

# Oscillation criteria for nonlinear second-order damped delay dynamic equations on time scales

Da-Xue Chen and Guang-Hui Liu

**Abstract**—In this paper, we establish several oscillation criteria for the nonlinear second-order damped delay dynamic equation

$$\left(r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t)\right)^\Delta + p(t)|x^{\Delta\sigma}(t)|^{\beta-1}x^{\Delta\sigma}(t) + q(t)f(x(\tau(t))) = 0$$

on an arbitrary time scale  $\mathbb{T}$ , where  $\beta > 0$  is a constant. Our results generalize and improve some known results in which  $\beta > 0$  is a quotient of odd positive integers. Some examples are given to illustrate our main results.

**Keywords**—Oscillation, Damped delay dynamic equation, Time scale.

## I. INTRODUCTION

**I**N this paper, we investigate the oscillation of the nonlinear second-order damped delay dynamic equation

$$\left(r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t)\right)^\Delta + p(t)|x^{\Delta\sigma}(t)|^{\beta-1}x^{\Delta\sigma}(t) + q(t)f(x(\tau(t))) = 0 \quad (1)$$

on an arbitrary time scale  $\mathbb{T}$ , where  $\beta > 0$  is a constant,  $r, p$  and  $q$  are positive rd-continuous functions on time scale interval  $[t_0, \infty)$ , and  $\tau : \mathbb{T} \rightarrow \mathbb{T}$  satisfies  $\tau(t) \leq t$  for  $t \in \mathbb{T}$  and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ . The function  $f \in C(\mathbb{R}, \mathbb{R})$  is assumed to satisfy  $uf(u) > 0$  and  $|f(u)| \geq K|u|^\beta$ , for  $u \neq 0$  and for some  $K > 0$ . Since the oscillatory behavior of solutions near infinity is our primary concern, we make the assumption that  $\sup \mathbb{T} = \infty$ .

Recall that a solution of (1) is a nontrivial real function  $x$  such that  $x \in C_{rd}^1[t_x, \infty)$  and  $r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t) \in C_{rd}^1[t_x, \infty)$  for a certain  $t_x \geq t_0$  and satisfying (1) for  $t \geq t_x$ . Our attention is restricted to those solutions of (1) which exist on the half-line  $[t_x, \infty)$  and satisfy  $\sup\{|x(t)| : t > t_*\} > 0$  for any  $t_* \geq t_x$ . A solution  $x$  of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Recently, much interest has focused on obtaining sufficient conditions for the oscillation and nonoscillation of solutions of different classes of dynamic equations on time scales, and we refer the reader to [1]–[27]. In particular, much work has been done on second-order damped dynamic equations. For example, Guseinov and Kaymakçalan [21] considered the linear damped dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^\Delta(t) + q(t)x(t) = 0, \quad (2)$$

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where  $p$  and  $q$  are positive rd-continuous functions, and established some sufficient conditions for nonoscillation. They proved that if

$$\int_{t_0}^{\infty} p(t)\Delta t < \infty \quad \text{and} \quad \int_{t_0}^{\infty} tq(t)\Delta t < \infty,$$

then (2) is nonoscillatory.

In [22], Erbe et al. considered (2) and the nonlinear damped dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\Delta\sigma}(t) + q(t)(f \circ x^\sigma)(t) = 0, \quad (3)$$

where  $p$  and  $q$  are positive rd-continuous functions and  $f \in C(\mathbb{R}, \mathbb{R})$  is assumed to satisfy

$$\begin{cases} xf(x) > 0 & \text{for } x \neq 0, \\ |f(x)| \geq K|x| & \text{for some } K > 0, \end{cases} \quad (4)$$

and established some sufficient conditions for oscillation by reducing the equations to the self-adjoint form and employing the generalized Riccati transformation technique.

In [23], Erbe and Peterson considered (3) and obtained an oscillation criterion when  $p$  is nonnegative rd-continuous function and

$$f'(x) \geq \frac{f(x)}{x} \geq \lambda > 0 \quad \text{for } |x| \geq L > 0. \quad (5)$$

No explicit sign assumptions are made with respect to the coefficient  $q$  and the oscillation criterion is obtained by comparing the oscillation of (3) with the self-adjoint equation

$$(e_p(t, t_0)x^\Delta(t))^\Delta + \lambda e_p(t, t_0)q(t)x^\sigma(t) = 0,$$

when  $\int_{t_0}^{\infty} e_{-p}(t, t_0)\Delta t = \infty$ .

Bohner et al. [24] considered (3) when

$$f'(x) > 0 \quad \text{and} \quad xf(x) > 0 \quad \text{for } x \neq 0 \quad (6)$$

and established some new oscillation criteria in which no explicit sign assumptions on  $p$  and  $q$  are required. The results are obtained by reducing the equation to the nonlinear equation

$$(e_p(t, t_0)x^\Delta(t))^\Delta + e_p(t, t_0)q(t)(f \circ x^\sigma)(t) = 0.$$

Saker et al. [25] gave some oscillation criteria for the second-order nonlinear damped dynamic equation

$$(r(t)x^\Delta(t))^\Delta + p(t)x^{\Delta\sigma}(t) + q(t)(f \circ x^\sigma)(t) = 0,$$

where  $r, p$  and  $q$  are positive rd-continuous functions and  $f \in C(\mathbb{R}, \mathbb{R})$  satisfies (4) or

$$f'(x) \geq K \quad \text{for } x \neq 0 \text{ and some } K > 0.$$

The results are essentially new and complement the nonoscillation conditions for (2) for the linear case that has been established in [21].

Very recently, Erbe et al. [26] presented several oscillation criteria for the second-order nonlinear damped delay dynamic equation

$$\left(r(t)(x^\Delta(t))^\beta\right)^\Delta + p(t)(x^\Delta(\sigma(t)))^\beta + q(t)f(x(\tau(t))) = 0,$$

where  $\beta$  is a quotient of odd positive integers.

However, all the results in [21]–[25] cannot be applied to (1) when  $\beta \neq 1$ . Also, the results in [26] cannot be applied to (1) when  $\beta$  is not equal to a quotient of odd positive integers. Furthermore, in the case when  $f(x) = x\left(\frac{1}{9} + \frac{1}{1+x^2}\right)$ , the conditions (5) and (6) do not hold and the results in [23], [24] cannot be applied, since  $f'(x) = \frac{(x^2-2)(x^2-5)}{9(1+x^2)^2}$  changes sign four times (see Saker et al. [25]). Therefore, it is of great interest to study (1) when  $\beta > 0$  is a constant. The main goal of this paper is to establish some new criteria for the oscillation of (1) when  $\beta > 0$  is a constant. Our results are essentially new and extend and improve the results in [21]–[26].

This paper is organized as follows: In the next section, we present some preliminaries on time scales and several lemmas which enable us to prove our main results. In Section 3, we establish several new oscillation criteria for (1). In the last section, we illustrate our results with some examples to which the oscillation criteria in [21]–[26] fail to apply.

In what follows, for convenience, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large  $t$ .

## II. PRELIMINARIES ON TIME SCALES AND LEMMAS

For completeness, we recall the following concepts related to the notion of time scales. More details can be found in [5], [6].

A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of the real numbers  $\mathbb{R}$ . We assume throughout that  $\mathbb{T}$  has the topology that it inherits from the standard topology on the real numbers  $\mathbb{R}$ . Some examples of time scales are as follows: the real numbers  $\mathbb{R}$ , the integers  $\mathbb{Z}$ , the positive integers  $\mathbb{N}$ , the nonnegative integers  $\mathbb{N}_0$ ,  $[0, 1] \cup [2, 3]$ ,  $[0, 1] \cup \mathbb{N}$ ,  $h\mathbb{Z} := \{hk : k \in \mathbb{Z}, h > 0\}$  and  $\overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}, q > 1\} \cup \{0\}$ . But the rational numbers  $\mathbb{Q}$ , the complex numbers  $\mathbb{C}$  and the open interval  $(0, 1)$  are no time scales. Many other interesting time scales exist, and they give rise to plenty of applications (see [5]).

For  $t \in \mathbb{T}$ , the forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where  $\inf \emptyset = \sup \mathbb{T}$  (i.e.,  $\sigma(t) = t$  if  $\mathbb{T}$  has a maximum  $t$ ) and  $\sup \emptyset = \inf \mathbb{T}$  (i.e.,  $\rho(t) = t$  if  $\mathbb{T}$  has a minimum  $t$ ), here  $\emptyset$  denotes the empty set.

Let  $t \in \mathbb{T}$ . If  $\sigma(t) > t$ , we say that  $t$  is right-scattered, while if  $\rho(t) < t$ , we say that  $t$  is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if  $t < \sup \mathbb{T}$  and  $\sigma(t) = t$ , then  $t$  is called right-dense,

and if  $t > \inf \mathbb{T}$  and  $\rho(t) = t$ , then  $t$  is called left-dense. The graininess function  $\mu : \mathbb{T} \rightarrow [0, \infty)$  is defined by

$$\mu(t) := \sigma(t) - t.$$

We also need below the set  $\mathbb{T}^\kappa$ : If  $\mathbb{T}$  has a left-scattered maximum  $m$ , then  $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$ . Otherwise,  $\mathbb{T}^\kappa = \mathbb{T}$ . Let  $f : \mathbb{T} \rightarrow \mathbb{R}$ , then we define the function  $f^\sigma : \mathbb{T}^\kappa \rightarrow \mathbb{R}$  by

$$f^\sigma(t) := f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}^\kappa,$$

i.e.,  $f^\sigma := f \circ \sigma$ .

For  $a, b \in \mathbb{T}$  with  $a < b$ , we define the interval  $[a, b]$  in  $\mathbb{T}$  by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals, etc. are defined accordingly.

Fix  $t \in \mathbb{T}^\kappa$  and let  $f : \mathbb{T} \rightarrow \mathbb{R}$ . Define  $f^\Delta(t)$  to be the number (provided it exists) with the property that given any  $\varepsilon > 0$ , there is a neighbourhood  $U$  of  $t$  such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, we say that  $f^\Delta(t)$  is the (delta) derivative of  $f$  at  $t$  and that  $f$  is (delta) differentiable at  $t$ .

Assume that  $f : \mathbb{T} \rightarrow \mathbb{R}$  and let  $t \in \mathbb{T}^\kappa$ . If  $f$  is (delta) differentiable at  $t$ , then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t). \quad (7)$$

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be right-dense continuous (rd-continuous) provided it is continuous at each right-dense point in  $\mathbb{T}$  and its left-sided limits exist (finite) at all left-dense points in  $\mathbb{T}$ . The set of all such rd-continuous functions is denoted by

$$C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions  $f : \mathbb{T} \rightarrow \mathbb{R}$  that are (delta) differentiable and whose (delta) derivative is rd-continuous is denoted by

$$C_{rd}^1(\mathbb{T}) = C_{rd}^1(\mathbb{T}, \mathbb{R}).$$

We will make use of the following product and quotient rules for the (delta) derivative of the product  $fg$  and the quotient  $f/g$  (where  $gg^\sigma \neq 0$ , here  $g^\sigma = g \circ \sigma$ ) of two (delta) differentiable functions  $f$  and  $g$ :

$$(fg)^\Delta = f^\Delta g^\sigma + fg^\Delta = f^\Delta g + f^\sigma g^\Delta \quad (8)$$

and

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \quad (9)$$

For  $a, b \in \mathbb{T}$  and a (delta) differentiable function  $f$ , the Cauchy (delta) integral of  $f^\Delta$  is defined by

$$\int_a^b f^\Delta(t) \Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f(t)g^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g^\sigma(t) \Delta t \quad (10)$$

or

$$\int_a^b f^\sigma(t)g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(t)\Delta t. \quad (11)$$

The infinite integral is defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t \rightarrow \infty} \int_a^t f(s)\Delta s.$$

Next we present some lemmas which we will need in the proof of our main results.

Throughout this paper, we let

$$E(t) := e_{\frac{p(t)}{r^\sigma(t)}}(t, t_0), \quad g(t) := \int_{\tau(t)}^\infty \left( \frac{1}{E(s)r(s)} \right)^{1/\beta} \Delta s,$$

and

$$\alpha(t, u) := \int_u^{\tau(t)} \left( \frac{1}{r(s)} \right)^{1/\beta} \Delta s / \int_u^{\sigma(t)} \left( \frac{1}{r(s)} \right)^{1/\beta} \Delta s.$$

**Lemma 1.** Suppose that

$$\int_{t_0}^\infty \left( \frac{1}{E(t)r(t)} \right)^{1/\beta} \Delta t = \infty \quad (12)$$

holds and  $x$  is an eventually positive solution of (1). Then  $x^\Delta(t) > 0$  and  $(r(t)(x^\Delta(t))^\beta)^\Delta < 0$ .

**Proof.** Take  $t_1 \geq t_0$  such that

$$x(\tau(t)) > 0, \quad t \in [t_1, \infty). \quad (13)$$

Therefore, from (1) we have

$$\begin{aligned} & (r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t))^\Delta + p(t)|x^{\Delta\sigma}(t)|^{\beta-1}x^{\Delta\sigma}(t) \\ & = -q(t)f(x(\tau(t))) < 0, \quad t \in [t_1, \infty), \end{aligned} \quad (14)$$

which implies

$$\begin{aligned} & (r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t))^\Delta E(t) \\ & + (r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t))^\sigma E(t) \frac{p(t)}{r^\sigma(t)} < 0, \quad t \in [t_1, \infty). \end{aligned}$$

Since  $E^\Delta(t) = E(t) \frac{p(t)}{r^\sigma(t)}$ , we obtain

$$(E(t)r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t))^\Delta < 0, \quad t \in [t_1, \infty). \quad (15)$$

Now, we claim  $x^\Delta(t) > 0$  for  $t \in [t_1, \infty)$ . If not, then there exists  $t_2 \geq t_1$  such that  $x^\Delta(t_2) \leq 0$ . Take  $t_3 > t_2$ . Since (15) implies that  $E(t)r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t)$  is strictly decreasing on  $[t_1, \infty)$ , we get, for  $t \geq t_3$

$$\begin{aligned} & E(t)r(t)|x^\Delta(t)|^{\beta-1}x^\Delta(t) \\ & \leq c_1 := E(t_3)r(t_3)|x^\Delta(t_3)|^{\beta-1}x^\Delta(t_3) \\ & < E(t_2)r(t_2)|x^\Delta(t_2)|^{\beta-1}x^\Delta(t_2) \\ & \leq 0. \end{aligned}$$

Thus, for  $t \geq t_3$  we conclude that  $x^\Delta(t) < 0$ ,  $-E(t)r(t)(-x^\Delta(t))^\beta \leq c_1$  and

$$x^\Delta(t) \leq -(-c_1)^{1/\beta} \left( \frac{1}{E(t)r(t)} \right)^{1/\beta}.$$

Integrating from  $t_3$  to  $t$ , we find

$$x(t) \leq x(t_3) - (-c_1)^{1/\beta} \int_{t_3}^t \left( \frac{1}{E(s)r(s)} \right)^{1/\beta} \Delta s, \quad t \in [t_3, \infty).$$

Therefore, from (12) we obtain that  $x(t)$  is eventually negative, which is a contradiction. Hence, we have  $x^\Delta(t) > 0$  for  $t \in [t_1, \infty)$  and thus, from (14) we get

$$(r(t)(x^\Delta(t))^\beta)^\Delta = -p(t)(x^{\Delta\sigma}(t))^\beta - q(t)f(x(\tau(t))) < 0$$

for  $t \in [t_1, \infty)$ . The proof is complete.  $\square$

**Lemma 2.** Suppose that

$$\int_{t_0}^\infty \left[ \frac{1}{E(v)r(v)} \int_{t_0}^v E(u)q(u)g^\beta(u)\Delta u \right]^{1/\beta} \Delta v = \infty \quad (16)$$

holds and  $x$  is an eventually positive solution of (1). Then  $x^\Delta(t) > 0$  and  $(r(t)(x^\Delta(t))^\beta)^\Delta < 0$ .

**Proof.** We proceed as in the proof of Lemma 1 to get that (13)–(15) hold. Now, we claim  $x^\Delta(t) > 0$  for  $t \in [t_1, \infty)$ . If not, then we proceed as in the proof of Lemma 1 to obtain that there exists  $t_3 \geq t_1$  such that  $x^\Delta(t) < 0$  for  $t \in [t_3, \infty)$ . Take  $t_4 \geq t_3$  such that  $\tau(t) \geq t_3$  for  $t \in [t_4, \infty)$ . Using the fact that  $-E(t)r(t)(-x^\Delta(t))^\beta$  is strictly decreasing on  $[t_1, \infty)$ , we have

$$\begin{aligned} & -x(\tau(t)) \leq x(\infty) - x(\tau(t)) \\ & = - \int_{\tau(t)}^\infty \frac{[E(s)r(s)(-x^\Delta(s))^\beta]^{1/\beta}}{[E(s)r(s)]^{1/\beta}} \Delta s \\ & \leq -[E(\tau(t))r(\tau(t))(-x^\Delta(\tau(t)))^\beta]^{1/\beta} \int_{\tau(t)}^\infty \frac{1}{[E(s)r(s)]^{1/\beta}} \Delta s \\ & \leq -[E(t_3)r(t_3)(-x^\Delta(t_3))^\beta]^{1/\beta} \int_{\tau(t)}^\infty \frac{1}{[E(s)r(s)]^{1/\beta}} \Delta s \\ & = c_2 g(t), \quad t \in [t_4, \infty), \end{aligned}$$

where  $x(\infty) := \lim_{t \rightarrow \infty} x(t) \geq 0$  and  $c_2 := -[E(t_3)r(t_3)(-x^\Delta(t_3))^\beta]^{1/\beta} < 0$ . Thus, from (14) we obtain

$$\begin{aligned} & -(E(t)r(t)(-x^\Delta(t))^\beta)^\Delta = -E(t)q(t)f(x(\tau(t))) \\ & \leq -KE(t)q(t)x^\beta(\tau(t)) \\ & \leq -K(-c_2)^\beta E(t)q(t)g^\beta(t) \end{aligned}$$

for  $t \in [t_4, \infty)$ . Integrating from  $t_4$  to  $t$ , we get

$$\begin{aligned} & -E(t)r(t)(-x^\Delta(t))^\beta \leq -E(t_4)r(t_4)(-x^\Delta(t_4))^\beta \\ & \quad - K(-c_2)^\beta \int_{t_4}^t E(u)q(u)g^\beta(u)\Delta u \\ & \leq -K(-c_2)^\beta \int_{t_4}^t E(u)q(u)g^\beta(u)\Delta u \end{aligned}$$

for  $t \in [t_4, \infty)$ . From the last inequality, we have

$$x^\Delta(t) \leq K^{\frac{1}{\beta}} c_2 \left[ \frac{1}{E(t)r(t)} \int_{t_4}^t E(u)q(u)g^\beta(u)\Delta u \right]^{1/\beta}$$

for  $t \in [t_4, \infty)$ . Integrating from  $t_4$  to  $t$ , we find

$$x(t) \leq x(t_4) + K^{\frac{1}{\beta}} c_2 \int_{t_4}^t \left[ \frac{1}{E(v)r(v)} \int_{t_4}^v E(u)q(u)g^\beta(u)\Delta u \right]^{\frac{1}{\beta}} \Delta v$$

for  $t \in [t_4, \infty)$ . Therefore by (16), we obtain  $\lim_{t \rightarrow \infty} x(t) = -\infty$ , which contradicts the fact that  $x$  is an eventually positive solution of (1). Hence, we have  $x^\Delta(t) > 0$  for  $t \in [t_1, \infty)$  and thus, from (14) we get

$$\left( r(t)(x^\Delta(t))^\beta \right)^\Delta = -p(t)(x^\Delta(t))^\beta - q(t)f(x(\tau(t))) < 0$$

for  $t \in [t_1, \infty)$ . The proof is complete.  $\square$

**Lemma 3.** Assume that

$$\int_{t_0}^{\infty} \left( \frac{1}{E(v)r(v)} \int_{t_0}^v E(u)q(u)\Delta u \right)^{1/\beta} \Delta v = \infty \quad (17)$$

holds and  $x$  is an eventually positive solution of (1). Then either

$$x^\Delta(t) > 0 \quad \text{and} \quad \left( r(t)(x^\Delta(t))^\beta \right)^\Delta < 0$$

or  $\lim_{t \rightarrow \infty} x(t) = 0$ .

**Proof.** We consider two cases: (i)  $x^\Delta(t) > 0$ ; (ii)  $x^\Delta(t) \leq 0$ .

Case (i): suppose  $x^\Delta(t) > 0$ . In this case, it follows from (1) that

$$\left( r(t)(x^\Delta(t))^\beta \right)^\Delta = -p(t)(x^\Delta(t))^\beta - q(t)f(x(\tau(t))) < 0.$$

Case (ii): suppose  $x^\Delta(t) \leq 0$ . In this case, we get  $\lim_{t \rightarrow \infty} x(t) := l \geq 0$  and  $x(\tau(t)) \geq l$ . Therefore, from (1) there exists  $t_1 \geq t_0$  such that

$$\begin{aligned} -\left( E(t)r(t)(-x^\Delta(t))^\beta \right)^\Delta &= -E(t)q(t)f(x(\tau(t))) \\ &\leq -KE(t)q(t)x^\beta(\tau(t)) \\ &\leq -Kl^\beta E(t)q(t), \quad t \in [t_1, \infty). \end{aligned}$$

Integrating from  $t_1$  to  $t$ , we have

$$\begin{aligned} &-E(t)r(t)(-x^\Delta(t))^\beta \\ &\leq -E(t_1)r(t_1)(-x^\Delta(t_1))^\beta - Kl^\beta \int_{t_1}^t E(u)q(u)\Delta u \\ &\leq -Kl^\beta \int_{t_1}^t E(u)q(u)\Delta u, \quad t \in [t_1, \infty). \end{aligned}$$

Thus, for  $t \in [t_1, \infty)$  we obtain

$$x^\Delta(t) \leq -K^{\frac{1}{\beta}} l \left( \frac{1}{E(t)r(t)} \int_{t_1}^t E(u)q(u)\Delta u \right)^{1/\beta}.$$

Integrating from  $t_1$  to  $t$ , we get

$$x(t) \leq x(t_1) - K^{\frac{1}{\beta}} l \int_{t_1}^t \left( \frac{1}{E(v)r(v)} \int_{t_1}^v E(u)q(u)\Delta u \right)^{1/\beta} \Delta v$$

for  $t \in [t_1, \infty)$ . If  $l > 0$ , then by (17) we obtain  $\lim_{t \rightarrow \infty} x(t) = -\infty$ , which contradicts the fact that  $x$  is an eventually positive solution of (1). Hence, we have  $l = 0$ , i.e.,  $\lim_{t \rightarrow \infty} x(t) = 0$ . The proof is complete.  $\square$

**Lemma 4.** Assume that there exists  $T \geq t_0$  such that

$$x(t) > 0, \quad x^\Delta(t) > 0 \quad \text{and} \quad \left( r(t)(x^\Delta(t))^\beta \right)^\Delta < 0$$

for  $t \in [T, \infty)$ . Then  $x(\tau(t)) \geq \alpha(t, T)x^\sigma(t)$  for  $t \in [T_1, \infty)$ , where  $T_1 > T$  satisfies that  $\tau(t) > T$  for  $t \in [T_1, \infty)$ .

**Proof.** Since  $r(t)(x^\Delta(t))^\beta$  is strictly decreasing on  $[T, \infty)$ , for  $t \in [T_1, \infty)$  we have

$$\begin{aligned} x^\sigma(t) - x(\tau(t)) &= \int_{\tau(t)}^{\sigma(t)} \frac{[r(s)(x^\Delta(s))^\beta]^{1/\beta}}{r^{1/\beta}(s)} \Delta s \\ &\leq [r(\tau(t))(x^\Delta(\tau(t)))^\beta]^{1/\beta} \int_{\tau(t)}^{\sigma(t)} \frac{1}{r^{1/\beta}(s)} \Delta s \end{aligned}$$

and

$$\frac{x^\sigma(t)}{x(\tau(t))} \leq 1 + \frac{[r(\tau(t))(x^\Delta(\tau(t)))^\beta]^{1/\beta}}{x(\tau(t))} \int_{\tau(t)}^{\sigma(t)} \frac{1}{r^{1/\beta}(s)} \Delta s. \quad (18)$$

Also, for  $t \in [T_1, \infty)$  we get

$$\begin{aligned} x(\tau(t)) &> x(\tau(t)) - x(T) = \int_T^{\tau(t)} \frac{[r(s)(x^\Delta(s))^\beta]^{1/\beta}}{r^{1/\beta}(s)} \Delta s \\ &\geq [r(\tau(t))(x^\Delta(\tau(t)))^\beta]^{1/\beta} \int_T^{\tau(t)} \frac{1}{r^{1/\beta}(s)} \Delta s \end{aligned}$$

and

$$\frac{[r(\tau(t))(x^\Delta(\tau(t)))^\beta]^{1/\beta}}{x(\tau(t))} < \left( \int_T^{\tau(t)} \frac{1}{r^{1/\beta}(s)} \Delta s \right)^{-1}. \quad (19)$$

Therefore, (18) and (19) imply

$$\begin{aligned} \frac{x^\sigma(t)}{x(\tau(t))} &\leq 1 + \int_{\tau(t)}^{\sigma(t)} \frac{1}{r^{1/\beta}(s)} \Delta s \left( \int_T^{\tau(t)} \frac{1}{r^{1/\beta}(s)} \Delta s \right)^{-1} \\ &= \int_T^{\sigma(t)} \frac{1}{r^{1/\beta}(s)} \Delta s \left( \int_T^{\tau(t)} \frac{1}{r^{1/\beta}(s)} \Delta s \right)^{-1}, \end{aligned}$$

for  $t \in [T_1, \infty)$ . Hence, we obtain  $x(\tau(t)) \geq \alpha(t, T)x^\sigma(t)$  for  $t \geq T_1$ . The proof is complete.  $\square$

**Lemma 5.** (Bohner and Peterson [5], p. 32, Theorem 1.87) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable and suppose  $g : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable. Then  $f \circ g : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable and satisfies

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh \right\} g^\Delta(t).$$

**Lemma 6.** (Hardy et al. [28]) If  $X$  and  $Y$  are nonnegative, then

$$\lambda XY^{\lambda-1} - X^\lambda \leq (\lambda-1)Y^\lambda \quad \text{when } \lambda > 1,$$

where the equality holds if and only if  $X = Y$ .

## III. MAIN RESULTS

**Theorem 1.** Suppose one of the conditions (12) or (16) holds. Furthermore, suppose that there exists a positive  $\Delta$ -differentiable function  $\varphi(t)$  such that for all sufficiently large  $T \geq t_0$ ,

$$\limsup_{t \rightarrow \infty} \int_{T_1}^t \left[ K\alpha^\beta(s, T)\varphi(s)q(s) - \frac{r(s)(Q_+(s))^{\beta+1}}{(\beta+1)^{\beta+1}\varphi^\beta(s)} \right] \Delta s = \infty, \quad (20)$$

where  $T_1 > T$  satisfies that  $\tau(t) > T$  for  $t \in [T_1, \infty)$  and  $Q_+(t) := \max\{0, Q(t)\}$ , here  $Q(t) := \varphi^\Delta(t) - \frac{\varphi(t)p(t)}{r^\sigma(t)}$ . Then every solution of (1) is oscillatory.

**Proof.** Suppose that  $x$  is a nonoscillatory solution of (1). Without loss of generality, we may assume that  $x$  is an eventually positive solution of (1). Then, by Lemma 1 and 2.2 there exists  $T \geq t_0$  such that

$$x(\tau(t)) > 0, \quad x^\Delta(t) > 0 \quad \text{and} \quad \left( r(t)(x^\Delta(t))^\beta \right)^\Delta < 0$$

for  $t \in [T, \infty)$ . Define the function  $w$  by the generalized Riccati substitution

$$w(t) = \varphi(t) \frac{r(t)(x^\Delta(t))^\beta}{x^\beta(t)}. \quad (21)$$

It is easy to see that  $w(t) > 0$  for  $t \in [T, \infty)$ . By the product and quotient rules (8) and (9) for the  $\Delta$ -derivative of two  $\Delta$ -differentiable functions, we have

$$\begin{aligned} w^\Delta(t) &= \varphi^\Delta(t) \left( \frac{r(t)(x^\Delta(t))^\beta}{x^\beta(t)} \right)^\sigma + \varphi(t) \left( \frac{r(t)(x^\Delta(t))^\beta}{x^\beta(t)} \right)^\Delta \\ &= \varphi^\Delta(t) \left( \frac{r(t)(x^\Delta(t))^\beta}{x^\beta(t)} \right)^\sigma + \varphi(t) \frac{[r(t)(x^\Delta(t))^\beta]^\Delta}{(x^\beta(t))^\sigma} \\ &\quad - \varphi(t) \frac{r(t)(x^\Delta(t))^\beta (x^\beta(t))^\Delta}{x^\beta(t)(x^\beta(t))^\sigma}, \quad t \in [T, \infty). \end{aligned}$$

Hence, from (1), (21) and the definition of  $Q(t)$  we obtain

$$\begin{aligned} w^\Delta(t) &\leq \frac{Q(t)}{\varphi^\sigma(t)} w^\sigma(t) - K\varphi(t)q(t) \left( \frac{x(\tau(t))}{x^\sigma(t)} \right)^\beta \\ &\quad - \varphi(t) \frac{r(t)(x^\Delta(t))^\beta (x^\beta(t))^\Delta}{x^\beta(t)(x^\beta(t))^\sigma}, \quad t \in [T, \infty). \end{aligned} \quad (22)$$

Since  $t \leq \sigma(t)$  and  $\left( r(t)(x^\Delta(t))^\beta \right)^\Delta$  is strictly decreasing on  $t \in [T, \infty)$ , we get  $r(t)(x^\Delta(t))^\beta \geq \left( r(t)(x^\Delta(t))^\beta \right)^\sigma$  for  $t \in [T, \infty)$ . Thus, from (21) and (22), we have

$$\begin{aligned} w^\Delta(t) &\leq \frac{Q(t)}{\varphi^\sigma(t)} w^\sigma(t) - K\varphi(t)q(t) \left( \frac{x(\tau(t))}{x^\sigma(t)} \right)^\beta \\ &\quad - \varphi(t) \frac{w^\sigma(t)}{\varphi^\sigma(t)} \frac{(x^\beta(t))^\Delta}{x^\beta(t)}, \quad t \in [T, \infty). \end{aligned}$$

Taking  $T_1 > T$  such that  $\tau(t) > T$  for  $t \in [T_1, \infty)$ , it follows from Lemma 4 that

$$\begin{aligned} w^\Delta(t) &\leq \frac{Q(t)}{\varphi^\sigma(t)} w^\sigma(t) - K\alpha^\beta(t, T)\varphi(t)q(t) \\ &\quad - \varphi(t) \frac{w^\sigma(t)}{\varphi^\sigma(t)} \frac{(x^\beta(t))^\Delta}{x^\beta(t)}, \quad t \in [T_1, \infty). \end{aligned}$$

By Lemma 5, for  $t \in [T_1, \infty)$  we obtain

$$\begin{aligned} (x^\beta(t))^\Delta &= \beta \left\{ \int_0^1 [x(t) + h\mu(t)x^\Delta(t)]^{\beta-1} dh \right\} x^\Delta(t) \\ &= \beta \left\{ \int_0^1 [(1-h)x(t) + hx^\sigma(t)]^{\beta-1} dh \right\} x^\Delta(t) \\ &\geq \begin{cases} \beta(x(t))^{\beta-1}x^\Delta(t), & \beta > 1, \\ \beta(x^\sigma(t))^{\beta-1}x^\Delta(t), & 0 < \beta \leq 1. \end{cases} \end{aligned}$$

Therefore, for  $t \in [T_1, \infty)$ , if  $0 < \beta \leq 1$ , we get

$$\begin{aligned} w^\Delta(t) &\leq \frac{Q(t)}{\varphi^\sigma(t)} w^\sigma(t) - K\alpha^\beta(t, T)\varphi(t)q(t) \\ &\quad - \beta\varphi(t) \frac{w^\sigma(t)}{\varphi^\sigma(t)} \frac{x^\Delta(t)}{x^\sigma(t)} \left( \frac{x^\sigma(t)}{x(t)} \right)^\beta, \end{aligned} \quad (23)$$

whereas if  $\beta > 1$ , we get

$$\begin{aligned} w^\Delta(t) &\leq \frac{Q(t)}{\varphi^\sigma(t)} w^\sigma(t) - K\alpha^\beta(t, T)\varphi(t)q(t) \\ &\quad - \beta\varphi(t) \frac{w^\sigma(t)}{\varphi^\sigma(t)} \frac{x^\Delta(t)}{x^\sigma(t)} \frac{x^\sigma(t)}{x(t)}. \end{aligned} \quad (24)$$

Using the fact that  $x(t)$  is strictly increasing and  $r(t)(x^\Delta(t))^\beta$  is strictly decreasing, we have

$$x^\sigma(t) \geq x(t), \quad x^\Delta(t) \geq \left( \frac{r^\sigma(t)}{r(t)} \right)^{1/\beta} (x^\Delta(t))^\sigma, \quad t \in [T_1, \infty). \quad (25)$$

From (23), (24) and (25), we obtain

$$\begin{aligned} w^\Delta(t) &\leq \frac{Q(t)}{\varphi^\sigma(t)} w^\sigma(t) - K\alpha^\beta(t, T)\varphi(t)q(t) \\ &\quad - \beta\varphi(t) \frac{w^\sigma(t)}{\varphi^\sigma(t)} \left( \frac{r^\sigma(t)}{r(t)} \right)^{1/\beta} \frac{(x^\Delta(t))^\sigma}{x^\sigma(t)}, \quad t \in [T_1, \infty). \end{aligned}$$

In view of (21), we get

$$\begin{aligned} w^\Delta(t) &\leq \frac{Q_+(t)}{\varphi^\sigma(t)} w^\sigma(t) - K\alpha^\beta(t, T)\varphi(t)q(t) \\ &\quad - \frac{\beta\varphi(t)(w^\sigma(t))^\lambda}{(\varphi^\sigma(t))^\lambda r^{1/\beta}(t)}, \quad t \in [T_1, \infty), \end{aligned} \quad (26)$$

where  $\lambda := 1 + \frac{1}{\beta}$ . Taking

$$X = \frac{(\beta\varphi(t))^{1/\lambda} w^\sigma(t)}{\varphi^\sigma(t) r^{1/(\beta+1)}(t)} \quad \text{and} \quad Y = \frac{(\beta r(t))^{1/\lambda} (Q_+(t))^\beta}{(\beta+1)^\beta \varphi^{\beta/\lambda}(t)},$$

by Lemma 6 and (26) we have

$$w^\Delta(t) \leq \frac{r(t)(Q_+(t))^{\beta+1}}{(\beta+1)^{\beta+1}\varphi^\beta(t)} - K\alpha^\beta(t, T)\varphi(t)q(t)$$

for  $t \in [T_1, \infty)$ . Integrating from  $T_1$  to  $t$ , we obtain

$$\begin{aligned} \int_{T_1}^t \left[ K\alpha^\beta(s, T)\varphi(s)q(s) - \frac{r(s)(Q_+(s))^{\beta+1}}{(\beta+1)^{\beta+1}\varphi^\beta(s)} \right] \Delta s \\ \leq w(T_1) - w(t) < w(T_1), \quad t \in [T_1, \infty), \end{aligned}$$

which implies a contradiction to (20). The proof is complete.  $\square$

The following theorem gives a Philos-type oscillation criterion for (1).

**Theorem 2.** Suppose one of the conditions (12) or (16) holds. Furthermore, suppose that there exist a positive function  $\varphi \in C_{rd}^1([t_0, \infty), \mathbb{R})$  and a function  $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ , where  $\mathbb{D} := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t \geq s \geq t_0\}$ , such that

$$H(t, t) = 0 \quad \text{for } t \geq t_0, \quad H(t, s) > 0 \quad \text{for } (t, s) \in \mathbb{D}_0,$$

where  $\mathbb{D}_0 := \{(t, s) \in \mathbb{T} \times \mathbb{T} : t > s \geq t_0\}$ , and  $H$  has a nonpositive rd-continuous delta partial derivative  $H^{\Delta_s}(t, s)$  on  $\mathbb{D}_0$  with respect to the second variable and satisfies, for all sufficiently large  $T \geq t_0$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, T_1)} \int_{T_1}^{t-1} \left[ KH(t, s) \alpha^\beta(s, T) \varphi(s) q(s) - \frac{r(s)(h_+(t, s) \varphi^\sigma(s))^{\beta+1}}{(\beta+1)^{\beta+1} (H(t, s) \varphi(s))^\beta} \right] \Delta s = \infty, \quad (27)$$

where  $T_1$  is defined as in Theorem 1 and  $h_+(t, s) := \max\{0, H^{\Delta_s}(t, s) + H(t, s) \frac{Q_+(s)}{\varphi^\sigma(s)}\}$ , here  $Q_+(t)$  is defined as in Theorem 1. Then all solutions of (1) are oscillatory.

**Proof.** Suppose that  $x$  is a nonoscillatory solution of (1). Without loss of generality, we may assume that  $x$  is an eventually positive solution of (1). We proceed as in the proof of Theorem 1 to get that (26) holds. Multiplying (26) by  $H(t, s)$  and integrating from  $T_1$  to  $t-1$ , we find

$$\begin{aligned} & \int_{T_1}^{t-1} H(t, s) K \alpha^\beta(s, T) \varphi(s) q(s) \Delta s \\ & \leq - \int_{T_1}^{t-1} H(t, s) w^\Delta(s) \Delta s + \int_{T_1}^{t-1} H(t, s) \frac{Q_+(s)}{\varphi^\sigma(s)} w^\sigma(s) \Delta s \\ & \quad - \int_{T_1}^{t-1} H(t, s) \frac{\beta \varphi(s) (w^\sigma(s))^\lambda}{(\varphi^\sigma(s))^\lambda r^{1/\beta}(s)} \Delta s, \quad t \in [T_1 + 1, \infty). \end{aligned} \quad (28)$$

Applying the integration by parts formula (10), for  $t \in [T_1 + 1, \infty)$  we get

$$\begin{aligned} & - \int_{T_1}^{t-1} H(t, s) w^\Delta(s) \Delta s \\ & = \left[ -H(t, s) w(s) \right]_{s=T_1}^{s=t-1} + \int_{T_1}^{t-1} H^{\Delta_s}(t, s) w^\sigma(s) \Delta s \\ & < H(t, T_1) w(T_1) + \int_{T_1}^{t-1} H^{\Delta_s}(t, s) w^\sigma(s) \Delta s. \end{aligned} \quad (29)$$

Substituting (29) in (28), for  $t \in [T_1 + 1, \infty)$  we obtain

$$\begin{aligned} & \int_{T_1}^{t-1} H(t, s) K \alpha^\beta(s, T) \varphi(s) q(s) \Delta s \\ & < H(t, T_1) w(T_1) \\ & \quad + \int_{T_1}^{t-1} \left\{ \left[ H^{\Delta_s}(t, s) + H(t, s) \frac{Q_+(s)}{\varphi^\sigma(s)} \right] w^\sigma(s) - H(t, s) \frac{\beta \varphi(s) (w^\sigma(s))^\lambda}{(\varphi^\sigma(s))^\lambda r^{1/\beta}(s)} \right\} \Delta s \\ & \leq H(t, T_1) w(T_1) \\ & \quad + \int_{T_1}^{t-1} \left[ h_+(t, s) w^\sigma(s) - H(t, s) \frac{\beta \varphi(s) (w^\sigma(s))^\lambda}{(\varphi^\sigma(s))^\lambda r^{1/\beta}(s)} \right] \Delta s. \end{aligned} \quad (30)$$

Therefore by using Lemma 6 in (30) with

$$X = \frac{(H(t, s) \beta \varphi(s))^{1/\lambda} w^\sigma(s)}{\varphi^\sigma(s) r^{1/(\beta+1)}(s)}$$

and

$$Y = \frac{r^{1/\lambda}(s) (h_+(t, s) \varphi^\sigma(s))^\beta}{\lambda^\beta (H(t, s) \beta \varphi(s))^{\beta/\lambda}},$$

we have for  $t \in [T_1 + 1, \infty)$ ,

$$\begin{aligned} & \int_{T_1}^{t-1} H(t, s) K \alpha^\beta(s, T) \varphi(s) q(s) \Delta s \\ & < H(t, T_1) w(T_1) + \int_{T_1}^{t-1} \frac{r(s) (h_+(t, s) \varphi^\sigma(s))^{\beta+1}}{(\beta+1)^{\beta+1} (H(t, s) \varphi(s))^\beta} \Delta s. \end{aligned}$$

Therefore, we obtain for  $t \in [T_1 + 1, \infty)$ ,

$$\begin{aligned} & \frac{1}{H(t, T_1)} \int_{T_1}^{t-1} \left[ H(t, s) K \alpha^\beta(s, T) \varphi(s) q(s) - \frac{r(s) (h_+(t, s) \varphi^\sigma(s))^{\beta+1}}{(\beta+1)^{\beta+1} (H(t, s) \varphi(s))^\beta} \right] \Delta s < w(T_1), \end{aligned}$$

which implies a contradiction to (27). Thus, this completes the proof.  $\square$

**Remark 1.** From Theorems 1 and 2, we can obtain many different sufficient conditions for the oscillation of (1) with different choices of  $\varphi(t)$  and  $H(t, s)$ .

For example, let  $\varphi(t) = t$ , then Theorem 1 yields the following results.

**Corollary 1.** Suppose one of the conditions (12) or (16) holds and for all sufficiently large  $T$ ,

$$\limsup_{t \rightarrow \infty} \int_{T_1}^t \left[ K s \alpha^\beta(s, T) q(s) - \frac{r(s) (V_+(s))^{\beta+1}}{(\beta+1)^{\beta+1} s^\beta} \right] \Delta s = \infty, \quad (31)$$

where  $T_1$  is defined as in Theorem 1 and  $V_+(t) := \max\{0, 1 - \frac{tp(t)}{r^\sigma(t)}\}$ . Then every solution of (1) is oscillatory.

Let  $\varphi(t) = 1$ , then from Theorem 1 we obtain the following results.

**Corollary 2.** Suppose one of the conditions (12) or (16) holds and for all sufficiently large  $T$ ,

$$\int_{T_1}^{\infty} \alpha^\beta(t, T) q(t) \Delta t = \infty, \quad (32)$$

where  $T_1$  is defined as in Theorem 1. Then every solution of (1) is oscillatory.

Let  $\varphi(t) = 1$  and  $H(t, s) = (t-s)^m$ ,  $(t, s) \in \mathbb{D}$ , where  $m \geq 1$  is a constant, then  $H^{\Delta_s}(t, s) \leq -m(t-s)^{m-1} \leq 0$  for  $(t, s) \in \mathbb{D}_0$  (see Saker [27]). Therefore, from Theorem 2 we obtain the following Kamenev-type oscillation criteria for (1).

**Corollary 3.** Suppose one of the conditions (12) or (16) holds and for all sufficiently large  $T$ ,

$$\limsup_{t \rightarrow \infty} \frac{1}{(t - T_1)^m} \int_{T_1}^t (t - s)^m \alpha^\beta(s, T) q(s) \Delta s = \infty, \quad (33)$$

where  $T_1$  is defined as in Theorem 1 and  $m \geq 1$  is a constant. Then all solutions of (1) are oscillatory.

Also, by Lemma 3, we obtain another oscillation criterion for (1) as in Theorems 1 and 2 and Corollaries 1–3 as follows.

**Corollary 4.** Assume that (17) and (20) hold. Then every solution of (1) is oscillatory or tends to zero as  $t \rightarrow \infty$ .

**Corollary 5.** Assume that (17) and (27) hold. Then every solution of (1) is oscillatory or tends to zero as  $t \rightarrow \infty$ .

**Corollary 6.** Assume that (17) and (31) hold. Then every solution of (1) is oscillatory or tends to zero as  $t \rightarrow \infty$ .

**Corollary 7.** Assume that (17) and (32) hold. Then every solution of (1) is oscillatory or tends to zero as  $t \rightarrow \infty$ .

**Corollary 8.** Assume that (17) and (33) hold. Then every solution of (1) is oscillatory or tends to zero as  $t \rightarrow \infty$ .

#### IV. SOME EXAMPLES

**Example 1.** Consider the second-order nonlinear damped delay dynamic equation

$$\begin{aligned} & \left( \frac{t^{\beta-1}}{b(t)} |x^\Delta(t)|^{\beta-1} x^\Delta(t) \right)^\Delta + \frac{(\sigma(t))^{\beta-1}}{tb^\sigma(t)} |x^\Delta(\sigma(t))|^{\beta-1} x^\Delta(\sigma(t)) \\ & + \frac{1}{\alpha^\beta(t, t^*) t^2} |x(\tau(t))|^{\beta-1} x(\tau(t)) = 0, \end{aligned} \quad (34)$$

where  $\beta > 0$  is a constant,  $t^*$  satisfies that  $t_0 > t^* > 0$  and  $\tau(t) > t^*$  for  $t \in [t_0, \infty)$ , and  $b(t) := e_{\frac{1}{t}}(t, t_0)$ . In (34),  $r(t) = \frac{t^{\beta-1}}{b(t)}$ ,  $p(t) = \frac{(\sigma(t))^{\beta-1}}{tb^\sigma(t)}$ ,  $q(t) = \frac{1}{\alpha^\beta(t, t^*) t^2}$  and  $f(u) = |u|^{\beta-1} u$ .

We will apply Corollary 1. We have

$$\int_{t_0}^{\infty} \left( \frac{1}{E(t)r(t)} \right)^{1/\beta} \Delta t = \int_{t_0}^{\infty} \frac{1}{t^{1-\frac{1}{\beta}}} \Delta t = \infty,$$

which implies (12) holds. Furthermore, we see that  $V_+(t) := \max\{0, 1 - \frac{tp(t)}{r^\sigma(t)}\} = 0$  and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{T_1}^t \left[ K s \alpha^\beta(s, T) q(s) - \frac{r(s)(V_+(s))^{\beta+1}}{(\beta+1)^{\beta+1} s^\beta} \right] \Delta s \\ & = \limsup_{t \rightarrow \infty} \int_{T_1}^t K \left( \frac{\alpha(s, T)}{\alpha(s, t^*)} \right)^\beta \frac{1}{s} \Delta s \\ & = \limsup_{t \rightarrow \infty} \int_{T_1}^t K \frac{1}{s} \Delta s \\ & = \infty, \end{aligned}$$

since  $\lim_{t \rightarrow \infty} \frac{\alpha(t, T)}{\alpha(t, t^*)} = 1$ . Therefore by Corollary 1, every solution of (34) is oscillatory.

**Example 2.** Consider the second-order nonlinear damped delay dynamic equation

$$\begin{aligned} & \left( \frac{(t\sigma(t))^\beta}{b(t)} |x^\Delta(t)|^{\beta-1} x^\Delta(t) \right)^\Delta + \frac{(\sigma(t))^\beta}{b^\sigma(t)} |x^\Delta(\sigma(t))|^{\beta-1} x^\Delta(\sigma(t)) \\ & + \frac{t^\beta}{\alpha^\beta(t, t^*)} |x(\tau(t))|^{\beta-1} x(\tau(t)) = 0, \end{aligned} \quad (35)$$

where  $0 < \beta \leq 1$  is a constant,  $t^*$  satisfies that  $t_0 > t^* > 0$  and  $\tau(t) > t^*$  for  $t \in [t_0, \infty)$ ,  $b(t) := e_{\frac{1}{(\sigma(\sigma(t)))^\beta}}(t, t_0)$  and we suppose

$$\int_{t_0}^{\infty} \frac{1}{t^{1-\frac{1}{\beta}} \sigma(t)} \Delta t = \infty, \quad (36)$$

for those time scales  $\mathbb{T}$ . This holds for many time scales, for example when  $\mathbb{T} = q_0^{\mathbb{N}_0} := \{q_0^k : k \in \mathbb{N}_0, q_0 > 1\}$ .

In (35),  $r(t) = \frac{(t\sigma(t))^\beta}{b(t)}$ ,  $p(t) = \frac{(\sigma(t))^\beta}{b^\sigma(t)}$ ,  $q(t) = \frac{t^\beta}{\alpha^\beta(t, t^*)}$  and  $f(u) = |u|^{\beta-1} u$ . It is clear that

$$\begin{aligned} \int_{t_0}^{\infty} \left( \frac{1}{E(t)r(t)} \right)^{1/\beta} \Delta t & = \int_{t_0}^{\infty} \frac{1}{t\sigma(t)} \Delta t = \int_{t_0}^{\infty} \left( -\frac{1}{t} \right)^\Delta \Delta t \\ & = \frac{1}{t_0} < \infty, \end{aligned}$$

which implies (12) does not hold. Now we prove that (16) holds. We have

$$\begin{aligned} & \int_{t_0}^{\infty} \left[ \frac{1}{E(v)r(v)} \int_{t_0}^v E(u)q(u)g^\beta(u)\Delta u \right]^{1/\beta} \Delta v \\ & \geq \int_{t_0}^{\infty} \left[ \frac{1}{(v\sigma(v))^\beta} \int_{t_0}^v \left( \frac{ug(u)}{\alpha(u, t^*)} \right)^\beta \Delta u \right]^{1/\beta} \Delta v \\ & \geq \int_{t_0}^{\infty} \left[ \frac{v - t_0}{(v\sigma(v))^\beta} \right]^{1/\beta} \Delta v, \end{aligned} \quad (37)$$

since for  $t \in [t_0, \infty)$ ,  $E(t) := e_{\frac{1}{(\sigma(\sigma(t)))^\beta}}(t, t_0) \geq 1$  and

$$\begin{aligned} g(t) & := \int_{\tau(t)}^{\infty} \left( \frac{1}{E(s)r(s)} \right)^{1/\beta} \Delta s = \int_{\tau(t)}^{\infty} \frac{1}{s\sigma(s)} \Delta s \\ & = \int_{\tau(t)}^{\infty} \left( -\frac{1}{s} \right)^\Delta \Delta s = \frac{1}{\tau(t)} \geq \frac{1}{t} \geq \frac{\alpha(t, t^*)}{t}. \end{aligned}$$

Take  $0 < c < 1$  such that  $t - t_0 > ct$  for  $t \geq t_c > t_0$ , then from (37) and (36) we obtain

$$\begin{aligned} & \int_{t_0}^{\infty} \left[ \frac{1}{E(v)r(v)} \int_{t_0}^v E(u)q(u)g^\beta(u)\Delta u \right]^{1/\beta} \Delta v \\ & \geq c^{\frac{1}{\beta}} \int_{t_c}^{\infty} \frac{1}{v^{1-\frac{1}{\beta}} \sigma(v)} \Delta v = \infty, \end{aligned}$$

which implies (16) holds. To apply Corollary 2, it remains to prove that (32) holds. We get

$$\begin{aligned} \int_{T_1}^{\infty} \alpha^\beta(t, T) q(t) \Delta t & = \int_{T_1}^{\infty} \left( \frac{\alpha(t, T)}{\alpha(t, t^*)} \right)^\beta t^\beta \Delta t \\ & = \int_{T_1}^{\infty} t^\beta \Delta t = \infty, \end{aligned}$$

since  $\lim_{t \rightarrow \infty} \frac{\alpha(t, T)}{\alpha(t, t^*)} = 1$ . Thus, for those time scales where (36) holds, every solution of (35) is oscillatory by Corollary 2.

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