

# Planning rigid body motions and optimal control problem on Lie group $SO(2, 1)$

Nemat Abazari, Ilgin Sager

**Abstract**—In this paper smooth trajectories are computed in the Lie group  $SO(2,1)$  as a motion planning problem by assigning a Frenet frame to the rigid body system to optimize the cost function of the elastic energy which is spent to track a timelike curve in Minkowski space. A method is proposed to solve a motion planning problem that minimizes the integral of the Lorentz inner product of Darboux vector of a timelike curve. This method uses the coordinate free Maximum Principle of Optimal control and results in the theory of integrable Hamiltonian systems. The presence of several conserved quantities inherent in these Hamiltonian systems aids in the explicit computation of the rigid body motions.

**Keywords**—Optimal control, Hamiltonian vector field, Darboux vector, Maximum Principle, Lie group, Rigid body motion, Lorentz metric

## I. INTRODUCTION

**A**N optimal control problem is used for rigid body motions and formulation on Lie group  $SO(2,1)$  where the cost function to be minimized is equal to the integral of the Lorentz inner product of Darboux vector of a timelike curve. This problem is analogous to the elastic problem in [1], [4]. The coordinate free maximum principle [2], [3] is applied to solve this problem. In [4] the author applied an integrable case where the necessary conditions for optimality can be expressed analytically and the corresponding optimal motions are expressed in a coordinate free manner. These optimal motions are showed to trace helical paths. In this study, the optimal control problem formulation is considered as the general theory of optimal control for the motion planning application, framed curves and left-invariant Hamiltonian systems are applied to this particular setting. Frenet frame of the curve is applied to the Lorentz-Minkowski space to solve the problem and a particular set of curves is analyzed that satisfies these necessary conditions and provides analytic solutions for the corresponding optimal motions. An application of the Maximum Principle to this problem results in a system of first order differential equations that yields coordinate free necessary conditions for optimality. This system minimizes the cost function of elastic energy which is spent to track a timelike curve in Minkowski space.

## II. FRENET FRAME

The Lorentz-Minkowski space is the metric space  $E^3 = (R^3, \langle, \rangle)$  where the metric is given by

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$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 \quad (1)$$

The metric  $\langle, \rangle$  is called as Lorentzian metric.

Let  $H^2$  denote the hyperboloid  $x_1^2 - (x_2^2 + x_3^2) = 1, x_1 > 0$

The isometry group for a hyperbolic plane  $H^2$  is denoted by  $SO(2, 1)$ . Recall that  $SO(2, 1)$  is the group that leaves the bilinear form  $\langle, \rangle$  in  $E^3$  invariant.

$$\langle Ax, y \rangle + \langle x, Ay \rangle = 0 \quad (2)$$

is satisfied for any  $3 \times 3$  matrix  $A$  on the Lie algebra  $L$  of  $SO(2, 1)$ .

It is verified that  $L$  is equal to the space of matrices

$$A = \begin{bmatrix} 0 & a_1 & a_2 \\ a_1 & 0 & -a_2 \\ a_2 & a_3 & 0 \end{bmatrix} \quad (3)$$

**Definition** A vector  $v \in E^3$  is called

$$1. \text{Spacelike if } \langle v, v \rangle > 0 \text{ or } v = 0 \quad (4)$$

$$2. \text{Timelike if } \langle v, v \rangle < 0$$

$$3. \text{Lightlike if } \langle v, v \rangle = 0 \text{ and } v \neq 0$$

**Definition** For a curve  $\alpha$  in  $E^3$ ,  $\alpha$  is spacelike (resp. timelike, lightlike) at  $t$  if  $\alpha'(t)$  is a spacelike (resp. timelike, lightlike) vector. If it is for any  $t \in I$ , the curve  $\alpha$  is called spacelike (resp. timelike, lightlike).

In this paper it is supposed that  $\alpha$  is a timelike curve parametrized by length-arc  $s$ . In this case  $T(s) = \alpha'(s)$  as the unitary tangent vector at  $s$  is a timelike vector and moreover  $T(s) \neq 0$  is the spacelike vector independent with  $T(s)$ .

The curvature of  $\alpha$  at  $s$  is defined as  $k_1(s) = |T'(s)|$ . The normal vector  $N(s)$  is defined by

$$N(s) = \frac{T'(s)}{k_1(s)} = \frac{\alpha''(s)}{|\alpha''(s)|} \quad (5)$$

Moreover  $k_1(s) = \langle T'(s), N(s) \rangle$  is the curvature of the curve  $\alpha$ . The binormal vector  $B(s)$  is defined by

$$B(s) = T(s) \times N(s) \quad (6)$$

where  $B(s)$  is unitary and spacelike vector. And  $k_2(s) = \langle N'(s), B(s) \rangle$  is defined as the torsion of the curve  $\alpha$ . For each  $s$ ,  $\{T, N, B\}$  is a unit orthonormal base of  $E^3$  which is called the Frenet trihedron of  $\alpha$ .

By differentiation each one of the vector functions of the frenet trihedron frame  $R = (T | N | B) \in L$  about the curve  $\alpha : I \rightarrow E^3$  described by the following differential equations:

$$\begin{aligned}
\alpha'(t) &= T \\
T' &= k_1 N \\
N' &= k_1 T + k_2 B \\
B' &= -k_2 N
\end{aligned} \quad (7)$$

where  $k_1$  curvature,  $k_2$  torsion of the timelike curve  $\alpha$ , [6].

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (8)$$

These equations form a rotation motion with Darboux vector  $w = -k_2 T - k_1 B$ , [5]. Also momentum rotation vector is expressed as follows:

$$\begin{aligned}
T' &= w \times T \\
N' &= w \times N \\
B' &= w \times B
\end{aligned} \quad (9)$$

Moreover

$$\langle w, w \rangle = k_2^2 \langle T, T \rangle + k_1^2 \langle B, B \rangle = k_1^2 - k_2^2.$$

Since  $T$  is timelike and  $B$  is spacelike unitary vectors then  $\langle T, T \rangle = -1$  and  $\langle B, B \rangle = 1$ .

In this study, this Frenet frame is used to plan rigid body motions by applying the Maximum Principle to optimal control systems defined on the Lie group [2]. An element  $g(t) \in M$  is defined as:

$$g(t) = \begin{pmatrix} 1 & 0 \\ \alpha(t) & R(t) \end{pmatrix} \quad (10)$$

where  $R(t) \in L$ . There is also associated with (7) via the relations

$$\begin{aligned}
[1 \ \alpha(t)]^T &= g(t) \vec{e}_1 \\
[0 \ T]^T &= g(t) \vec{e}_2 \\
[0 \ N]^T &= g(t) \vec{e}_3 \\
[0 \ B]^T &= g(t) \vec{e}_4
\end{aligned} \quad (11)$$

where  $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$  is the standard orthonormal frame in  $E^4$ .

**Proposition 1** The left-invariant differential equation:

$$\frac{dg(t)}{dt} = g(t) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & 0 \\ 1 & k_1 & 0 & -k_2 \\ 0 & 0 & k_2 & 0 \end{pmatrix} \quad (12)$$

where  $g(t) \in M$  is equivalent to the Frenet frame (7).

**Proof:** It follows from differentiating (11) w.r.t to  $t$  that

$$\begin{aligned}
[1 \ \alpha(t)]^T &= \frac{dg(t)}{dt} \vec{e}_1 = g(t) \vec{e}_2 = [0 \ T]^T \\
[0 \ T]^T &= \frac{dg(t)}{dt} \vec{e}_2 = g(t) (k_1 \vec{e}_3) = k_1 [0 \ N]^T \\
[0 \ N]^T &= \frac{dg(t)}{dt} \vec{e}_3 = g(t) (k_1 \vec{e}_2 + k_2 \vec{e}_4) \\
&= k_1 [0 \ T]^T + k_2 [0 \ B]^T \\
[0 \ B]^T &= \frac{dg(t)}{dt} \vec{e}_4 = g(t) (-k_2 \vec{e}_3) = -k_2 [0 \ N]^T
\end{aligned} \quad (13)$$

then equating the L.H.S to the R.H.S yields (7).

The system (12) can be expressed conveniently in coordinate form by defining the following basis for the Lie algebra of  $M$  denoted by  $m$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (14)$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

Using these notations, it follows that (12) can be expressed as:

$$\frac{dg(t)}{dt} = g(t) (B_2 + k_1 A_1 - k_2 A_3) \quad (17)$$

To minimize the elastic energy of the curve, this is equivalent minimizing the function:

$$J = \frac{1}{2} \int \langle w, w \rangle dt = \frac{1}{2} \int (k_1^2 - k_2^2) dt \quad (18)$$

where  $w = -k_2 T - k_1 B$  is the Darboux vector.

The motion  $g(t) \in M$  of the left-invariant differential system (15) which minimizes the expression (16) is computed on a given interval  $[0, T]$  subject to the given boundary conditions  $g(0) = g_0, g(T) = g_T$  on the next section.

### III. HAMILTONIAN LIFT ON $M$

Due to the similarity in between optimal control problem and elastic problem, this optimal control problem is considered as elastic problem and the applicability of Maximum Principle is obvious. The Maximum Principle states that the optimal paths are the projections of the extremal curves onto the base manifold, where the extremal curves are solutions of the certain Hamiltonian systems on the cotangent bundle. For the problem, the manifold is  $M$  and the cotangent bundle is  $T^*M$ . The appropriate pseudo-Hamiltonian on  $T^*M$  is defined as:

$$\begin{aligned}
H(p, u, g) &= p(g(t) B_2) + k_1 p(g(t) A_1) - k_2 p(g(t) A_3) \\
&\quad - p_0 \frac{1}{2} (k_1^2 - k_2^2)
\end{aligned} \quad (19)$$

where  $p(\cdot) : TM \rightarrow \mathbb{R}$ . In this study, the regular extremals where  $p_0 = 1$  (ignoring abnormal extremals where  $p_0 = 0$ ) is carried.

The cotangent bundle  $T^*M$  can be written as the direct product  $M \times m^*$  where  $m^*$  is the dual of the Lie algebra  $m$  of  $M$ .

The original Hamiltonian defined on  $T^*M$  can be expressed as a reduced Hamiltonian on the dual of the Lie algebra  $m^*$ . The linear functions  $M_i = \hat{p}(A_i), p_i = \hat{p}(B_i)$  for  $i = 1, 2, 3$  where  $\hat{p} : m \rightarrow \mathbb{R}$  are the Hamiltonian lifts of left-invariant

vector fields on  $M$ , because  $p(g(t)A_i) = \hat{p}(A_i)$  for any  $P = (g(t), \hat{p})$  and any  $A_i \in m$ . If  $M_i, p_i$  is a collection of linear functions generated by the basis  $A_i, B_i$  in  $m$  then the vector  $(M_1, M_2, M_3, p_1, p_2, p_3)$  is the coordinate vector of  $\hat{p}$  relative to the dual basis  $A_i^*, B_i^*$ . The Hamiltonian (17) becomes

$$H = p_2 + k_1 M_1 - k_2 M_3 - \frac{1}{2} (k_1^2 - k_2^2) \quad (22)$$

It follows from [2] that calculating  $\frac{\partial H}{\partial k_1} = \frac{\partial H}{\partial k_2} = 0$  yields the optimal controls:

$$k_1 = M_1, k_2 = M_3 \quad (23)$$

substituting (19) into (18) gives the optimal Hamiltonian:

$$H = p_2 + \frac{1}{2} (M_1^2 - M_3^2) \quad (24)$$

In addition substituting the expressions (19) into (12) the optimal motions are the solutions  $g(t) \in M$  of the differential equation:

$$\frac{dg(t)}{dt} = g(t) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 1 & M_1 & 0 & -M_3 \\ 0 & 0 & M_3 & 0 \end{pmatrix} \quad (25)$$

To solve the equation (21) for  $g(t) \in M$ , it is necessary to solve the ekstremal curves  $M_1, M_2, M_3$  for a special case.

#### IV. SOLVING THE EXTREMAL CURVES

To compute the corresponding Hamiltonian vector fields from the left-invariant Hamiltonian (20) the Lie bracket table (a) obtained for the basis (14):

$[\cdot, \cdot]$	$A_1$	$A_2$	$A_3$	$B_1$	$B_2$	$B_3$
$A_1$	0	$A_3$	$A_2$	0	$B_3$	$B_2$
$A_2$	$-A_3$	0	$-A_1$	$B_3$	0	$B_1$
$A_3$	$-A_2$	$A_1$	0	$B_2$	$-B_1$	0
$B_1$	0	$-B_3$	$-B_2$	0	0	0
$B_2$	$-B_3$	0	$B_1$	0	0	0
$B_3$	$-B_2$	$-B_1$	0	0	0	0

table (a)

where the Lie Bracket is defined as  $[X, Y] = XY - YX$ .

The time derivatives of  $M_i, p_i$  along the Hamiltonian flow are described by the Poisson bracket given by the equation:

$$\left\{ \hat{p}(\cdot), \hat{p}(\cdot) \right\} = -\hat{p}([\cdot, \cdot]) \quad (26)$$

$$\begin{aligned} M_1' &= \{M_1, H\} = \left\{ M_1, p_2 + \frac{1}{2} (M_1^2 - M_3^2) \right\} \\ &= \{M_1, p_2\} + M_1 \{M_1, M_1\} - M_3 \{M_1, M_3\} \\ &= -p_3 + M_3 M_2 \\ M_2' &= \{M_2, H\} = \left\{ M_2, p_2 + \frac{1}{2} (M_1^2 - M_3^2) \right\} \\ &= \{M_2, p_2\} + M_1 \{M_2, M_1\} - M_3 \{M_2, M_3\} \\ &= 0 \\ M_3' &= \{M_3, H\} = \left\{ M_3, p_2 + \frac{1}{2} (M_1^2 - M_3^2) \right\} \\ &= \{M_3, p_2\} + M_1 \{M_3, M_1\} - M_3 \{M_3, M_3\} \\ &= p_1 + M_1 M_2 \\ p_1' &= \{p_1, H\} = \left\{ p_1, p_2 + \frac{1}{2} (M_1^2 - M_3^2) \right\} \\ &= \{p_1, p_2\} + M_1 \{p_1, M_1\} - M_3 \{p_1, M_3\} \\ &= -p_2 M_3 \\ p_2' &= \{p_2, H\} = \left\{ p_2, p_2 + \frac{1}{2} (M_1^2 - M_3^2) \right\} \\ &= \{p_2, p_2\} + M_1 \{p_2, M_1\} - M_3 \{p_2, M_3\} \\ &= p_1 M_3 + p_3 M_1 \\ p_3' &= \{p_3, H\} = \left\{ p_3, p_2 + \frac{1}{2} (M_1^2 - M_3^2) \right\} \\ &= \{p_3, p_2\} + M_1 \{p_3, M_1\} - M_3 \{p_3, M_3\} \\ &= p_2 M_1 \end{aligned} \quad (27)$$

The Hamiltonian vector fields are

$$\begin{aligned} M_1' &= -p_3 + M_3 M_2 \\ M_2' &= 0 \\ M_3' &= p_1 + M_1 M_2 \\ p_1' &= -p_2 M_3 \\ p_2' &= p_1 M_3 + p_3 M_1 \\ p_3' &= p_2 M_1 \end{aligned} \quad (28)$$

A trivial example of an integrable case of vector fields (24) occurs when  $p_1 = p_2 = p_3 = M_1 = M_2 = M_3 = 0$ . Moreover, for these values  $p_1 = p_2 = p_3 = M_1 = M_2 = M_3$  are constant  $\forall t$  and therefore the system is integrable. Substituting these values into (15)

$$\frac{dg(t)}{dt} = g(t) B_2 \quad (29)$$

This is easily integrated to yield  $\alpha(t) = [t, 0, 0]^T$  with  $R$  equal to a  $3 \times 3$  matrix with zero entries. Therefore, a straight line motion with zero rotation about this line is an optimal rigid body motion. In addition there exists a nontrivial integrable case of the Hamiltonian vector fields (24). This case is considered nontrivial as it gives rise to time-dependent extremal curves. It is observed that  $p_1 = p_2 = p_3 = 0$  is an invariant surface for the Hamiltonian vector fields (24). Explicitly, for  $p_1 = p_2 = p_3 = 0$  the equations (24) degenerate

to:

$$\begin{aligned} M_1' &= M_3 M_2 \\ M_2' &= 0 \\ M_3' &= M_1 M_2 \\ p_1' &= 0 \\ p_2' &= 0 \\ p_3' &= 0 \end{aligned} \quad (30)$$

This implies that  $M_2$  is constant that will be denoted by  $c$ . In addition  $p_1 = p_2 = p_3 = 0 \forall t$ . It follows that the Hamiltonian vector fields (26) are completely integrable. For these particular curves the Hamiltonian (20) reduces to

$$H = M_1^2 - M_3^2 \quad (31)$$

It follows that the differential equations (26) are satisfied that the extremal curves are:

$$\begin{aligned} M_2 &= c \\ M_1 &= r \sinh ct \\ M_3 &= r \cosh ct \end{aligned} \quad (32)$$

To compute the optimal motions corresponding to the extremal curves (28) is not trivial as the elements of the Lie algebra are time-dependent.

#### V. OPTIMAL MOTIONS FOR THE RIGID BODY

The geodesic frame (21) is splitted into its translational and rotational part:

$$\frac{d\alpha(t)}{dt} = R\vec{e}_2 \quad (33)$$

and

$$\frac{dR}{dt} = R \begin{bmatrix} 0 & M_1 & 0 \\ M_1 & 0 & -M_3 \\ 0 & M_3 & 0 \end{bmatrix} \quad (34)$$

where  $R^{-1} = R^T$ . A basis is described for the Lie algebra  $m$  as:

$$\begin{aligned} E_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ E_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned} \quad (35)$$

The quantities

$$RPR^{-1} = \text{constant} \quad (36)$$

and

$$RMR^{-1} + [X, RPR^{-1}] = \text{constant} \quad (37)$$

are conversed for all left-invariant Hamiltonian systems on  $M$  where

$$\begin{aligned} M &= M_1 E_1 + M_2 E_2 + M_3 E_3 \\ P &= p_1 E_1 + p_2 E_2 + p_3 E_3 \\ X &= x_1 E_1 + x_2 E_2 + x_3 E_3 \end{aligned} \quad (38)$$

where  $x_1, x_2, x_3$  are the position coordinates of the vector  $\alpha(t) = [x_1, x_2, x_3]^T$ .

Using these constants of motion (30) is integrated which is stated in the following theorem:

**Theorem 3:**  $R = (T | N | B) \in L$  is the optimal rotation matrix corresponding to the extremals (28) which relates the Frenet frame to a fixed inertial frame where:

$$\begin{aligned} T &= \begin{bmatrix} -\sinh Kt \sinh ct + \frac{c}{K} \cosh Kt \cosh ct \\ \frac{r}{K} \cosh ct \\ -\cosh Kt \sinh ct + \frac{c}{K} \sinh Kt \cosh ct \end{bmatrix} \\ N &= \begin{bmatrix} \frac{r}{K} \cosh Kt \\ \frac{c}{K} \\ \frac{r}{K} \sinh Kt \end{bmatrix} \\ B &= \begin{bmatrix} \sinh Kt \cosh ct - \frac{c}{K} \cosh Kt \sinh ct \\ -\frac{r}{K} \sinh ct \\ \cosh Kt \cosh ct - \frac{c}{K} \sinh Kt \sinh ct \end{bmatrix} \end{aligned} \quad (39)$$

where  $K^2 = c^2 - r^2$  and  $r, c$  are the constant parameters of the curvatures (28).

*Proof* For these particular curves  $p_1 = p_2 = p_3 = 0$  the conversation laws (32) and (33) reduce to:

$$RMR^{-1} = \text{constant}t \quad (40)$$

this constant matrix  $RMR^{-1}$  is then conjugated for a particular solution  $R$  such that:

$$RMR^{-1} = \sqrt{M_1^2 + M_2^2 - M_3^2} E_2 \quad (41)$$

substituting (24) into (37) gives

$$RMR^{-1} = \sqrt{c^2 - r^2} E_2 \quad (42)$$

The constant  $K$  is defined with the equation:  $K^2 = c^2 - r^2$ . Therefore

$$M = KR^{-1} E_2 R \quad (43)$$

is verified. Expressing  $R$  in a convenient coordinate from [12]:

$$R = \exp(\varphi_1 E_2) \exp(\varphi_2 E_1) \exp(\varphi_3 E_2) \quad (44)$$

and substituting (40) into (39) yields:

$$M = K \exp(-\varphi_3 E_2) \exp(-\varphi_2 E_1) E_2 \exp(\varphi_2 E_1) \exp(\varphi_3 E_2) \quad (45)$$

It is shown that:

$$M = K \begin{bmatrix} 0 & -\sinh \varphi_2 \sinh \varphi_3 & \cosh \varphi_2 \\ -\sinh \varphi_2 \sinh \varphi_3 & 0 & -\sinh \varphi_2 \cosh \varphi_3 \\ \cosh \varphi_2 & \sinh \varphi_2 \cosh \varphi_3 & 0 \end{bmatrix} \quad (46)$$

equating  $M$  in (34) to (42) gives:

$$M_1 = -K \sinh \varphi_2 \sinh \varphi_3 \quad (47)$$

$$M_2 = K \cosh \varphi_2$$

$$M_3 = K \sinh \varphi_2 \cosh \varphi_3$$

So it is easily shown that:

$$\begin{aligned} \cosh \varphi_2 &= \frac{M_2}{K} = \frac{c}{K} \\ \sinh \varphi_2 &= \pm \sqrt{\frac{c^2}{K^2} - 1} = \pm \frac{r}{K} \end{aligned} \quad (48)$$

in addition form (43):

$$\tanh \varphi_3 = -\frac{M_1}{M_3} \quad (49)$$

therefore

$$\begin{aligned} \sinh \varphi_3 &= \mp \frac{M_1}{\sqrt{M_3^2 - M_1^2}} = \mp \sinh ct \\ \cosh \varphi_3 &= \pm \frac{M_3}{\sqrt{M_3^2 - M_1^2}} = \cosh ct \end{aligned} \quad (50)$$

in order to obtain an expression for  $\varphi_1$ , it is substituted (40) into (30) yields:

$$\begin{aligned} \frac{dR}{dt} &= \varphi_1 E_2 \exp(\varphi_1 E_2) \exp(\varphi_2 E_1) \exp(\varphi_3 E_2) \\ &+ \varphi_2' \exp(\varphi_1 E_2) E_1 \exp(\varphi_2 E_1) \exp(\varphi_3 E_2) \\ &+ \varphi_3' \exp(\varphi_1 E_2) \exp(\varphi_2 E_1) E_2 \exp(\varphi_3 E_2) \end{aligned} \quad (51)$$

therefore

$$\begin{aligned} R^{-1} \frac{dR}{dt} &= \varphi_1' \exp(-\varphi_3 E_2) \exp(-\varphi_2 E_1) E_2 \exp(\varphi_2 E_1) \exp(\varphi_3 E_2) \\ &+ \varphi_2' \exp(-\varphi_3 E_2) E_1 \exp(\varphi_3 E_2) \\ &+ \varphi_3' E_2 \\ &= \varphi_1' \begin{bmatrix} 0 & -\sinh \varphi_2 \sinh \varphi_3 & \cosh \varphi_2 \\ -\sinh \varphi_2 \sinh \varphi_3 & 0 & -\sinh \varphi_2 \cosh \varphi_3 \\ \cosh \varphi_2 & \sinh \varphi_2 \cosh \varphi_3 & 0 \end{bmatrix} \\ &+ \varphi_2' \begin{bmatrix} 0 & \cosh \varphi_3 & 0 \\ \cosh \varphi_3 & 0 & \sinh \varphi_3 \\ 0 & -\sinh \varphi_3 & 0 \end{bmatrix} \\ &+ \varphi_3' \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & M_1 & 0 \\ M_1 & 0 & -M_3 \\ 0 & M_3 & 0 \end{bmatrix} \end{aligned} \quad (52)$$

which leads to

$$M_1 = -\varphi_1' \sinh \varphi_2 \sinh \varphi_3 + \varphi_2' \cosh \varphi_3 \quad (53)$$

$$M_3 = \varphi_1' \sinh \varphi_2 \cosh \varphi_3 - \varphi_2' \sinh \varphi_3 \quad (54)$$

therefore

$$\varphi_1' = \frac{M_1 \sinh \varphi_3 + M_3 \cosh \varphi_3}{\sinh \varphi_2} \quad (55)$$

substituting (28), (44) and (46) into (50) yields:

$$\varphi_1' = K \quad (56)$$

and integrating with respect to  $t$  yields:

$$\varphi_1 = Kt + \beta \quad (57)$$

where  $\beta$  is a constant of integration and for  $\beta = 0$  yields:

$$\varphi_1 = Kt \quad (58)$$

An other hand from (40) yields :

$$\begin{aligned} T &= \begin{bmatrix} \sinh \varphi_1 \sinh \varphi_3 + \cosh \varphi_1 \cosh \varphi_2 \cosh \varphi_3 \\ \sinh \varphi_2 \cosh \varphi_3 \\ \cosh \varphi_1 \sinh \varphi_3 + \sinh \varphi_1 \cosh \varphi_2 \cosh \varphi_3 \end{bmatrix} \\ N &= \begin{bmatrix} \cosh \varphi_1 \sinh \varphi_2 \\ \cosh \varphi_2 \\ \sinh \varphi_1 \sinh \varphi_2 \end{bmatrix} \\ B &= \begin{bmatrix} \sinh \varphi_1 \cosh \varphi_3 + \cosh \varphi_1 \cosh \varphi_2 \sinh \varphi_3 \\ \sinh \varphi_2 \sinh \varphi_3 \\ \cosh \varphi_1 \cosh \varphi_3 + \sinh \varphi_1 \sinh \varphi_3 \cosh \varphi_2 \end{bmatrix} \end{aligned} \quad (59)$$

substituting (44) , (46) and (52) into (55) yields

$$R = (T \mid N \mid B).$$

*Lemma:* The optimal path  $\alpha(t) \in E^3$  defined by the differential equation (29), with  $M_1 = r \sinh ct$ ,  $M_2 = c$  and  $M_3 = r \cosh ct$  are described by:

$$\begin{aligned} \frac{d\alpha(t)}{dt} &= \frac{1}{K} \begin{bmatrix} r \cosh Kt \\ c \\ r \sinh Kt \end{bmatrix} \\ \alpha(t) &= \frac{1}{K} \int \begin{bmatrix} r \cosh Kt \\ c \\ r \sinh Kt \end{bmatrix} dt = \frac{1}{K^2} [r \sinh Kt, ct, r \cosh Kt]^T \end{aligned}$$

## VI. CONCLUSION

An application of the Maximum Principle to this optimal control problem results in a system of first order differential equations that yields coordinate free necessary conditions for optimality. In this paper, Frenet frame of the curve is applied to the Lorentz-Minkowski space to solve optimal control problem. A particular set of curves is analyzed that satisfies these necessary conditions and provides analytic solutions for the corresponding optimal motions. In this study, the coordinate free maximum principle and the theory of integrable Hamiltonian systems are used to minimize the cost function which is equivalent to integrate the Lorentz inner product of Darboux vector with respect to frenet frame of the curve.

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