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Output Regulation of Perturbed Nonlinear Systems by Nested Sliding Mode Control

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Abstract—In this paper, we consider nested sliding mode control of SISO nonlinear systems, perturbed by bounded matched and unmatched uncertainties. The systems are assumed to be in strict-feedback form. A step wise procedure is introduced to obtain the controller. In each step, a continuous sliding mode controller is designed as virtual control law. Then the next step sliding surface is defined by using this virtual controller. These sliding surfaces are selected as nonlinear static functions of the system states. Finally in the last step, smooth static state feedback control law is determined such that the output reaches the desired set-point while the system is forced arbitrary close to the intersection of sliding surfaces and the states remain bounded.

Keywords—**.** Sliding mode control, Strict-feedback form, Unmatched uncertainty, output regulation**.**

I. Introduction

In the conventional Sliding Mode Control (SMC), the control of an n^{th} order dynamical system is effectively replaced by the controllers design for two lower order systems. Thus each sliding mode controller design contains two steps: 1) Design a discontinuous control term to force the system states onto the sliding mode in finite time (reaching phase), 2) design the sliding surface such that the reached states slide to the origin (sliding motion). Sliding surfaces in SMC can be chosen static linear as in [1]-[4], dynamic linear as in [7], static nonlinear as in [5] and [6], and dynamic nonlinear as in [8].

The ideal sliding mode control provides robustness against matched uncertainties; it means that if a discontinuous switch is used in the SMC (ideal SMC), regardless of any matched uncertainties and disturbances, system states slide to the origin, [1] and [2]. In continuous SMC, effects of matched disturbances and uncertainties in the closed-loop system can be reduced arbitrarily by tuning the continuous switch sharpness, [1] – [3] and [5].

Generally this desirable sliding action can not be maintained, if the uncertainties and disturbances are unmatched. Some control methodologies such as [7] and [8] have been developed to combat this problem. In [7] the problem is addressed for linear systems with the dynamic sliding surface, whereby the so called 'reaching phase' dynamics are of higher order. In [8], higher order sliding mode control is considered for nonlinear systems in strict-feedback form, perturbed by bounded and unmatched disturbances. These methods have the advantage that any external disturbances only affect the system reaching phase, and not the sliding motion. But these methods only concern the external disturbances and do not guarantee the closed-loop stability or any other performance in the presence of unmatched uncertainties.

This paper introduces a new methodology to achieve set-point regulation with arbitrary attenuation level against any bounded matched and unmatched uncertainties for the nonlinear systems in strict-feedback form. The method is based on the continuous sliding mode control and backstepping design. In each step of the design procedure, a continuous sliding mode controller is designed as virtual control law for the next state. In this method any unmatched uncertainty can be seen as a matched uncertainty due to its virtual control input. In the last step, the obtained control law attenuates the effects of all uncertainties. The attenuation level of each uncertainty depends on the sharpness of its related continuous switch. All the sliding surfaces in this procedure are selected as nonlinear static functions as in [5].

The reminder of the paper is organized as follows: section II contains the problem formulation and used notations. In section III the design procedure of the nested sliding mode controller is introduced and discussed. Finally, in section IV an analytic example is employed to show the effectiveness of the proposed method which clearly indicates the advantages gained by the nested SMC. The paper ends with some concluding remarks in Section V.

II. Problem Statement

Suppose the perturbed nonlinear system is in the strict-feedback form (1).

$$\begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 + \delta_1(x_1) \\ \quad\vdots \\ \dot{x}_i = f_i(x_1,\ldots,x_i) + g_i(x_1,\ldots,x_i)x_{i+1} + \delta_i(x_1,\ldots,x_i) \\ \quad\vdots \\ \dot{x}_n = f_n(x_1,\ldots,x_n) + g_n(x_1,\ldots,x_n)u + \delta_n(x_1,\ldots,x_n) \\ y = x_1 \end{cases} \tag{1}$$

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where $x = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T \in R^n$ is the state vector, $y \in R$ and $u \in R$ are the output and control input, respectively. $f_i, g_i : R^i \to R$ for $i = 1,\ldots,n$ are known and sufficiently smooth functions, and $g_i(x_1,\ldots,x_i) \neq 0$, $\forall \begin{bmatrix} x_1 & \ldots & x_i \end{bmatrix}^T \in R^i$. $\delta_i : R^i \to R$; $i = 1,\ldots,n$ are bounded uncertainties. δ_i ; $i = 1,\ldots,n-1$ are called unmatched uncertainties and δ_n is called matched uncertainties.

The goal is to design the control law that regulates the output y of system (1) to arbitrary small neighborhood of the set-point y_d. Also state boundedness is desired.

Before proceeding, it is important to formally list the notation and terminology used through the paper.

- The i^{th} subsystem will be presented by the following functions and vectors definitions:

$$x_i = \begin{bmatrix} x_1 & \cdots & x_i \end{bmatrix}^T$$

$$\delta_i = \begin{bmatrix} \delta_1 & \cdots & \delta_i \end{bmatrix}^T$$

$$f_i = \begin{bmatrix} f_1 + g_1 x_2 & \cdots & f_{i-1} + g_{i-1} x_i & f_i \end{bmatrix}^T$$

$$g_i = \begin{bmatrix} 0 & 0 & \cdots & g_i \end{bmatrix}^T$$

- Continuous switch function used instead of signum in design steps is defined by (2) and depicted in Fig.1.

$$sigm(x) = \frac{2}{\pi} \tan^{-1}(x) \tag{2}$$

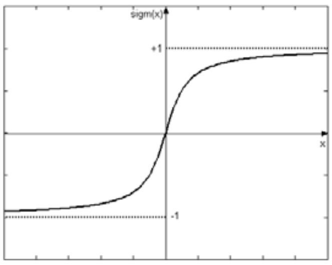

Fig1. $sigm(x)$ as a continuous switch function

III. Design Procedure

Design of Nested SMC for system (1) consists of n steps. In the first step sliding surface is chosen as the difference between output and desired set-point to guarantee output regulation. In each of the other steps, continuous sliding mode controller with nonlinear sliding surface is designed as virtual controllers. The difference between this virtual control law and the next state forms the next step sliding surface.

Step1: Suppose the dynamic of $x_1 = x_1$ subsystem (3) with virtual control signal $x_2 = \varphi_1(x_1)$ and sliding surface (4).

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\varphi_1(x_1) + \delta_1(x_1) \tag{3}$$

$$s_1 = y - y_d = x_1 - y_d \tag{4}$$

in this system δ_1 is matched uncertainty. If a sufficiently smooth function $\beta_1(x_1)$ that satisfies inequality (5) exists, the continuous sliding mode control law for this system will be (6).

$$\beta_1(x_1) \geq \left|\delta_1(x_1)\right| + b_1 \tag{5}$$

where b_1 is an arbitrary positive scalar.

$$\varphi_1(x_1) = g_1^{-1}(x_1)\left(-f_1(x_1) - \beta_1(x_1) sigm(\varepsilon_1 s_1)\right) \tag{6}$$

where $\varepsilon_1 > 0$ is a parameter for tuning the sharpness and smoothness of the continuous switching function. It means by increasing ε_1, the continuous function $sigm(\varepsilon_1 s_1)$ approaches the ideal discontinuous signum nonlinearity $\operatorname{sgn}(\varepsilon_1 s_1)$.

Lyapunov function $V_1 = 1/2 s_1^2$ shows the robust stability of system (3) with controller (6).

$$\dot{V}_1 = s_1(f_1 + g_1\varphi_1 + \delta_1) \tag{7}$$

By replacing (6) in (7),

$$\dot{V}_1 = s_1(-\beta_1 sigm(\varepsilon_1 s_1) + \delta_1) \tag{8}$$

if ε_1 are chosen sufficiently large, $sigm(\varepsilon_1 s_1)$ could be approximated by $\operatorname{sgn}(x_1)$. Thus:

$$\begin{aligned} \dot{V}_1 &\simeq s_1\left(-\beta_1 \operatorname{sgn}(s_1) + \delta_1\right) \\ &\leq \left|s_1\right|\left(-\beta_1 + \left|\delta_1\right|\right) \end{aligned}$$

Using inequality (5) leads to:

$$\dot{V}_1 \leq -b_1 \left|s_1\right| \tag{9}$$

(9) shows that y converges to y_d (x_1 met the sliding surface s_1) in finite time. It can be easily shown this time is less than $\left|x_i(0)\right|/b_1$.

Now if $x_2 \equiv \varphi_1(x_1)$ the x_1 subsystem will be robustly regulated. Thus $s_2 = x_2 - \varphi_1(x_1)$ is selected for the next sliding surface (step2).

Step i: Suppose x_i subsystem (10) and set $x_{i+1} = \varphi_i(x_i)$ as virtual control.

$$\begin{cases} \dot{x}_{i-1} = f_{i-1}(x_{i-1}) + g_{i-1}(x_{i-1})\, x_i + \delta_{i-1}(x_{i-1}) \\ \dot{x}_i = f_i(x_i, x_{i-1}) + g_i(x_i, x_{i-1})\, \varphi_i(x_i, x_{i-1}) + \delta_i(x_i, x_{i-1}) \end{cases} \tag{10}$$

in system (10) δ_{i-1} and δ_i are unmatched and matched uncertainties, respectively.

In step *i-1* virtual control law $\varphi_{i-1}(x_{i-1})$ has been designed for x_i, thus (11) is selected for i^{th} sliding surface.

$$s_i = x_i - \varphi_{i-1}(x_{i-1}) \tag{11}$$

by this surface definition the robust stabilizing controller (12) is obtained.

$$\begin{aligned} \varphi_i(x_i) = g_i^{-1}&\left(-f_i + \frac{\partial \varphi_{i-1}}{\partial x_{i-1}}\left(f_{i-1} + g_{i-1} x_i\right)\right) \\ &- g_i^{-1}\beta_i(x_{i-1}, x_i) sigm(\varepsilon_i S_i) \end{aligned} \tag{12}$$

where β_i is a sufficiently smooth function which satisfies (13).

$$\left|\delta_i - \frac{\partial \varphi_{i-1}}{\partial x_{i-1}} \delta_{i-1}\right| + b_i \le \beta_i (x_{i-1}, x_i) \tag{13}$$

b_i is a positive scalar constant.

Lyapunov function $V_i = S_i^2/2$ can show the states of (10) reach the sliding manifold (11).

$$\dot{V}_i = S_i \left(f_i + g_i \varphi_i + \delta_i - \frac{\partial \varphi_{i-1}}{\partial x_{i-1}} \left(f_{i-1} + g_{i-1} x_i + \delta_{i-1} \right) \right) \tag{14}$$

Replacing $\varphi_i(x_i)$ by (12) after some simple manipulations, (14) leads to:

$$\dot{V}_i = S_i \left(-\beta_i \, sigm(\varepsilon_i S_i) + \delta_i - \frac{\partial \varphi_{i-1}}{\partial x_{i-1}} \delta_{i-1} \right) \tag{15}$$

If ε_i is sufficiently large (15),

$$\dot{V}_i \le |S_i| \left(-\beta_i + \left| \delta_i - \frac{\partial \varphi_{i-1}}{\partial x_{i-1}} \delta_{i-1} \right| \right) \tag{16}$$

then with respect to (13), we have

$$\dot{V}_i \le -b_i |S_i| \tag{17}$$

(17) shows that the states of x_i subsystem converge to surface (11) in a time less than $t_{i\max} = |x_i(0)|/b_i$ and remain on it. This means that after this time $x_i \equiv \varphi_{i-1}(x_{i-1})$ and the effect of δ_i is rejected (when $\varepsilon \to \infty$).

Step n: The design procedure is completed in this step by setting $i = n$ in *step i*. the control law (18) is obtained in this step to force the overall system states to sliding surface $S_n = x_n - \varphi_n(x_{n-1})$.

$$u(x_n) = g_n^{-1} \left(-f_n + \frac{\partial \varphi_{n-1}}{\partial x_{n-1}} \left(f_{n-1} + g_{n-1} x_n \right) \right) - g_n^{-1} \beta_n (x_{n-1}, x_n) sigm(\varepsilon_n S_n) \tag{18}$$

where β_n is a function which satisfies (19) with scalar constant b_n.

$$\left|\delta_n - \frac{\partial \varphi_{n-1}}{\partial x_{n-1}} \delta_{n-1}\right| + b_n \le \beta_n (x_{n-1}, x_n) \tag{19}$$

Now the desired control law is obtained. Some remarks and hints should be noticed.

- Since φ_i s in the first $n-2$ steps of the proposed method should have $n-i^{th}$ bounded derivatives, unlike the sliding mode control, discontinuous switch can not be used even theoretically. Also, in the last two steps discontinuous switches are not used to avoid chattering problem.
- In the i^{th} step of this method, δ_i is seen as matched uncertainty, thus all these uncertainties are rejected by obtained control law (18), when $\varepsilon_i \to \infty$ $i = 1, \ldots, n$. For actual $\varepsilon_i < \infty$ effect of these uncertainties are reduced arbitrarily depend on $\left\| [\varepsilon_1 \ \cdots \ \varepsilon_n] \right\|$.

IV. Example

Consider the third order nonlinear system in strict feedback form, [5].

$$\begin{cases} \dot{x}_1 = \theta_1 x_1^2 + x_2 \\ \dot{x}_2 = \theta_2 \sin x_2 + x_3 \\ \dot{x}_3 = x_1 - x_3 + u \end{cases} \tag{20}$$

Suppose the parameters $\theta_1 \in [0, 2]$, $\theta_2 \in [-1, 1]$ are uncertain, and let $\hat{\theta}_1 = 1$, $\hat{\theta}_2 = 0$ be their known nominal values. The system can be represented in the form

$$\begin{cases} \dot{x}_1 = x_1^2 + x_2 + \delta_1(x_1) \\ \dot{x}_2 = x_3 + \delta_2(x_2) \\ \dot{x}_3 = x_1 - x_3 + u \end{cases} \tag{21}$$

where $\delta_1 = (\theta_1 - 1)x_1^2$ and $\delta_2 = \theta_2 \sin x_2$ are bounded unmatched uncertainties.

Using the design procedure, we have in *step1*:

$$\beta_1 = x_1^2 + b_1$$

$$\varphi_1 = -x_1^2 - \beta_1 sigm(\varepsilon_1 (x_1 - y_d))$$

step2:

$$\beta_2 = b_2 + 1 + x_1^2 \left[2\left(x_1^2 + 1\right) + 2x_1 sigm\left(\varepsilon_1 (x_1 - y_d)\right) + 2|y_d| + \frac{2\varepsilon_1}{\pi} \frac{x_1^2 + b_1}{1 + \varepsilon_1^2 x_1^2} \right]$$

$$\varphi_2 = \frac{\partial \varphi_1}{\partial x_1}\left(x_1^2 + x_2\right) - \beta_2 sigm\left(\varepsilon_2 \left(x_2 - \varphi_1\right)\right)$$

step 3:

$$\beta_3 = b_3 + x_1^2 \left| \frac{\partial \varphi_2}{\partial x_1} \right| + \left| \frac{\partial \varphi_2}{\partial x_2} \right|$$

and nested sliding mode control law is obtained as (22).

$$u = -x_1 + x_3 + \frac{\partial \varphi_2}{\partial x_1}\left(x_1^2 + x_2\right) + x_3 \frac{\partial \varphi_2}{\partial x_2} - \beta_3 sigm\left(\varepsilon_3 \left(x_3 - \varphi_2\right)\right) \tag{22}$$

Fig2. shows the closed-loop response when $y_d = 1$, $\delta_1 = \delta_2 = 0$ and $x(0) = [0 \ \ 0 \ \ 0]^T$. The controller parameters are chosen as $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 10$ and $b_1 = b_2 = b_3 = 1$. This figure illustrates the effectiveness of nested sliding ode controller for the nominal system. The required control cost is $\int_0^4 |u(t)| dt = 18.15$.

Response of the perturbed system by $\delta_1 = x_1^2$ and $\delta_2 = -\sin x_2$ ($\theta_1 = 2, \theta_2 = -1$), are showed in Fig.3. Control parameter and initial conditions are the same as the first simulation. In this situation, output reaches to $y_\infty = 1.17$. the control signal energy is $\int_0^4 |u(t)| dt = 37.8$.

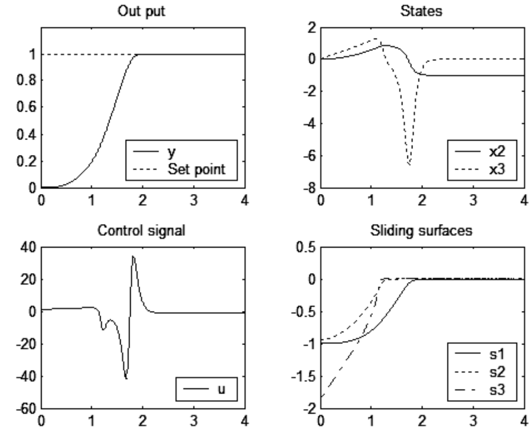

Fig2. Nested sliding mode control response for nominal system with $\varepsilon_i = 10$.

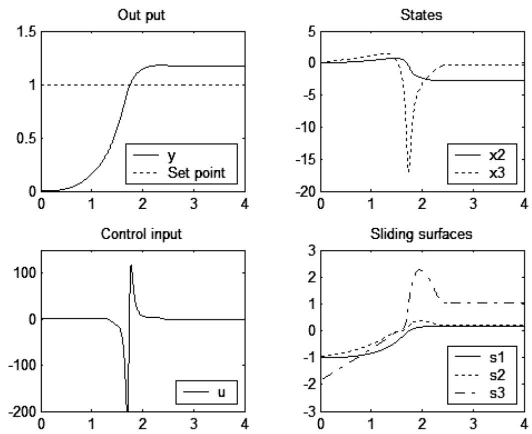

Fig3. Nested sliding mode control response for perturbed system with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 10$.

Second simulation is repeated by setting $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 50$, the results are shown in Fig.4. Final output and required control signal energy are $y_\infty = 1.025$ and $\int_0^4 |u(t)| dt = 172$.

Comparison of second and third simulations clearly illustrates the role of ε_i s in the closed-loop response. Larger ε_i s leads to lower tracking error and Required control cost increases in this situation. Chattering (high frequency signal with limited amplitude) in control signal may occur when too large ε_i s are used. Thus ε_i s should be tuned to make a trade off between control energy and tracking error.

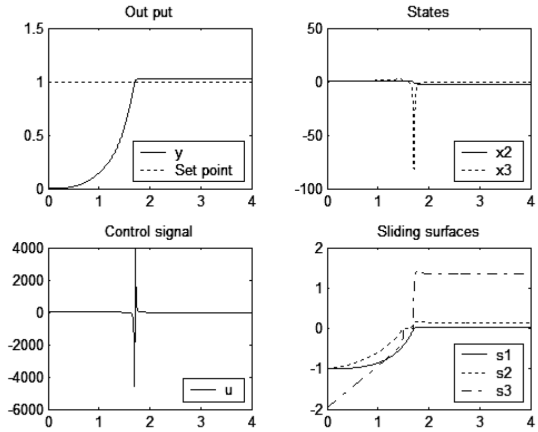

Fig4. Nested sliding mode control response for perturbed system with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 50$.

V. CONCLUSION

Robust control of perturbed nonlinear system in strict feedback form is achieved via introduced method called nested sliding mode control. Step wise algorithm is followed to obtain the controller. In each step, the unmatched uncertainties are seen as a matched one for its virtual controller, thus using virtual sliding mode controller rejects them. The obtained controller in the last step, contains n (n is the system dimension) nested continuous switch functions where each of them reduces the effect of the related unmatched uncertainty. For each switch function and virtual sliding control law, an static nonlinear sliding surface is designed.

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