

# Performance analysis of a discrete-time $Geo^X/G/1$ queue with single working vacation

Shan Gao, and Zaiming Liu

**Abstract**—This paper treats a discrete-time batch arrival queue with single working vacation. The main purpose of this paper is to present a performance analysis of this system by using the supplementary variable technique. For this purpose, we first analyze the Markov chain underlying the queueing system and obtain its ergodicity condition. Next, we present the stationary distributions of the system length as well as some performance measures at random epochs by using the supplementary variable method. Thirdly, still based on the supplementary variable method we give the probability generating function (PGF) of the number of customers at the beginning of a busy period and give a stochastic decomposition formulae for the PGF of the stationary system length at the departure epochs. Additionally, we investigate the relation between our discrete-time system and its continuous counterpart. Finally, some numerical examples show the influence of the parameters on some crucial performance characteristics of the system.

**Keywords**—Discrete-time queue; Batch arrival; Working vacation; Supplementary variable technique; Stochastic decomposition

## I. INTRODUCTION

**D**URING the last two decades, discrete-time queues with vacations have been well investigated because these systems are more appropriate than their continuous-time counterparts for modeling computer and telecommunication systems since the basic units in these systems are digital such as a machine cycle time, bits and packets, etc. Past work may be divided into two categories: (i) the case of server vacation and (ii) the case of working vacation. In the case of server vacation, the readers are referred to the survey paper by Chang and Choi[2], Doshi [3], Takagi [13], Tian and Zhang [14] and Zhang and Tian [19], Shanthikumar[12] and references therein. In the case of working vacations, Servi and Finn [11] first examined an  $M/M/1$  queue with multiple working vacations (Such model is denoted by  $M/M/1/WV$  queue) and modeled a wavelength division multiplexing (WDM) optical access network using multiple wavelengths which can be reconfigured. The work of [11] is rooted in performance analysis of gateway router in fiber communication networks. On the other hand, working vacation policy has practical application background in optimal design of the system. When the number of customers in the system is relatively few, we set a lower speed operating period in order to economize operating cost and energy consumption. Furthermore, Liu et al. [10] obtained the stochastic decomposition structures of the system indices in the  $M/M/1$  queue with working vacations. Later Kim et al. [4], Wu and Takagi [16], Li et al. [8] generalized results

in Servi and Finn [11] to an  $M/G/1$  queue with working vacations. Yi et al.[17] presented a  $Geo/G/1$  queue with disasters and multiple working vacations and Li et al. [9] analyzed a  $Geo^X/G/1$  queue with working vacations. Wu and Takagi [16] used the methods of embedded Markov Chain and plural function. Embedded Markov chain method and  $M/G/1$ -type matrix analytic technique were mainly used in Li et al. [8] and Li et al. [9]. One thing to be mentioned is that Li et al. [9] applied the connection of the  $M/G/1$ -type matrix analytic approach and the stochastic decomposition method. Based on the supplementary variable method, Yi et al.[17] firstly considered a  $Geo/G/1$  queue with disasters and then used its results to analyze the working vacation queue. Baba [1] and Li et al.[5] respectively discussed a continuous-time  $GI/M/1$  and a discrete-time  $GI/Geo/1$  queue with working vacations. Recently, Li and Tian [6] analyzed a  $GI/Geo/1$  queue with working vacations and introduced a new policy: vacation interruption. Under such a policy, the server can come back to the normal working level no matter whether the vacation ends. They obtained the steady-state distributions for the number of customers in the system at arrival epochs and waiting time for an arbitrary customer using matrix-geometric solution method. Subsequently, its continuous counterpart, the  $GI/M/1$  queue with working vacations and vacation interruption, was studied by Li et al.[7]. Lately, Zhang and Hou [18] discussed an  $M/G/1$  queue with multiple working vacations and vacation interruption. To the best of our knowledge, most of the existing literatures focus on multiple working vacation queues and there are no works on discrete-time batch arrival queueing systems with single working vacation. In order to fill the existing gap in the literature about the working vacations in discrete-time systems, this article studies a  $Geo^X/G/1$  queue with single working vacation which is denoted by  $Geo^X/G/1/SWV$ .

The rest of this paper is organized as follows. In Section 2, we give the description of the model and give the ergodicity condition for the Markov chain underlying the queueing system. Section 3 analyzes the joint distribution of the server state and the system length together with the main performance measures by using supplementary variable method. In section 4, we obtain the stochastic decomposition formulae for the PGF of the stationary queue length at the departure epochs. Moreover, section 5 focuses on the relationship between our discrete-time queue system and its corresponding continuous-time queue system. Section 6 gives some numerical examples to illustrate the effect of varying parameters on some crucial performance measures.

S.Gao is with the Department of Mathematics, Fuyang Normal College, Fuyang 236032, China, e-mail: (sgao\_09@yeah.net)

Z.M. Liu is with the Department of Mathematics, Central South University, Changsha 410075, China, e-mail: (zmliu@csu.edu.cn)

## II. MODEL FORMULATIONS AND THE MARKOV CHAIN

Thereinafter, we denote  $\bar{x} = 1 - x$  for any real number  $x \in (0, 1)$ . The  $Geom^X/G/1$  queue with single working vacation we considered here is an early arrival system that is, a potential arrival can only take place in  $(n, n^+)$  and a potential departure can only take place in  $(n^-, n)$ . We assume that the beginning and ending of vacations occurs at the instant  $n^+$ . Arriving customers are queued according to the first-come, first-served (FCFS) discipline. The server can serve only one customer at a time. Various stochastic processes involved in the system are independent of each other.

The detailed description of the model is given as follows:

(1). Batches of customers arrive at the system according to a Bernoulli arrival process with parameter  $p(0 < p < 1)$ ,  $p$  is the probability that a batch of customers arrives in the interval  $(n, n^+)$ . The batch size sequence  $\{X_i\}_{i=1}^{\infty}$  consists of independent and identically distributed (iid) random variables distributed as  $X$  having the probability mass function (PMF)  $P(X = j) = x_j, j = 1, 2, \dots$ , and  $n$ -th factorial moments  $\xi_n, n = 1, 2$ .

(2). The service time  $S_b$  in a regular busy period has a general PMF  $P(S_b = i) = s_i^b, i \geq 1$ , and PGF  $\tilde{S}_b(z) = \sum_{i=1}^{\infty} s_i^b z^i$  and  $n$ -th factorial moments  $\beta_{b,n}, n \geq 1$ , (Obviously  $\beta_{b,1} = E[S_b] \triangleq \frac{1}{\mu_b}$ ).

(3). The Working vacation is an operating period in a lower rate, the service time  $S_v$  in a working vacation period has a general PMF  $P(S_v = i) = s_i^v, i \geq 1$ , and PGF  $\tilde{S}_v(z) = \sum_{i=1}^{\infty} s_i^v z^i$  and mean  $E[S_v] = \frac{1}{\mu_v}, \mu_v < \mu_b$ .

(4). The server begins a working vacation at the epoch when the system becomes empty, the distribution of vacation time  $V$  is geometrically distributed with rate  $\theta(0 < \theta < 1)$ , i.e.,  $P(V = j) = \theta\theta^{j-1}, j \geq 1, 0 < \theta < 1$ .

If a customer arrives during a working vacation period then the server serves the customer at the lower rate  $\mu_v$ . At a vacation completion instant, if there are customers in the system, the server will immediately come back to the normal working level and restart to serve the interruptive customer and a regular busy period starts. Otherwise, the server stays in a idle period, the server begin a new busy period immediately once there are customers arrived in the system.

Next, we will study the Markov chain underlying the queueing system at random epochs.

At time  $n^+$ , the system can be described by the process  $Y_n = \{J(n), \zeta(n), N(n)\}$ , where  $J(n)$  denotes the state of the server (0 or 1 according whether or not the system stays in a working vacation period) and  $N(n)$  is the number of customers in the system. If  $J(n) = i, N(n) \geq 1, \zeta(n)$  represents the remaining service time of the customer being served immediately after the  $n$ -th slot,  $i=0,1$ . It can be shown that  $\{Y_n, n \in \mathbb{N}\}$  is the Markov chain of our queueing system with state space  $\mathcal{S} = \{(0, 0), (1, 0)\} \cup \{(i, j, k) : i = 0, 1, j \geq 1, k \geq 1\}$ .

For the Markov chain  $\{Y_n, n \in \mathbb{N}\}$  we have the following results.

**Theorem 1.** Define  $\rho = \frac{p\xi_1}{\mu_b}$ . The Markov chain  $\{Y_n, n \in \mathbb{N}\}$  is ergodic if and only if  $\rho < 1$

**Proof** It is not difficult to see that  $\{Y_n, n \in \mathbb{N}\}$  is irreducible

and aperiodic. To prove it is also positive recurrent, we shall use the following Foster's criterion: An irreducible and aperiodic Markov chain is ergodic if there exists a nonnegative function  $f(s), s \in \mathcal{S}$ , called test function, and  $\varepsilon > 0$  such that the mean drift  $x_s = E[f(Y_{n+1}) - f(Y_n) | Y_n = s]$  is finite for all  $s \in \mathcal{S}$  and  $x_s \leq -\varepsilon$  for all  $s \in \mathcal{S}$  except perhaps a finite number. In our case, we choose the following test function on the state space  $\mathcal{S}$ :

$$f(s) = \begin{cases} \frac{1}{\mu_b} - 1, & s = (0, 0), (1, 0), \\ \frac{k+1}{\mu_b} + \frac{\bar{\theta}}{\theta}, & s = (0, j, k), j \geq 1, k \geq 1, \\ j - 1 + \frac{k}{\mu_b}, & s = (1, j, k), j \geq 1, k \geq 1. \end{cases} \quad (1)$$

To obtain its mean drift  $x_s$ , it is necessary to specify the one-step transition probability probabilities of  $\{Y_n, n \in \mathbb{N}\}$ :

If  $Y_n = (0, 0)$ :

$$Y_{n+1} = \begin{cases} (0, 0), & \text{with probability } \bar{p}\bar{\theta}, \\ (0, j, k), & \text{with probability } p\theta x_k s_j^v, \\ & j \geq 1, k \geq 1, \\ (1, 0), & \text{with probability } \bar{p}\theta, \\ (1, j, k), & \text{with probability } p\theta x_k s_j^b, \\ & j \geq 1, k \geq 1. \end{cases}$$

If  $Y_n = (1, 0)$ :

$$Y_{n+1} = \begin{cases} (1, 0), & \text{with probability } \bar{p}, \\ (1, j, k), & \text{with probability } px_k s_j^b, \\ & j \geq 1, k \geq 1. \end{cases}$$

If  $Y_n = (0, 1, k), k \geq 1$ :

$$Y_{n+1} = \begin{cases} (0, 0), & \text{with probability } \bar{p}\bar{\theta}, k = 1, \\ (1, 0), & \text{with probability } \bar{p}\theta, k = 1, \\ (0, j, k-1), & \text{with probability } \bar{p}\theta s_j^v, \\ & j \geq 1, k \geq 2, \\ (1, j, k-1), & \text{with probability } \bar{p}\theta s_j^b, \\ & j \geq 1, k \geq 2, \\ (0, j, k+m-1), & \text{with probability } \bar{p}\theta x_m s_j^v, \\ & j \geq 1, m \geq 1, \\ (1, j, k+m-1), & \text{with probability } p\theta x_m s_j^b, \\ & j \geq 1, m \geq 1. \end{cases}$$

If  $Y_n = (0, i, k), i \geq 2, k \geq 1$ :

$$Y_{n+1} = \begin{cases} (0, i-1, k), & \text{with probability } \bar{p}\bar{\theta}, \\ (0, i-1, k+m), & \text{with probability } \bar{p}\theta x_m, \\ & m \geq 1, \\ (1, j, k), & \text{with probability } \bar{p}\theta s_j^b, \\ & j \geq 1, \\ (1, j, k+m), & \text{with probability } p\theta x_m s_j^b, \\ & j \geq 1, m \geq 1. \end{cases}$$

If  $Y_n = (1, 1, k), k \geq 1$ :

$$Y_{n+1} = \begin{cases} (0, 0), & \text{with probability } \bar{p}, k = 1, \\ (1, j, k-1), & \text{with probability } \bar{p}s_j^b, k \geq 2, \\ (1, j, k+m-1), & \text{with probability } px_m s_j^b. \end{cases}$$

If  $Y_n = (1, i, k), k \geq 1, i \geq 2$ :

$$Y_{n+1} = \begin{cases} (1, i-1, k), & \text{with probability } \bar{p}, \\ (1, i-1, k+m), & \text{with probability } px_m. \end{cases}$$

Therefore, the mean drift is given by

$$x_s = \begin{cases} \rho + \delta_{i,0} \frac{p\bar{\theta}}{\theta}, & s = (i, 0), i = 0, 1, \\ \rho - 1 - \frac{1}{\mu_b} - \delta_{k,1} \frac{\bar{p}\bar{\theta}}{\theta}, & s = (0, 1, k), k \geq 1, \\ \rho - 1, & s = (0, i, k) \text{ or } (1, j, k), \\ & i \geq 2, j \geq 1, k \geq 1, \end{cases} \quad (2)$$

where  $\delta_{i,k}$  denotes the Kronecker delta.

If  $\rho < 1$ , taking  $\varepsilon = (1 - \rho)/2 > 0$  and it follows from (2)(3) and (4), we know that  $x_s < -\varepsilon$  for all states except for states  $(i, 0), i = 0, 1$ . Therefore,  $\rho < 1$  is a sufficient condition for the ergodicity of the Markov chain  $\{Y_n, n \in \mathbb{N}\}$ . This condition is also necessary since  $p_{0,0} > 0$  (the exact expression of  $p_{0,0}$  is obtained in the next section).  $\square$

### III. STEADY-STATE DISTRIBUTION

In this section, we study the steady-state distribution for the system under consideration. First, we introduce some generating functions to be used later.

Let  $A_x$  be the number of the batches arriving during the random length  $x$ ,

$$\begin{aligned} a_k &= P(A_{S_b} = k) = \sum_{j=\max(1,k)} s_j^b \binom{j}{k} p^k \bar{p}^{j-k}, \quad k \geq 0, \\ b_k &= P(A_{S_v} = k, S_v < V) \\ &= \sum_{j=\max(1,k)} s_j^v \binom{j}{k} p^k \bar{p}^{j-k} \bar{\theta}^j, \quad k \geq 0, \\ c_k &= P(A_{S_v} = k, S_v = V) \\ &= \sum_{j=\max(1,k)} s_j^v \binom{j}{k} p^k \bar{p}^{j-k} \theta \bar{\theta}^{j-1}, \quad k \geq 0, \\ v_k &= P(A_V = k, V < S_v) \\ &= \sum_{j=\max(1,k)} \theta \bar{\theta}^{j-1} \binom{j}{k} p^k \bar{p}^{j-k} \sum_{i=j+1}^{\infty} s_i^v, \quad k \geq 0. \end{aligned}$$

Then  $a_k$  is the probability that there are  $k$  batches arriving during  $S_b$  (normal service time),  $b_k$  is the probability that  $S_v < V$  and  $k$  batches arrive during  $S_v$  (vacation service time),  $c_k$  is the probability that  $S_v = V$  and  $k$  batches arrive during  $S_v$ , and  $v_k$  is the probability that  $V < S_v$  and  $k$  batches arrive during  $V$ . The z-transforms of  $\{a_k, k \geq 0\}$ ,  $\{b_k, k \geq 0\}$ ,  $\{c_k, k \geq 0\}$  and  $\{v_k, k \geq 0\}$  are given, respectively, as follows:

$$\begin{aligned} A(z) &= \sum_{k=0}^{\infty} a_k z^k = \tilde{S}_b(\bar{p} + pz), \\ B(z) &= \sum_{k=0}^{\infty} b_k z^k = \tilde{S}_v(\bar{\theta}(\bar{p} + pz)), \\ C(z) &= \sum_{k=0}^{\infty} c_k z^k = \frac{\theta B(z)}{\bar{\theta}}, \\ V(z) &= \sum_{k=0}^{\infty} v_k z^k = \frac{\theta}{\bar{\theta}(1 - \bar{\theta}(\bar{p} + pz))} (\bar{\theta}(\bar{p} + pz) - B(z)). \end{aligned}$$

Putting  $\alpha_k = \sum_{j=0}^k a_j x_k^{(j)}$ ,  $\beta_k = \sum_{j=0}^k b_j x_k^{(j)}$ ,  $\gamma_k = \sum_{j=0}^k c_j x_k^{(j)}$ ,  $\delta_k = \sum_{j=0}^k v_j x_k^{(j)}$ ,  $k \geq 0$ , where  $x_k^{(j)}$  is the probability that  $k$  customers arrive in  $j$  batches and is the  $j$ -fold convolution of  $x_k$ , and  $x_0^{(0)} = 1$ . Then  $\alpha_k$  is the probability that there are  $k$  customers arriving during  $S_b$ ,  $\beta_k$  is the probability that  $S_v < V$  and  $k$  customers arrive during  $S_v$ ,  $\gamma_k$  is the probability that  $S_v = V$  and  $k$  customers arrive during  $S_v$ , and  $\delta_k$  is the probability that  $V < S_v$  and  $k$  customers arrive during  $V$ .

Let  $\eta(z) = \bar{p} + pX(z)$ , the PGFs of  $\{\alpha_k\}_{k=0}^{\infty}$ ,  $\{\beta_k\}_{k=0}^{\infty}$ ,  $\{\gamma_k\}_{k=0}^{\infty}$  and  $\{\delta_k\}_{k=0}^{\infty}$  are given by as follows:

$$\begin{aligned} \alpha(z) &= \sum_{k=0}^{\infty} \alpha_k z^k = A(X(z)) = \tilde{S}_b(\eta(z)), \\ \beta(z) &= \sum_{k=0}^{\infty} \beta_k z^k = B(X(z)) = \tilde{S}_v(\bar{\theta}\eta(z)), \\ \gamma(z) &= \sum_{k=0}^{\infty} \gamma_k z^k = C(X(z)) = \frac{\theta\beta(z)}{\bar{\theta}}, \\ \delta(z) &= \sum_{k=0}^{\infty} \delta_k z^k = V(X(z)) = \frac{\theta}{1 - \bar{\theta}\eta(z)} \left( \eta(z) - \frac{\beta(z)}{\bar{\theta}} \right). \end{aligned}$$

Evidently,

$$\alpha(1) = 1, \beta(1) = \tilde{S}_v(\bar{\theta}), \gamma(1) = \frac{\theta\beta(1)}{\bar{\theta}}, \delta(1) = 1 - \frac{\beta(1)}{\bar{\theta}}.$$

Thus  $\{\beta_k, k \geq 0\}$ ,  $\{\gamma_k, k \geq 0\}$  and  $\{\delta_k, k \geq 0\}$  are three non-complete probability distributions.

Let  $h_k = \sum_{j=0}^k \delta_j a_{k-j}$ ,  $k \geq 0$ , then  $h_k$  represents the probability that the vacation time  $V$  is smaller than the vacation service time  $S_v$  and  $k$  customers arrive during  $V$  plus  $S_b$  and  $H(z) = \sum_{k=0}^{\infty} h_k z^k = \delta(z)\alpha(z)$ . For these PGFs, we have

$$\bar{\theta}(\beta(z) + \gamma(z)) = \beta(z),$$

$$\alpha'(1) = \frac{p\xi_1}{\mu_b} = \rho, \quad \beta'(1) = \bar{\theta}p\xi_1\tilde{S}_v'(\bar{\theta}),$$

$$\beta'(1) + \bar{\theta}\delta'(1) = \bar{\theta}(\beta'(1) + \gamma'(1) + \delta'(1)) = \frac{\bar{\theta}p\xi_1}{\bar{\theta}}(1 - \beta(1)),$$

$$H'(1) + \beta'(1) + \gamma'(1) = \frac{p\xi_1}{\bar{\theta}}(1 - \beta(1)) + \rho(1 - \beta(1)/\bar{\theta}),$$

We also need the following lemma whose proof is omitted.

**Lemma 1** If  $\rho < 1$ , the equation  $z = \tilde{S}_b(\eta(z))$  has the minimal nonnegative root  $z = 1$  and the equation  $z = \tilde{S}_v(\bar{\theta}\eta(z))$  has the unique root  $r$  in the interval  $(0, 1)$ .

**Remark 1** (1) If  $\theta \rightarrow 1$ , our model is approximately reduced to the classical discrete-time  $Geo^X/G/1$  queue (without vacation), in this case,  $r \rightarrow 0$ .

(2) If no service exists during vacation, i.e.,  $\tilde{S}_v(z) = 0$ , the root of the equation  $z = \tilde{S}_v(\bar{\theta}\eta(z))$  is  $r = 0$  and our model is changed to the classical single vacation model.

Based on the above results, we derive the steady-state distribution of the process  $\{Y_n, n \in \mathbb{N}\}$ . Under the stability condition  $\rho < 1$ , we define

$$p_{j,0} = \lim_{n \rightarrow \infty} P(J(n) = j, N(n) = 0), \quad j = 0, 1,$$

$$p_{j,i,k} = \lim_{n \rightarrow \infty} P(J(n) = j, \zeta(n) = i, N(n) = k), \\ j = 0, 1, \quad i \geq 1, k \geq 1.$$

Then we have the following theorem.

**Theorem 2** Under the stationary condition  $\rho < 1$ , the generating functions of the stationary joint distribution of the Markov chain  $\{Y_n, n \in \mathbb{N}\}$  are given by:

$$\begin{aligned} p_{0,0} &= [\theta p \xi_1 (1 - \rho)(1 - \beta(1))] \times \\ &\quad \{\theta \xi_1 [p + \theta(\bar{p} + pX(r))](1 - \beta(1)) \\ &\quad + p(1 - X(r))(\bar{\theta} p \xi_1 (1 - \beta(1)) - \theta \rho \beta(1))\}^{-1}, \\ p_{1,0} &= \frac{\theta(\bar{p} + pX(r))}{p} p_{0,0}, \\ \phi_0(x, z) &= \frac{\bar{\theta} p [\tilde{S}_v(x) - \beta(z)]}{x - \bar{\theta} \eta(z)} \frac{X(z) - X(r)}{z - \beta(z)} x z p_{0,0}, \\ \phi_1(x, z) &= \frac{\tilde{S}_b(x) - \alpha(z)}{p(1 - X(z))} x z p_{0,0} \frac{1}{(z - \alpha(z))(z - \beta(z))} \times \\ &\quad \{(p + \theta \eta(r))(X(z) - 1)(z - \beta(z)) + \\ &\quad p[X(z) - X(r)](\beta(z) + \bar{\theta} z \delta(z) - \bar{\theta} z)\}. \end{aligned}$$

**Proof** Based on the one-step transition probabilities, we have the following set of equilibrium equations

$$p_{0,0} = \bar{p} \bar{\theta} p_{0,0} + \bar{p} \bar{\theta} p_{0,1,1} + \bar{p} p_{1,1,1}, \quad (5)$$

$$p_{1,0} = \bar{p} p_{1,0} + \bar{p} \theta p_{0,0} + \bar{p} \theta p_{0,1,1}, \quad (6)$$

$$\begin{aligned} p_{0,i,k} &= p \bar{\theta} x_k s_i^v p_{0,0} \\ &\quad + \bar{\theta} \left( \bar{p} p_{0,i+1,k} + p \bar{\theta} \sum_{m=1}^{k-1} x_m p_{0,i+1,k-m} \right) \\ &\quad + \bar{\theta} \left( \bar{p} p_{0,1,k+1} + p \sum_{m=1}^k x_m p_{0,1,k-m+1} \right) s_i^v, \\ i \geq 1, k \geq 1, \end{aligned} \quad (7)$$

$$\begin{aligned} p_{1,i,k} &= p \theta x_k s_i^b p_{0,0} + p x_k s_i^b p_{1,0} + \bar{p} p_{1,i+1,k} \\ &\quad + p \sum_{m=1}^{k-1} x_m p_{1,i+1,k-m} \\ &\quad + \left( \bar{p} p_{1,1,k+1} + p \sum_{m=1}^k x_m p_{1,1,k-m+1} \right) s_i^b \\ &\quad + \theta \left( \bar{p} p_{0,1,k+1} + p \sum_{m=1}^k x_m p_{0,1,k-m+1} \right) s_i^b \\ &\quad + \theta \left( \bar{p} \sum_{j=2}^{\infty} p_{0,j,k} + p \sum_{m=1}^{k-1} \sum_{j=2}^{\infty} x_m p_{0,j,k-m} \right) s_i^b, \\ i \geq 1, k \geq 1, \end{aligned} \quad (8)$$

(by convention  $\sum_{m=1}^0 = 0$ ) and the normalization condition is

$$p_{0,0} + p_{1,0} + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \pi_{0,i,k} + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} p_{1,i,k} = 1.$$

In order to obtain the solution of (5)-(8), we define the

following generating functions:

$$\begin{aligned} \phi_{0,i}(z) &= \sum_{k=1}^{\infty} p_{0,i,k} z^k, i \geq 1, \quad \phi_0(x, z) = \sum_{i=1}^{\infty} x^i \phi_{0,i}(z), \\ \phi_{1,i}(z) &= \sum_{k=0}^{\infty} p_{1,i,k} z^k, i \geq 1, \quad \phi_1(x, z) = \sum_{i=1}^{\infty} x^i \phi_{1,i}(z). \end{aligned}$$

Multiplying (7) by  $z^k$  and summing over  $k$  leads to

$$\begin{aligned} \phi_{0,i}(z) &= \bar{\theta} p X(z) s_i^v p_{0,0} + \bar{\theta} \eta(z) \phi_{0,i+1}(z) + \frac{\bar{\theta} \eta(z)}{z} s_i^v \phi_{0,1}(z) \\ &\quad - \bar{p} \bar{\theta} s_i^v p_{0,1,1}, i \geq 1. \end{aligned} \quad (9)$$

Multiplying (9) by  $x^i$  and summing over  $i$  yields

$$\begin{aligned} \frac{x - \bar{\theta} \eta(z)}{x} \phi_0(x, z) &= \bar{\theta} \tilde{S}_v(x) (p X(z) p_{0,0} - \bar{p} p_{0,1,1}) \\ &\quad - \frac{\bar{\theta} \eta(z)}{z} (z - \tilde{S}_v(x)) \phi_{0,1}(z). \end{aligned} \quad (10)$$

Choosing  $x = \bar{\theta} \eta(z)$  in (10), the left hand side of this equation vanishes and therefore we have

$$\phi_{0,1}(z) = \frac{z \beta(z) (p X(z) p_{0,0} - \bar{p} p_{0,1,1})}{\eta(z) (z - \beta(z))}. \quad (11)$$

From lemma 1, the denominator at the right-side of (11) is equal to 0 if  $z = r$ , so does the numerator. Then we have

$$\bar{p} p_{0,1,1} = p X(r) p_{0,0}.$$

Substituting the above expression into (11) yields

$$\phi_{0,1}(z) = \frac{p z \beta(z) (X(z) - X(r))}{\eta(z) (z - \beta(z))} p_{0,0}. \quad (12)$$

Substituting (12) into (10), we obtain

$$\phi_0(x, z) = \frac{\bar{\theta} p [\tilde{S}_v(x) - \beta(z)]}{x - \bar{\theta} \eta(z)} \frac{X(z) - X(r)}{z - \beta(z)} x z p_{0,0}. \quad (13)$$

Similarly, multiplying (8) by  $z^k$  and  $x^i$ , summing over  $k$  and  $i$  leads to

$$\begin{aligned} \frac{x - \eta(z)}{x} \phi_1(x, z) &= \left( p X(z) (\theta p_{0,0} + p_{1,0}) - \bar{p} (\theta p_{0,1,1} + p_{1,1,1}) + \left( \sum_{j=2}^{\infty} \phi_{0,j}(z) \right. \right. \\ &\quad \left. \left. + \frac{\phi_{0,1}(z)}{z} \right) \theta \eta(z) \right) \tilde{S}_b(x) + \frac{\eta(z) [\tilde{S}_b(x) - z]}{z} \phi_{1,1}(z). \end{aligned} \quad (14)$$

Inserting  $\bar{p} p_{0,1,1} = p X(r) p_{0,0}$  into (5) and (6) we can get

$$\bar{p} p_{1,1,1} = (1 - \bar{\theta}(\bar{p} + pX(r))) p_{0,0}, \quad (15)$$

$$\bar{p} p_{1,0} = \theta(\bar{p} + pX(r)) p_{0,0}. \quad (16)$$

From (12) and (13), after some calculations we can obtain that

$$\begin{aligned} \sum_{j=2}^{\infty} \phi_{0,j}(z) + \frac{\phi_{0,1}(z)}{z} &= \phi_0(1, z) + \frac{(1 - z)}{z} \phi_{0,1}(z) \\ &= \frac{p \bar{\theta} [X(z) - X(r)]}{z - \beta(z)} \frac{\gamma(z) + z \delta(z)}{\theta \eta(z)} p_{0,0}. \end{aligned} \quad (17)$$

Inserting (15)-(17) into (14) and using the equality  $\theta + \bar{\theta}p = p + \bar{\theta}p$  yields

$$\frac{x - \eta(z)}{x} \phi_1(x, z) = \frac{\eta(z)[\tilde{S}_b(x) - z]}{z} \phi_{1,1}(z) + p_{0,0} \frac{\tilde{S}_b(x)}{z - \beta(z)} \times \{(p + \theta\eta(r))(X(z) - 1)(z - \beta(z)) + p[X(z) - X(r)] \times (\beta(z) + \bar{\theta}z\delta(z) - \bar{\theta}z)\}. \quad (18)$$

Taking  $x = \eta(z)$  in (18), the left hand side of (18) vanishes and then

$$\phi_{1,1}(z) = \frac{z\alpha(z)}{\eta(z)(z - \alpha(z))(z - \beta(z))} p_{0,0} \times \{(p + \theta\eta(r))(X(z) - 1)(z - \beta(z)) + p[X(z) - X(r)] \times (\beta(z) + \bar{\theta}z\delta(z) - \bar{\theta}z)\}. \quad (19)$$

Combining (18) with (19), we have

$$\phi_1(x, z) = \frac{\tilde{S}_b(x) - \alpha(z)}{p(1 - X(z))} x z p_{0,0} \frac{1}{(z - \alpha(z))(z - \beta(z))} \times \{(p + \theta\eta(r))(X(z) - 1)(z - \beta(z)) + p[X(z) - X(r)](\beta(z) + \bar{\theta}z\delta(z) - \bar{\theta}z)\}.$$

Finally, from the normalization condition  $p_{0,0} + p_{1,0} + \phi_0(1, 1) + \phi_1(1, 1) = 1$  and (16), we can obtain the expression of  $p_{0,0}$  and  $p_{1,0}$ .  $\square$

**Remark 2** In (5)-(8), if we take  $\theta = 1$ , then these equilibrium equations are corresponding to the classical discrete-time  $Geo^X/G/1$  queue (without vacation), in this case, the state that the server is busy with lower service doesn't exist any more, and  $P_{0,0} + P_{1,0}$  is the probability that the server is idle, denoted as  $P_I$ . Then we can get the generating functions of the stationary joint distribution function for the classical discrete-time  $Geo^X/G/1$  queue without vacation as follows:

$$p_I = 1 - \rho, \\ \phi_1(x, z) = \frac{\tilde{S}_b(x) - \alpha(z)}{\alpha(z) - z} x z (1 - \rho)$$

and the generating function of the number of customers in system at arbitrary epochs is

$$L_0(z) \triangleq p_I + \phi_1(1, z) = \frac{(1 - \rho)(1 - z)\alpha(z)}{\alpha(z)},$$

which is the wellknown result.

**Corollary 1** (1) The marginal generating function of the system size when the server is busy with lower rate service is:

$$\phi_0(1, z) = \frac{\bar{\theta}p[1 - \beta(z)]}{1 - \bar{\theta}\eta(z)} \frac{X(z) - X(r)}{z - \beta(z)} z p_{0,0}.$$

(2) The marginal generating function of the system size when the server is busy with normal service is:

$$\phi_1(1, z) = \frac{1 - \alpha(z)}{p(1 - X(z))(z - \alpha(z))(z - \beta(z))} z p_{0,0} \times \{(p + \theta\eta(r))(X(z) - 1)(z - \beta(z)) + p[X(z) - X(r)](\beta(z) + \bar{\theta}z\delta(z) - \bar{\theta}z)\}.$$

(3) The PGF of the system size, denoted as  $P(z)$ , is given by:

$$P(z) = p_{0,0} + p_{1,0} + \phi_0(1, z) + \phi_1(1, z) \\ = \frac{1 - z}{1 - X(z)} \frac{\mathcal{N}(z)}{\mathcal{D}(z)} \frac{p_{0,0}}{p}, \quad (20)$$

where

$$\mathcal{N}(z) = (p + \theta\eta(r))(X(z) - 1)(z - \beta(z))\alpha(z) \\ + p[X(z) - X(r)](\bar{\theta}H(z) + \beta(z) - \bar{\theta}\alpha(z)), \\ \mathcal{D}(z) = (z - \alpha(z))(z - \beta(z)).$$

In the following corollary, we provide some performance measures at the stationary regime.

**Corollary 2** (1) The probability that the server is in working vacation period is :

$$p_{0,0} + \phi_0(1, 1) = \frac{\theta + \bar{\theta}p(1 - X(r))}{\theta} p_{0,0}.$$

(2) The probability that the server is idle is

$$p_{1,0} = \frac{\theta\eta(r)}{p} p_{0,0}.$$

(3) The probability that the server is busy with normal service is

$$\phi_1(1, 1) = \frac{\rho p_{0,0}}{\theta p \xi_1(1 - \rho)(1 - \beta(1))} \times \{\theta \xi_1(1 - \beta(1))(p + \theta\eta(r)) + p(1 - X(r)) \times (\bar{\theta} p \xi_1(1 - \beta(1)) - \theta \beta(1))\}.$$

(4) The probability that the system is empty is

$$P(\text{empty}) = p_{0,0} + p_{1,0} = \frac{p + \theta\eta(r)}{p} p_{0,0}.$$

#### IV. ANALYSIS OF SYSTEM SIZE AT THE DEPARTURE EPOCHS

In this section, we mainly use the supplementary variable method to obtain the expression for the PGF of the system size at the departure epochs and its stochastic decomposition. To this end, we firstly derive the queue length at the beginning of the regular busy period, and the expected busy period and expected busy cycle.

**4.1 The queue length at the beginning of the busy period and expected busy cycle**

In this subsection, we will use the supplementary variable method to obtain the PGF of the queue length at the beginning of the regular busy period.

The duration in which the server works at the normal service rate continuously is called a (regular) busy period, denoted by  $L_b$ . Denote the number of customer at the beginning of a busy period  $Q_b$ , and its PGF  $Q_b(z) = \sum_{n=1}^{\infty} z^n P(Q_b = n)$ . Let  $L_v$  be the working vacation which begins when a busy period expires and ends at the expire instant of a vacation,  $L_I$  be the length of the period in which the server is not in working period but available which begins when the vacation ends without customers waiting in the queue until the new

customers arrive. A busy cycle  $\Theta$  is then defined as the sum of  $L_V, L_I$  and  $L_b$ . Let  $\tilde{D}$ , with PGF  $\tilde{D}_b(z)$ , represent the length of busy period beginning with only one customer in non-vacation classic  $Geo^X/G/1$ , Tian et al. (2008) showed that  $\tilde{D}_b(z)$  fulfils the implicit equation  $\tilde{D}_b(z) = \tilde{S}_b(z(\bar{p} + pX(\tilde{D}_b(z))))$  and  $E[\tilde{D}] = \frac{E[S_b]}{1-\rho}$ . Thus in our mode, we have that

$$E[L_b] = E[\tilde{D}]E[Q_b]. \quad (21)$$

In the following, we first derive the PGF  $Q_b(z)$ .

Conditioning on the system states at the epoch prior to the beginning of the regular busy period, we have that

$$\begin{aligned} P(Q_b = n) &= K \left[ p\theta x_n p_{0,0} + \bar{p}\theta \sum_{i=2}^{\infty} p_{0,i,n} + p\theta \sum_{m=1}^{n-1} \sum_{i=2}^{\infty} p_{0,i,n-m} \right. \\ &\quad \left. + \bar{p}\theta p_{0,1,n+1} + p\theta \sum_{m=1}^n x_m p_{0,1,n-m+1} + p x_n p_{1,0} \right], n \geq 1. \end{aligned}$$

Multiplying both sides of the above expression by  $z^n$  and summing over  $n$  from 1 to  $\infty$ , we obtain after simplification

$$\begin{aligned} Q_b(z) &= K[\theta + \theta(1 + pX(r))(X(z) - 1)] \\ &\quad + K \left[ \frac{\bar{\theta}p(X(z) - X(r))(\gamma(z) + z\delta(z))}{z - \beta(z)} \right] p_{0,0}. \end{aligned} \quad (22)$$

The unknown constant  $K$  can be determined by using the normalization condition  $Q_b(1) = \sum_{n=1}^{\infty} P(Q_b = n) = 1$ , which leads to

$$K = \frac{1}{(\theta + \bar{\theta}p(1 - X(r))p_{0,0}}.$$

Then we obtain

$$\begin{aligned} E[Q_b] &= Q'_b(1) = \frac{1}{\theta(1 - \beta(1))(\theta + \bar{\theta}p(1 - X(r)))} \times \\ &\quad \{ \theta \xi_1 (p + \theta(\bar{p} + pX(r))) (1 - \beta(1)) \\ &\quad + p(1 - X(r)) (\bar{\theta}p\xi_1(1 - \beta(1)) - \theta\beta(1)) \}. \end{aligned} \quad (23)$$

Therefore,

$$\begin{aligned} E[L_b] &= \frac{E[S_b]}{[\theta(1 - \beta(1))(\theta + \bar{\theta}p(1 - X(r)))(1 - \rho)]} \times \\ &\quad \{ \theta \xi_1 (p + \theta(\bar{p} + pX(r))) (1 - \beta(1)) \\ &\quad + p(1 - X(r)) (\bar{\theta}p\xi_1(1 - \beta(1)) - \theta\beta(1)) \}. \end{aligned} \quad (24)$$

By using the alternating renewal theorem, we may write

$$\phi_1(1, 1) = \frac{E[L_b]}{E[\Theta]}, p_{1,0} = \frac{E[L_I]}{E[\Theta]} \text{ and } p_{0,0} + \phi_0(1, 1) = \frac{E[L_v]}{E[\Theta]}.$$

By Corollary 2 and (24), we can get  $E[\Theta]$ ,  $E[L_I]$  and  $E[L_v]$  as follows:

$$E[\Theta] = \frac{1}{(\theta + \bar{\theta}p(1 - X(r))p_{0,0}}, \quad (25)$$

$$E[L_I] = \frac{\theta(\bar{p} + pX(r))}{p(\theta + \bar{\theta}p(1 - X(r))), \quad (26)$$

$$E[L_v] = \frac{\theta + \bar{\theta}p(1 - X(r))}{\theta(\theta + \bar{\theta}p(1 - X(r)))} = \frac{1}{\theta}. \quad (27)$$

#### 4.2 The PGF of the queue length at departure epochs and its stochastic decomposition

In this subsection we will derive the PGF  $\Pi(z)$  of the stationary distribution  $\{\pi_n\}_{n=0}^{\infty}$  of the queue size at the departure epochs. Let  $P_v(P_b)$  be the probabilities that an arbitrary customer is served completely at the lower service rate (at the normal service rate), denoted by  $S = 0(S = 1)$ , and  $P_{v,n}, (P_{b,n})$  be the probability that there are  $n$  customers in the system immediately after the slot when one customer is served completely at the lower service rate (at the normal service rate). Define  $P_v(z) = \sum_{n=0}^{\infty} P_{v,n}z^n, P_b(z) = \sum_{n=0}^{\infty} P_{b,n}z^n$ .

By analyzing relations between arbitrary and departure epochs and considering various possibilities one can easily see that  $\pi_n, P_{v,n}, P_{b,n}$  and  $p_{0,1,n}, p_{0,1,n}$  are connected by the relation

$$\begin{aligned} \pi_n &= P_{v,n} + P_{b,n} \\ &= K_1 \left[ \bar{p}p_{0,1,n+1} + (1 - \delta_{0,n})p \sum_{m=1}^n x_m p_{0,1,n-m+1} \right] \\ &\quad + K_1 \left[ \bar{p}p_{1,1,n+1} + (1 - \delta_{0,n})p \sum_{m=1}^n x_m p_{1,1,n-m+1} \right], \end{aligned}$$

which leads to

$$\Pi(z) = P_v(z) + P_b(z) = K_1 \frac{\eta(z)}{z} (\phi_{0,1}(z) + \phi_{1,1}(z)),$$

$$P_v(z) = K_1 \frac{\eta(z)}{z} \phi_{0,1}(z), \quad P_b(z) = K_1 \frac{\eta(z)}{z} \phi_{1,1}(z).$$

From (12) and (19), we know that

$$P_v(z) = K_1 \frac{p\beta(z)(X(z) - X(r))}{z - \beta(z)} p_{0,0}, \quad (28)$$

$$\begin{aligned} P_b(z) &= K_1 \frac{\alpha(z)}{(z - \alpha(z))(z - \beta(z))} p_{0,0} \times \\ &\quad \{ (p + \theta(\bar{p} + \theta pX(r)))(X(z) - 1)(z - \beta(z)) \\ &\quad + p[X(z) - X(r)](\beta(z) + \bar{\theta}z\delta(z) - \bar{\theta}z) \}. \end{aligned} \quad (29)$$

Then we can obtain that

$$\Pi(z) = K_1 p_{0,0} \frac{\mathcal{N}(z)}{\mathcal{D}(z)}.$$

The unknown constant  $K_1$  can be determined by using the normalization condition  $\Pi(1) = P_v(1) + P_b(1) = 1$ , then we can obtain  $K_1 = \frac{1}{p\xi_1}$ .

We summarize the preceding results in the following theorem.

**Theorem 3** For the PGFs  $\Pi(z), P_v(z), P_b(z)$ , we have

$$\Pi(z) = \frac{p_{0,0} \mathcal{N}(z)}{p\xi_1 \mathcal{D}(z)}, \quad (30)$$

$$P_v(z) = \frac{\beta(z)(X(z) - X(r)) p_{0,0}}{z - \beta(z) \xi_1}, \quad (31)$$

$$\begin{aligned} P_b(z) &= \frac{\alpha(z)}{(z - \alpha(z))(z - \beta(z))} p_{0,0} \times \\ &\quad \{ (p + \theta(\bar{p} + \theta pX(r)))(X(z) - 1)(z - \beta(z)) \\ &\quad + p[X(z) - X(r)](\beta(z) + \bar{\theta}z\delta(z) - \bar{\theta}z) \}. \end{aligned} \quad (32)$$

**Remark 3** Comparing (20) with (30), we find that

$$\Pi(z) = P(z) \frac{1 - X(z)}{\xi_1(1 - z)}. \quad (33)$$

It is not surprising that this is same as that in the standard  $Geo^X/G/1$  queueing system.

**Corollary 3** (1) The probability that an arbitrary customer is served completely at the lower service rate (at the normal service rate) is

$$P_v = P_v(1) = \frac{\beta(1)(1 - X(r))}{1 - \beta(1)} \frac{p_{0,0}}{\xi_1}.$$

(2) The probability that an arbitrary customer is served completely at the normal service rate is

$$P_b = P_b(1) = 1 - P_v.$$

**Remark 4** If there is no service during the vacation period, i.e.  $\beta(1) = \tilde{S}_v(\theta) = 0$ , we have  $P_v = 0$  and  $P_b = 1$ , which means that all customers are served completely by the normal service rate.

By using the conditional argument

$$\begin{aligned} \Pi(z) &= P_v E[z^L | S = 0] + P_b E[z^L | S = 1] \\ &= P_v \frac{P_v(z)}{P_v} + P_b \frac{P_b(z)}{P_b}, \end{aligned}$$

After manipulating, we can easily get that

$$\begin{aligned} \Pi(z) &= P_v \frac{\beta(z)(X(z) - X(r))}{z - \beta(z)} \frac{1 - \beta(1)}{\beta(1)(1 - X(r))} \\ &\quad + P_b \frac{(1 - \rho)(1 - z)\alpha(z)}{\alpha(z) - z} L_d(z), \end{aligned}$$

where

$$L_d(z) = \frac{1 - Q_b(z)}{E[Q_b](1 - z)}.$$

Then we get the following theorem.

**Theorem 4** If  $\rho < 1$ , the probability generating function  $\Pi(z)$  of the stationary queue length  $L$  at departure epochs is

$$\begin{aligned} \Pi(z) &= P_v \frac{\beta(z)(X(z) - X(r))}{z - \beta(z)} \frac{1 - \beta(1)}{\beta(1)(1 - X(r))} \\ &\quad + P_b \frac{(1 - \rho)(1 - z)\alpha(z)}{\alpha(z) - z} \frac{1 - Q_b(z)}{E[Q_b](1 - z)}. \end{aligned}$$

Denote by  $L_1 = E[L | S = 1]$ . From above discussion, we can obtain the following conditional stochastic decomposition discipline.

**Theorem 5** If  $\rho < 1$ , the conditional queue length  $L_1$  can be decomposed into the sum of two independent random variables:  $L_1 = L_0 + L_d$ , where  $L_0$  is the stationary queue length of a classic  $Geo^X/G/1$  queue without vacations with PGF

$$L_0(z) = \frac{(1 - \rho)(1 - z)\alpha(z)}{\alpha(z) - z}.$$

From the above theorem, we can get the expected queue length at the departure epochs, denoted by  $\Pi_s$ , as follows

$$\begin{aligned} \Pi_s &= P_v \left( \frac{\beta'(1)}{\beta(1)} + \frac{\xi_1}{1 - X(r)} - \frac{1 - \beta'(1)}{1 - \beta(1)} \right) \\ &\quad + P_b \left( \rho + \frac{(p\xi_1)^2 \beta_{b,2} + p\xi_2 \beta_{b,1}}{2(1 - \rho)} + L'_d(1) \right). \end{aligned}$$

Let  $L_s$  be the expected system length at random epochs, then  $L_s = P'(1)$ , by (33) we have

$$L_s = P'(1) = \Pi_s - \frac{\xi_2}{2\xi_1}.$$

and by Little's law, the mean sojourn time in the system (denoted by  $W_s$ ) is given by

$$W_s = \frac{L_s}{p\xi_1}.$$

## V. RELATIONSHIP TO THE CONTINUOUS-TIME SYSTEM

This section is dedicated to the analysis of the relationship between our discrete-time system and its continuous-time counterpart. More specifically, we will show that the continuous-time  $M^X/G/1$  queue with single exponentially working vacation can be approximated by the corresponding discrete-time system. To this end, time is slotted into intervals of equal length, so the approximation tends to the exact value when the length of the intervals goes to zero.

We consider the continuous-time  $M^X/G/1$  queue with single working vacation where batches of the customers arrive according to a Poisson process with rate  $\lambda > 0$  and the batch size  $X$  has PMF  $P(X = j) = x_j, j = 1, 2, \dots$ , and PGF  $X(z)$  and  $n$ -th factorial moments  $\xi_n, n = 1, 2$ . Normal service times (lower rate service times during a vacation period) in a regular busy period are independent and identically distributed with distribution function  $G_b(x)$  ( $G_v(x)$ ), Laplace-Stieltjes transform  $G_b^*(s)$  ( $G_v^*(s)$ ) and a finite mean  $\mu_1^{-1}$  ( $\mu_2^{-1}, \mu_2 < \mu_1$ ). Vacation time  $V$  is exponentially distributed with parameter  $\delta$ . Various stochastic processes involved in the system are independent of each other. If we suppose that time is divided into intervals of length  $\Delta \in (0, 1/\lambda)$ , the previous continuous-time system can be approximated by our discrete-time model choosing the parameters as follows:

$$p = \lambda\Delta, \quad \theta = \delta\Delta,$$

$$s_i^b = \int_{(i-1)\Delta}^{i\Delta} dG_b(x), \quad s_i^v = \int_{(i-1)\Delta}^{i\Delta} dG_v(x), \quad i \geq 1$$

where  $\Delta$  is sufficiently small so that  $p$  is a probability. It is not difficult to check the following equalities using the definition of the Lebesgue integral:

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \rho &= \lim_{\Delta \rightarrow 0} p\xi_1\beta_{b,1} \\ &= \xi_1 \lim_{\Delta \rightarrow 0} \lambda\Delta \sum_{i=1}^{\infty} i \int_{(i-1)\Delta}^{i\Delta} dG_b(x) \\ &= \lambda\xi_1 \int_0^{\infty} x dG_b(x) = \frac{\lambda\xi_1}{\mu_1} \triangleq \tilde{\rho}, \end{aligned}$$

$$\lim_{\Delta \rightarrow 0} \tilde{S}_b(\bar{p} + pX(z)) = \int_0^\infty e^{-\lambda(1-X(z))} dG_b(x) = G_b^*(\lambda(1-X(z))).$$

$$\lim_{\Delta \rightarrow 0} \tilde{S}_v(\bar{\theta}(\bar{p} + pX(z))) = \int_0^\infty e^{-(\delta + \lambda(1-X(z)))} dG_v(x) = G_v^*(\delta + \lambda(1-X(z))).$$

Therefore we have

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \alpha(z) &= G_b^*(\lambda(1-X(z))) \triangleq \tilde{\alpha}(z), \\ \lim_{\Delta \rightarrow 0} \beta(z) &= G_v^*(\delta + \lambda(1-X(z))) \triangleq \tilde{\beta}(z), \\ \lim_{\Delta \rightarrow 0} \delta(z) &= \lim_{\Delta \rightarrow 0} \frac{\theta}{1 - \theta\eta(z)} \left( \eta(z) - \frac{\beta(z)}{\theta} \right) \\ &= \frac{\delta}{\delta + \lambda(1-X(z))} [1 - G_v^*(\delta + \lambda(1-X(z)))], \\ \lim_{\Delta \rightarrow 0} H(z) &= \lim_{\Delta \rightarrow 0} \delta(z)\alpha(z) \\ &= \frac{\delta}{\delta + \lambda(1-X(z))} \times \\ &\quad [1 - G_v^*(\delta + \lambda(1-X(z)))] G_b^*(\lambda(1-X(z))) \\ &\triangleq \tilde{H}(z), \end{aligned}$$

and

$$\begin{aligned} \lim_{\Delta \rightarrow 0} p_{0,0} &= [\delta\lambda\xi_1(1-\tilde{\rho})(1-G_v^*(\delta))] \times \\ &\quad \{\lambda(1-X(r))[\lambda\xi_1(1-G_v^*(\delta)) - \delta\tilde{\rho}G_v^*(\delta)] \\ &\quad + \delta\xi_1(\lambda + \delta)(1-G_v^*(\delta))\}^{-1} \triangleq \psi_{0,0}, \end{aligned}$$

where  $r$  is the unique root in the range  $(0, 1)$  of the equation  $z = G_v^*(\delta + \lambda(1-X(z)))$ . Taking into account these results, we achieve

$$\begin{aligned} \lim_{\Delta \rightarrow 0} P(z) &= \frac{1-z}{1-X(z)} \frac{1}{(z-\tilde{\alpha}(z))(z-\tilde{\beta}(z))} \psi_{0,0} \times \\ &\quad \{(\lambda + \delta)(1-X(z))(\tilde{\beta}(z) - z)\tilde{\alpha}(z) \\ &\quad + \lambda z(X(z) - X(r))(\tilde{H}(z) + \tilde{\beta}(z) - \tilde{\alpha}(z))\}, \end{aligned} \quad (34)$$

and

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \Pi(z) &= \frac{\psi_{0,0}}{\xi_1} \frac{1}{(z-\tilde{\alpha}(z))(z-\tilde{\beta}(z))} \times \\ &\quad \{(\lambda + \delta)(1-X(z))(\tilde{\beta}(z) - z)\tilde{\alpha}(z) \\ &\quad + \lambda z(X(z) - X(r))(\tilde{H}(z) + \tilde{\beta}(z) - \tilde{\alpha}(z))\}, \end{aligned} \quad (35)$$

which are respectively the PGFs of the system sizes in the  $M^X/G/1$  queue with single working vacation at random times and departure times.

## VI. NUMERICAL RESULTS

In this section, we present some numerical examples to study the effect of the varying parameters on the main performance characteristics of our system. For simplicity, it is assumed that the service times  $S_b$  and  $S_v$  follow the geometrical distributions with parameters  $\mu_b$  and  $\mu_v$ , respectively, and  $\mu_b = 0.8$ ,  $\mu_v$  varies from 0 to 0.8, the batches size follows a

geometrical distribution with mean  $\xi_1 = 2$  and  $p = 0.2$ . The model is denoted as  $Geo^{Geo}/(Geo_1, Geo_2)/1/SWV$ .

We will concentrate our attention on four important performance descriptors:  $L_s$ , the mean system size,  $P(\text{empty})$ , the probability that the system is empty,  $E[\Theta]$ , the expected busy cycle and  $P_b$ , the probability that a customer is served completely by the normal service rate.

In Fig. 1,  $L_s$  is plotted against the parameter  $\theta$ . Obviously, the system size  $L_s$  is decreasing as function of  $\theta$ , that is, the bigger the probability  $\theta$ , the shorter the vacation time, and then the chance that the customer is served by the normal service rate (which can also be seen in Fig.4) is increased which leads to the decrease of  $L_s$ . Additionally, we study the influence of the lower service rate  $\mu_v$  on  $L_s$ . Specially, three curves are presented corresponding to  $\mu_v = 0, 0.15, 0.25, 0.45, 0.65$ . As is to be expected,  $L_s$  decrease with increasing values of  $\mu_v$ , which also agrees with the intuitive. We should note that when  $\theta$  approaches to 1,  $L_s$  will achieve a fixed value, i.e., the system size without vacation.

Figs. 2-4 depict the behavior of the  $P(\text{empty})$ ,  $E[\Theta]$  and  $P_b$  against  $\mu_v$ . Obviously,  $P(\text{empty})$  is increasing as function of  $\mu_v$  and  $E[\Theta]$  and  $P_b$  decrease with increasing values of  $\mu_v$ . Meanwhile, in Figs. 2-4,  $P(\text{empty})$ ,  $E[\Theta]$  and  $P_b$  are compared with varying values  $\theta = 0.25, 0.45, 0.65, 1$ . As we expected,  $P(\text{empty})$  and  $P_b$  increase and  $E[\Theta]$  decreases with increasing values of  $\theta$ . It can be observed from Figs.2-3 that the effect of  $\theta$  can be ignored when  $\mu_v = \mu_b = 0.8$ , in this case,  $P(\text{empty})$  and  $E[\Theta]$  achieve fixed values, respectively, i.e., the probability that the system is empty and the mean busy cycle in the queueing system without vacation corresponding to  $\theta = 1$ . However, when  $\mu_v = 0$ , that is, the model is reduced to the classical discrete-time queue with single vacation, the effect of  $\theta$  can't be ignored any more. On the other hand, Fig.4 shows that when  $\mu_v = 0$ ,  $\theta$  has no effect on  $P_b$ , because customers can only be served by the normal service rate, i.e.,  $P_b = 1$ .

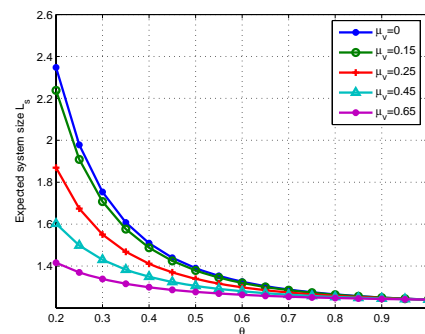
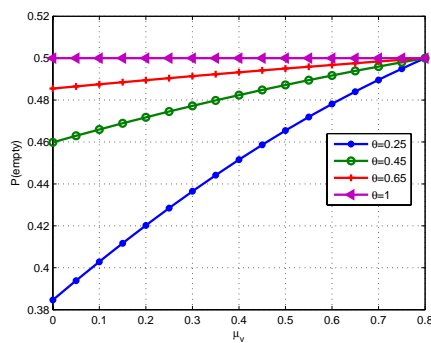
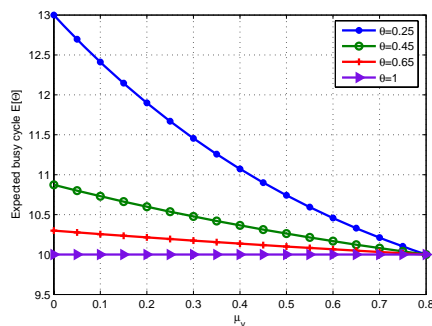
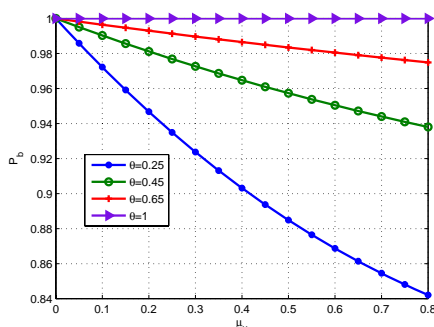


Fig.1  $L_s$  vs.  $\theta$  for different  $\mu_v$ .



Fig.2  $P(\text{empty})$  vs.  $\mu_v$  for different  $\theta$ .Fig.3  $E[\Theta]$  vs.  $\mu_v$  for different  $\theta$ .Fig.4  $P_b$  vs.  $\mu_v$  for different  $\theta$ .

## VII. CONCLUSION

In this paper, we consider a discrete-time  $Geo^X/G/1$  queue with single working vacation. For this model, we obtain the distributions for queue length at departure epochs and at arbitrary epoch, also we discuss some important system characteristics. With these indices of the system, we can model some practical problems in the communication networks and computer and evaluate the performance of those systems. Meanwhile, we also present the stochastic conditional decomposition result for the queue size at a departure epoch and establish the theoretical framework for the  $Geo^X/G/1$  queue with single working vacation. For future research, one could consider the case with single working vacation and vacation interruption.

## ACKNOWLEDGMENT

The authors would like to thank the National Natural Science Foundation of China (No. 10971230).

## REFERENCES

- [1] Baba Y. Analysis of a GI/M/1 queue with multiple working vacations. Oper. Res. Letters 2005; 33: 201-209.
- [2] Chang SH, Choi DW. Performance analysis of a finite-buffer discrete-time queue with bulk arrival, bulk service and vacations. Comput. Oper. Res 2005; 32: 2213-2234.
- [3] Doshi BT. Queueing systems with vacations - a survey. Queueing Syst 1986; (1): 29-66.
- [4] Kim JD, Choi DW, Chae KC. Analysis of queue-length distribution of the M/G/1 queue with working vacations. International Conference on Statistics and Related Fields, Hawaii; 2003.
- [5] Li J, Tian N, Liu W. Discrete-time GI/Geo/1 queue with working vacations. Queueing Syst 2007; 56: 53-63.
- [6] Li J, Tian N. The discrete-time GI/Geo/1 queue with working vacations and vacation interruption. Appl. Math. Comput 2007; 185: 1-10.
- [7] Li J, Tian N, Ma Z. Performance analysis of GI/M/1 queue with working vacations and vacation interruption. Appl. Math. Model 2008; 32: 2715-2730.
- [8] Li J, Tian N, Zhang ZG, Luh H. Analysis of the M/G/1 queue with exponentially working vacations-a matrix analytic approach. Queueing Syst 2009; 61: 139-166.
- [9] Li J, Liu W, Tian N. Steady-state analysis of a discrete-time batch arrival queue with working vacations. Perform. Eval 2010; 67: 897-912.
- [10] Liu W, Xu X, Tian N. Stochastic decompositions in the M/M/1 queue with working vacations. Oper. Res. Lett 2007; 35: 595-600.
- [11] Servi LD, Finn SG. M/M/1 queue with working vacations (M/M/1/WV). Perform. Eval 2002; 50: 41-52.
- [12] Shanthikumar J.G. On stochastic decomposition in M/G/1 type queues with generalized server vacations. Operations Research 1988; 36: 566-569.
- [13] Takagi H. Queueing Analysis - A Foundation of Performance Evaluation Vacation and Priority Systems. vol. 1, North-Holland, New York; 1991.
- [14] Tian N, Zhang ZG. Vacation Queueing Models: Theory and Applications, Springer-Verlag, New York; 2006.
- [15] Tian N, Xu X, Ma Z. Discrete-time queueing theory. Science press, Beijing; 2008.
- [16] Wu D., Takagi H. M/G/1 queue with multiple working vacations. Perform. Eval 2006; 63: 654-681.
- [17] Yi X.W. et al. The Geo/G/1 queue with disasters and multiple working vacations. Stochastic Models 2007; 23: 537-549.
- [18] Zhang M, Hou Z. Performance analysis of M/G/1 queue with working vacations and vacation interruption. J. Comput. Appl. Math 2010; 234(10): 2977-2985.
- [19] Zhang ZG, Tian N. Discrete time Geo/G/1 queue with multiple adaptive vacations. Queueing Syst 2001; 38 (4): 419-430.