, y_\alpha(|\alpha| \le n), z) - H_2(x, y_\alpha(|\alpha| \le n), z)$ is a mapping from

$$\Omega_1 \times \prod_{|\alpha| \le n} \Omega_2 \times \Omega_3$$

to Ω_4. For any $x \in \Omega_1, \varphi \in G_1$,

$$S(x,\varphi)=$$
$$H_1[x,D^\alpha\varphi(x)(|\alpha|\le n),\int_\Omega F_1(x,t,D^\alpha\varphi(x)(|\alpha|\le n))dt],$$
$$T(x,\varphi)=$$
$$H_2[x,D^\alpha\varphi(x)(|\alpha|\le n),\int_\Omega F_2(x,t,D^\alpha\varphi(x)(|\alpha|\le n))dt].$$
if
$$T[\Omega_1\times G_1]\cup\{0\}\subseteq B,\ S[\Omega_1\times G_1]\supseteq A\supseteq\{a\}+B,$$
$$A\cap\partial S[\Omega_1\times G_1]=\phi,$$
for any contractible set P satisfying
$$P\subseteq S[\Omega_1\times G_1]-B\subseteq G_2,$$
G_2/P is a nonempty set, then differential integral equation
$$H_1[x,D^\alpha\varphi(x)(|\alpha|\le n),\int_\Omega F_1(x,t,D^\alpha\varphi(x)(|\alpha|\le n))dt]$$
$$=a(t)+$$
$$H_2[x,D^\alpha\varphi(x)(|\alpha|\le n),\int_\Omega F_2(x,t,D^\alpha\varphi(x)(|\alpha|\le n))dt]$$
has (positive,nonnegative) continuous solution $\varphi\in G_1$.

II. Main Result

In order to prove our main result, firstly we give the Green's function for the homogenous boundary value problem
$$\begin{cases}\Delta^3x(t)=0,t\in Z[t_1,t_3-1],\\ x(t_1)=0,\alpha x(t_2)-\beta\Delta x(t_2)=0,\gamma x(t_3)-\delta\Delta^2x(t_3)=0,\end{cases}\tag{2.1}$$
where $\alpha,\gamma,\beta,\delta$ satisfy (H1).

The following lemmas are due to [6].

Lemma 2.1. Assume that (H2) holds. Then the Green's function for the homogeneous problem (2.1) is given by
$$G(t,s)=\begin{cases} s\in Z[t_1,t_2-1]:\begin{cases}u_1(t,s),t\le s+1,\\ v_1(t,s),t\ge s+1,\end{cases}\\ s\in Z[t_2-1,t_3-1]:\begin{cases}u_2(t,s),t\le s+1,\\ v_2(t,s),t\ge s+1\end{cases}\end{cases}$$
for $t\in Z[t_1,t_3+2],s\in Z[t_1,t_3-1]$, where
$$u_1(t,s)=$$
$$\frac{t-t_1}{2}(2s-t-t_1+3)+\frac{(t-t_1)(t_1-s-1)(t_1-s-2)}{2k}\times$$
$$\{\alpha[\alpha(t_2-t_3)(t+t_1-t_2-t_3)+2\delta]-\beta\gamma(t+t_1-2t_2-1)\},$$
$$v_1(t,s)=$$
$$\frac{(t_1-s-1)(t_1-s-2)}{2k}\times$$
$$\{k+(t-t_1)[\alpha(\gamma(t_2-t_3)(t+t_1-t_2-t_3)+2\delta)$$
$$-\beta\gamma(t+t_1-2t_2-1)]\},$$
$$u_2(t,s)$$
$$=\frac{t-t_1}{2k}[\alpha(t_2-t_1)(t-t_2)+\beta(2t_2-t-t_1+1)]\times$$
$$[2\delta+\gamma(t_3-s-1)(t_3-s-2)],$$
$$v_2(t,s)=\frac{1}{2}(t-s-1)(t-s-2)+u_2(t,s).$$

Lemma 2.2. Assume that (H1)-- (H3) hold. Then the Green's function $G(t,s)$ is positive on $Z[t_1,t_3+2]$ $\times Z[t_1,t_3-1]$.

Let
$$0<M:=\max G(t,s),0<m:=\min G(t,s)$$
for $t\in Z[t_1+1,t_3+2]$, $s\in Z[t_1,t_3-1]$. Let
$$B=\{x:Z[t_1,t_3+2]\to R:x(t_1)=0,$$
$$\alpha x(t_2)-\beta\Delta x(t_2)=0,\gamma x(t_3)-\beta\Delta^2x(t_3)=0\}$$
with the norm $\|x\|=\max\{|x(t)|,t\in Z[t_1+1,t_3+2]\}$,

and cone P in B given by
$$P=\left\{x\in B:x(t)\ge 0,t\in Z[t_1+1,t_3+2],\min_{t\in Z[t_1+1,t_3+2]}x(t)\ge\frac{m}{M}\|x\|\right\}$$
By Lemma 2.2, solving the BVP (1.1) is reduced to solving the following summation equation in P:
$$x(t)=\sum_{s=t_1}^{t_3-1}G(t,s)f(s,x(s+1)),t\in Z[t_1+1,t_3+2],$$
and consequently, it is reduced to finding fixed points of the operator $\Psi:B\to B$ defined by
$$\Psi x(t)=\sum_{s=t_1}^{t_3-1}G(t,s)f(s,x(s+1)),t\in Z[t_1+1,t_3+2].\tag{2.2}$$

An operator acting on a Banach space is said to completely continuous if it is continuous and maps bounded sets to relatively compact sets. From the continuity of $f(t,x)$ in x and $G(t,s)$, it follows that the operator ψ defined by (2.2) is completely continuous in B.

Lemma 2.3. Under the hypotheses (H1)—(H4), the operator ψ leaves the cone P invariant, i.e., $\psi(P)\subset P$.

The following theorem is our main result.

Theorem 2.1. Suppose that (H1)—(H4) hold. Moreover, $a_{i+1}>mb_i>Ma_i>0$ with $M>m>1$ and for $a_i<u_i<b_i$, we have
$$\frac{1}{m(t_3-t_1)}a_i<f(t,u_i)<\frac{1}{M(t_3-t_1)}b_i.$$

Then BVP (1.1) has infinitely many solutions satisfying $a_i \le u_i^* \le b_i$.

Proof. Let $\Omega = Z[t_1, t_3 - 1]$, the integral measure on Ω is $\sum_{k\in\Omega}\delta_k$, where δ_k is Dirac function,

$$\Omega_1 = Z[t_1, t_3+2],\ \Omega_2 = [a_i, b_i],\ \Omega_3 = \Omega_4 = R,$$

$$F_1(x,t,u) = 0,\ F_2(x,t,u) = G(t,s)f(u),$$

$$H_1(x,u,z) = u - \frac{a_i+b_i}{2},\ H_2(x,u,z) = z - \frac{a_i+b_i}{2},$$

$$G_1 = C(\Omega_1, \Omega_2),\ G_2 = C(\Omega_1, \Omega_4),$$

$$A = B = C\left(\Omega_1, \left(-\frac{b_i-a_i}{2}, \frac{b_i-a_i}{2}\right)\right),\ a = 0.$$

For any $u \in G_1$, when $k \in \Omega_1$, we have

$$\begin{aligned} a_i &< m\sum_{s=t_1}^{t_3-1} f(s, u(s+1)) \\ &\le \Psi u(k) \\ &\le M\sum_{s=t_1}^{t_3-1} f(s, u(s+1)) < b_i, \end{aligned}$$

therefore

$$\left|\Psi u(k) - \frac{a_i+b_i}{2}\right| < \frac{b_i-a_i}{2}.$$

Under the norm $\|\cdot\|$, we have

$$T[\Omega_1 \times G_1] \cup \{0\} \subseteq B,\ S[\Omega_1 \times G_1] = A = \{a\} + B,$$

$$A \cap \partial S[\Omega_1 \times G_1] = \phi,$$

for any contractible set P satisfying

$$P \subseteq S[\Omega_1 \times G_1] - B \subseteq G_2,$$

G_2 / P is a nonempty set, by Theorem 1.1, difference map

$$\Psi u(k) - \frac{a_i+b_i}{2} = u(k) - \frac{a_i+b_i}{2}$$

has a fixed point u_i^* satisfying $a_i \le u_i^* \le b_i$. Hence BVP (1.1) has also the solution u_i^* satisfying $a_i \le u_i^* \le b_i$.

Remark 2.1. Our method can be summarized as follows: firstly we turn the equation to integral form by using Green's function, then apply Theorem 1.1 to obtain the solutions. If the Green's function is positive, then we can prove the existence of positive solutions. Therefore, our method can be applied to find positive solutions of other higher-order multi-point boundary value problem.

Example 2.1. Let $t_1 = 0, t_2 = 3, t_3 = 4, \alpha = \frac{1}{3}, \beta = 2, \gamma = \frac{1}{4}$ and $\delta = \frac{3}{2}$, then $k = 1$. The corresponding Green's function for the homogenous problem (2.1) is given by

$$G(t,s) = \begin{cases} s \in Z[0,2]: \begin{cases} u_1(t,s), t \le s+1, \\ v_1(t,s), t \ge s+1, \end{cases} \\ s \in Z[2,3]: \begin{cases} u_2(t,s), t \le s+1, \\ v_2(t,s), t \ge s+1, \end{cases} \end{cases}$$

where

$$u_1(t,s) = \frac{t}{12}\left[13s^2 - ts^2 + 51s - 8t - 3st + 44\right],$$

$$v_1(t,s) = u_1(t,s) + \frac{1}{2}(t_1 - s - 1)(t_1 - s - 2),$$

$$u_2(t,s) = \frac{t}{28}\left[11s^2 - ts^2 - 55s - 48t + 5st + 528\right],$$

$$v_2(t,s) = u_2(t,s) + \frac{1}{2}(t_1 - s - 1)(t_1 - s - 2).$$

Thus, $m = \min G(t,s) = 3, M = \max G(t,s) = 48$ for $t \in Z[1,6], s \in Z[0,3]$.

Let $a_i = 60^{2i}, b_i = 60^{2i+1}, i \in N$,

$$f(t,x) = \begin{cases} \frac{1}{12}a_i, x \in (-\infty, a_i], \\ \frac{1}{12}\frac{b_i - x}{b_i - a_i}a_i + \frac{1}{192}\left(1 - \frac{b_i - x}{b_i - a_i}\right)b_i, x \in [a_i, b_i], \\ \frac{1}{12}\frac{b_i - x}{b_i - a_i}b_i + \frac{1}{192}\left(1 - \frac{b_i - x}{b_i - a_i}\right)a_{i+1}, x \in [b_i, a_{i+1}]. \end{cases}$$

Obviously all the conditions of Theorem 2.1 are satisfied. Hence the result of Theorem 2.1 is true.

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