International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:9, No:5, 2015

Out-of-Plane Free Vibrations of Circular Rods

Faruk Fırat Çalım, Nurullah Karaca, Hakan Tacettin Türker

Abstract**—**In this study, out-of-plane free vibrations of a circular rods is investigated theoretically. The governing equations for naturally twisted and curved spatial rods are obtained using Timoshenko beam theory and rewritten for circular rods. Effects of the axial and shear deformations are considered in the formulations. Ordinary differential equations in scalar form are solved analytically by using transfer matrix method. The circular rods of the mass matrix are obtained by using straight rod of consistent mass matrix. Free vibrations frequencies obtained by solving eigenvalue problem. A computer program coded in MATHEMATICA language is prepared. Circular beams are analyzed through various examples for free vibrations analysis. Results are compared with ANSYS results based on finite element method and available in the literature.

Keywords**—**Circular rod, Out-of-plane free vibration analysis, Transfer Matrix Method.

I. Introduction

CURVED rods are important structural elements that are commonly used for many engineering applications. There are many study about analysis of static and dynamic with straight rod but there is not enough study about analysis of circular rod.

Haktanır and Kıral [1], [2] investigated static and free vibration analysis of helical structures by the transfer and stiffness matrix method. Haktanır [3] investigated static, dynamic and buckling behavior of the helical systems by the transfer and stiffness matrix methods. Lee et al. [4] have presented out-of-plane free vibrations of curved beams with variable curvature. The effects of the rotary and torsional inertias and shear deformation were included. Doğruer and Tüfekçi [5], [6] investigated out-of-plane free vibration of a circular arch with uniform cross-section by using the initial value method. The frequency coefficients are obtained for the first five modes of arches with various slenderness ratios and opening angles. Fang [7] investigated dynamic analysis of structures with uncertain parameters using the transfer matrix method.

Kang et al. [8] applied the differential quadrature method in the computation of the eigenvalues for the in-plane and out-of-plane vibrations of circular arches. Howson and Jennah [9] investigated a method for finding the exact out-of-plane frequencies of curved Timoshenko beams is presented. The effects of shear deformation and rotary inertia due to both torsional and flexural vibrations are included in the equations. Irie et al. [10], investigated the transfer matrix method was used to study the out-of-plane free vibration of Timoshenko arches of constant radius. The results for clamped-clamped arches with circular and square cross-sections are given.

In this study, out-of-plane free vibrations of a circular rod are investigated. The governing equations for circular arch obtained using Timoshenko beam theory. Ordinary differential equations in scalar form are solved analytically by using transfer matrix method. The circular rods of the mass matrix are obtained by using straight rod of consistent mass matrix. The frequency coefficients are obtained for the first three modes of arches with various slenderness ratios and opening angles.

II. Formulation

The material of the rod is assumed to be homogeneous, linear elastic and isotropic. Principal axes of inertia of cross-section are always assumed coincident with **n, b.** The governing equations for circular rod can be obtained from [11], [12].

The out-of-plane behavior of curved rod is formulated except external load by several authors [3], [11], [12]

$$\begin{aligned} &\frac{dU_b}{d\varphi} = -R\Omega_n + R\frac{T_b}{C_{bb}} && \frac{dM_n}{d\varphi} = -M_t + RT_b \\ &\frac{d\Omega_n}{d\varphi} = -\Omega_t + \frac{R}{D_{nn}}M_n && \frac{dT_b}{d\varphi} = 0 \\ &\frac{d\Omega_t}{d\varphi} = \Omega_n + \frac{R}{D_{tt}}M_t && \frac{dM_t}{d\varphi} = M_n \end{aligned} \tag{1}$$

where $C_{bb} = GA/\alpha_b$, $D_{nn} = EI_n$, $D_{tt} = GI_b$ are the material properties. G and E are shear and Young moduli, A the cross-section area, I_n the principal inertia moments and I_b the torsional constant. The six scalar quantities present at any section of the curved rods constitute the elements of the column vector. $\{S(\phi)\}$, known as the state vector

$$\{S(\phi)\} = \{U_b(\phi), \Omega_n(\phi), \Omega_t(\phi), T_b(\phi), M_n(\phi), M_t(\phi)\}^T \tag{2}$$

The system of first-order differential as in (1) for the homogeneous case can be written in matrix notation as

$$\frac{d\{S(\phi)\}}{d\phi} = [D]\ \{S(\phi)\} \tag{3}$$

where [D] is the known differential transfer matrix.

A. Transfer Matrix Method

For the homogeneous case, the matrix relating state vector from $\phi = 0$ to that of any other section defined by ϕ is known as the transfer matrix [F], as given blow [11]

F. F. Çalim is with the Faculty of Engineering, Mustafa Kemal University, CO 31000 Turkey (corresponding author to provide phone: +90 555 271 86 70; e-mail: ffcalim@mku.edu.tr).

N. Karaca, Mr., and H. T. Türker, Mr., are with the Department of Engineering, Mustafa Kemal University, Hatay, Turkey (e-mail: nkaraca@gmail.com, hakantturker@ gmail.com).

$$\{S(\phi)\} = [F(\phi)]\ \{S(0)\} \tag{4}$$

where $\{S(0)\}$ is the state vector $\phi = 0$ and $\{S(\phi)\}$ is the state vector ($\phi = \phi$). If U_b is selected basic variable in (1) and other functions are expressed in terms of its derivatives, the differential equations below which are included axial and shear deformations,

$$\Omega_n = \frac{1}{R}\left[\frac{D}{1+D}\frac{d^5U_b}{d\phi^5} + \frac{2D+1}{1+D}\frac{d^3U_b}{d\phi^3}\right] - \frac{RC_{bb}}{R^2C_{bb}+D_{tt}}\left[\left(\frac{D}{1+D} - \frac{D_{tt}D_{nn}}{R^2C_{bb}(D_{nn}+D_{tt})}\right)\frac{d^5U_b}{d\phi^5} + \left(\frac{2D+1}{1+D} - \frac{D_{tt}D_{nn}}{R^2C_{bb}(D_{nn}+D_{tt})}\right)\frac{d^3U_b}{d\phi^3} + \frac{dU_b}{d\phi}\right] \tag{5a}$$

$$\Omega_t = \frac{1}{R}\left[\frac{D}{1+D}\frac{d^4U_b}{d\phi^4} + \frac{2D+1}{1+D}\frac{d^2U_b}{d\phi^2}\right] \tag{5b}$$

$$T_b = \frac{D_{nn}}{R^3}\frac{D}{1+D}\left[\frac{d^5U_b}{d\phi^5} + \frac{d^3U_b}{d\phi^3}\right] + \frac{D_{tt}C_{bb}}{R^3C_{bb}+RD_{tt}}\left[\left(\frac{D}{1+D} - \frac{D_{tt}D_{nn}}{R^2C_{bb}(D_{nn}+D_{tt})}\right)\frac{d^5U_b}{d\phi^5} + \left(\frac{2D+1}{1+D} - \frac{D_{tt}D_{nn}}{R^2C_{bb}(D_{nn}+D_{tt})}\right)\frac{d^3U_b}{d\phi^3} + \frac{dU_b}{d\phi}\right] \tag{5c}$$

$$M_n = \frac{D}{1+D}\frac{D_{nn}}{R^2}\left[\frac{d^4U_b}{d\phi^4} + \frac{d^2U_b}{d\phi^2}\right] \tag{5d}$$

$$M_t = \frac{D_{tt}C_{bb}}{R^2C_{bb}+D_{tt}}\frac{dU_b}{d\phi} + \frac{D_{tt}C_{bb}}{R^2C_{bb}+D_{tt}}\left[\left(\frac{D}{1+D} - \frac{D_{tt}D_{nn}}{R^2C_{bb}(D_{nn}+D_{tt})}\right)\frac{d^5U_b}{d\phi^5} + \left(\frac{2D+1}{1+D} - \frac{D_{tt}D_{nn}}{R^2C_{bb}(D_{nn}+D_{tt})}\right)\frac{d^3U_b}{d\phi^3}\right] \tag{5e}$$

where D demonstrates D_{tt}/D_{nn}. The following functions have been selected for the homogeneous parts of the solution [11].

$$U_b = U_t = C_1 + C_2\,\phi + C_3\,\mathrm{Sin}\,\phi + C_4\,\mathrm{Cos}\,\phi + C_5\,\phi\,\mathrm{Sin}\phi + C_6\,\phi\,\mathrm{Cos}\,\phi \tag{6}$$

where C_i is the constant of integration. By substituting (6) in (5a-e) with help of a computer program coded in MATHEMATICA language was prepared, transfer matrix [F] is obtained analytically. Hence, the relationship between initial $\phi = 0$, ends $\phi = \phi$ section is obtained transfer matrix analytically. For the nonhomogeneous case, the particular solution due to intermediate discontinuities such as single loads and supports are added to the homogeneous solution, as in (4)

$$\{S(\phi)\} = [F(\phi)]\ \{S(0)\} + \sum_{i=1}^{n}[F(\phi - \beta)]\ \{K(\beta)\} + \int_0^{\phi}[F(\phi - \alpha)]\ \{K(\phi)\}\,d\alpha \tag{7}$$

where n is the number of single external forces and moments acting from the beginning up to the section of concern.

B. Free Vibration Analysis

The undamped free vibration of a linearly elastic system in the form

$$[M]\,\{\ddot{X}\} + [K]\{X\} = \{0\} \tag{8}$$

where K is the system stiffness matrix, M is the system mass matrix and X is the system node displacement. Assuming the following harmonic solution,

$$\{X_i\} = \{a\}\ \mathrm{Sin}\ \omega\ t \tag{9}$$

Equation (8) is reduces to the general eigenvalue problem, by substituting (9) in (8), the solution of the free vibration problem is given by

$$([K] - \omega_i^2\,[M])\,\{a\} = \{0\} \tag{10}$$

a is the amplitude vector of nodal displacements and ω is the angular frequency. A nontrivial solution of set (10) is possible only if the characteristic determinant of the coefficients is vanished.

$$\left|[K] - \omega_i^2\,[M]\right| = 0 \tag{11}$$

The values making the determinant zero are the naturel frequencies of the circular rods.

C. Mass Matrix

The circular rods of the mass matrix are created by using straight rod of consistent mass matrix (Fig. 1).

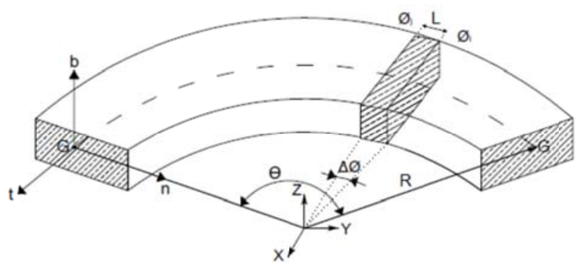

Fig. 1 The geometry of circular rod

The straight rod of consistent mass matrix as

$$[m] = \frac{\rho A L}{420}\begin{bmatrix} 156 & 0 & -22\,L & 54 & 0 & 13\,L \\ 0 & 140\,I_p/A & 0 & 0 & 70\,I_p/A & 0 \\ -22\,L & 0 & 4\,L^2 & -13\,L & 0 & -3\,L^2 \\ 54 & 0 & -13\,L & 156 & 0 & 22\,L \\ 0 & 70\,I_p/A & 0 & 0 & 140\,I_p/A & 0 \\ 13\,L & 0 & -3\,L^2 & 22\,L & 0 & 4\,L^2 \end{bmatrix} \tag{12}$$

where A is the cross-section area, ρ is mass density, L is rod length and I_p is the polar moment of inertia.

In the computation of the element mass matrix, the total length of the element L is determined approximately as (Fig. 1)

$$L = R(\Delta\phi) \qquad \Delta\phi = \emptyset_j - \emptyset_i \tag{13}$$

The transformation from the local coordinate axes to the fixed reference system (X, Y, Z) for the element stiffness and mass matrix is

$$[k]_{XYZ} = [T]^T[k][T] \qquad [m]_{XYZ} = [T]^T[m][T] \tag{14}$$

As in (14), [T] is defined as

$$[T]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mathrm{Sin}[\phi_1] & \mathrm{Cos}[\phi_1] & 0 & 0 & 0 \\ 0 & -\mathrm{Cos}[\phi_1] & -\mathrm{Sin}[\phi_1] & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathrm{Sin}[\phi_2] & \mathrm{Cos}[\phi_2] \\ 0 & 0 & 0 & 0 & -\mathrm{Cos}[\phi_2] & -\mathrm{Sin}[\phi_2] \end{bmatrix} \tag{15}$$

TABLE I
THE NON-DIMENSIONAL FREQUENCIES THE FREQUENCY PARAMETERS $\omega R^2\sqrt{\mu/(EI_n)}$

λ	ϴ	Modes	[6]	[8]	[9]	[10]
20	60	1	16.744	16.744	16.743	16.880
		2	36.946	-	36.921	39.700
		3	40.451	-	40.449	40.900
	120	1	4.283	4.283	4.282	4.309
		2	11.691	-	11.690	11.790
		3	22.054	-	22.045	22.500
	180	1	1.777	1.777	1.776	1.791
		2	4.982	-	4.981	5.032
		3	10.134	-	10.133	10.230
λ	**ϴ**	**Modes**	**Yıldırım [15]**		**ANSYS**	**This study**
20	60	1	16.744		16.874	16.832
		2	39.670		39.701	39.901
		3	40.350		40.913	41.04
	120	1	4.300		4.306	4.419
		2	11.738		11.788	11.991
		3	22.333		22.497	22.397
	180	1	1.789		1.789	1.862
		2	5.022		5.029	5.268
		3	10.194		10.225	10.489
λ	**ϴ**	**Modes**	**[6]**	**[8]**	**[9]**	**[10]**
100	60	1	19.402	19.402	19.401	19.450
		2	54.030	-	54.029	54.100
		3	105.648	-	105.650	105.690
	120	1	4.451	4.452	4.451	4.473
		2	12.826	-	12.826	12.890
		3	25.989	-	26.988	26.080
	180	1	1.804	1.805	1.804	1.818
		2	5.198	-	5.198	5.242
		3	10.918	-	10.917	10.990
λ	**ϴ**	**Modes**	**[15]**		**ANSYS**	**This study**
100	60	1	19.450		19.438	19.693
		2	54.100		54.105	54.401
		3	105.690		105.776	105.685
	120	1	4.473		4.469	4.604
		2	12.888		12.881	13.239
		3	26.069		26.059	26.489
	180	1	1.818		1.817	1.894
		2	5.241		5.237	5.506
		3	10.987		10.980	11.374

III. APPLICATIONS

A. Example 1

Circular rod of uniform cross-section with fixed at both ends is analyzed. The properties of the arc as follows $A = 4\times10^{-4}$ m^2, $E = 2.11\times10^{11}$ t/m^2, $\rho = 7850$ kg/m^3, $\nu = 0.3$, $\alpha_b = 1.1$, variable opening angles (ϴ= 60^o, 120^o, 180^o) and slenderness ratio ($\lambda = 20,100$).

Comparison of the results obtained here and literature has been presented Table I.

$$\lambda = 2\frac{D}{d} \tag{16}$$

where d is diameter of circular section and D is diameter of centroid axis of arc, respectively.

The results are obtained here found to be in good agreement with those available results in the literature and ANSYS. The frequency coefficient increases sharply for small opening angle and then decreases slowly for larger opening angles. If the slenderness ratio increases, the frequency coefficient is also increases.

B. Example 2

Circular rod of uniform cross-section with clamped-free ends is treated here. The properties of the arc as follows: ϴ=180^o, R = 0.305 m, $A = 1.1718\times10^{-4}$ m^2, $E = 68.13\times10^9$ N/m^2, $\rho = 7850$ kg/m^3, $\nu = 0.33$, $\alpha_b = 1.2$, $I_t = 1.22\times10^{-9}$ m^4, $I_n = 3.4882\times10^{-9}$ m^4, $I_b = 3.75367\times10^{-9}$ m^4.

Comparison of the results obtained here and literature has been presented Table II.

TABLE II
FREE VIBRATION FREQUENCIES (RAD/S)

Modes	[13]	[13] (Experimental)	[14]	[15]	ANSYS	This study
1	59.60	56.00	54.77	54.80	54.938	54.950
2	-	-	262.63	263.93	267.42	267.49
3	-	-	946.32	953.24	963.34	963.53
4	-	-	2325.78	2341.78	2351.04	2351.17

The results are obtained here found to be in good agreement with those available results in the literature and ANSYS.

IV. CONCLUSION

This paper presents the solution of free out-of-plane vibrations of circular rods theoretically. The effects of the axial and shear deformations are included in the analysis. The frequency coefficient increases sharply for small opening angle and then decreases slowly for larger opening angles. If the slenderness ratio increases, the frequency coefficient is also increases. The examples in the literature are also solved. The solutions are obtained by using the same assumptions in the literature and ANSYS a very good agreement between the results is observed.

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