International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:6, No:8, 2012

Orthogonal Functions Approach to LQG Control

B. M. Mohan *Senior Member, IEEE,* and Sanjeeb Kumar Kar

Abstract—In this paper a unified approach via block-pulse functions (BPFs) or shifted Legendre polynomials (SLPs) is presented to solve the linear-quadratic-Gaussian (LQG) control problem. Also a recursive algorithm is proposed to solve the above problem via BPFs. By using the elegant operational properties of orthogonal functions (BPFs or SLPs) these computationally attractive algorithms are developed. To demonstrate the validity of the proposed approaches a numerical example is included.

Keywords—Linear quadratic Gaussian control; Linear quadratic estimator; Linear quadratic regulator; Time-invariant systems; Orthogonal functions; Block-pulse functions; Shifted Legendre polynomials.

I. Introduction

THE LQG control problem [1] concerns linear systems disturbed by additive white Gaussian noise, incomplete state information and quadratic costs. The LQG controller is simply the combination of a linear-quadratic estimator (LQE), i.e. Kalman filter with a linear quadratic regulator (LQR). The separation principle guarantees that these can be designed and computed independently.

Orthogonal functions approach [8], [9] has been recognized as an efficient and useful approach computationally to solve variety of problems in systems and control. In [6] the solution of the LQG control design problem was obtained by employing general orthogonal polynomials. In [10] the authors considered the problem of LQG control system and showed its application as an information transmission problem. The discrete time LQG problem was considered in [11] and showed its applications over lossy data networks.

Very recently, applications of orthogonal function approach is extended to different type of systems, i.e. systems described by integro-differential equations [12], multi-delay systems [13], [17], distributed parameter systems [14], delay systems with reverse time functions [15], singular systems [16] and to nonlinear systems [18].

In this paper, we consider linear time-invariant systems and propose a unified approach, based on using BPFs or SLPs, to solve LQG control problem of such systems. We call this approach unified approach because it can be used via SLPs or BPFs. In addition to the unified approach a recursive algorithm is proposed using BPFs. It is very important to note that the LQG control problem is not yet studied via BPFs.

The paper is organized as follows : The next section deals with BPFs and SLPs, and their properties. The LQG control design problem is discussed in Section 3. The method of obtaining solution of the LQG control design problem is presented in Section 4. A numerical example is considered in Section 5. The last section concludes the paper.

B. M. Mohan is with the Department of Electrical Engineering, IIT Kharagpur, India e-mail:(mohan@ee.iitkgp.ernet.in.)
Sanjeeb Kumar Kar is with the Department of Electrical Engineering, Institute of Technical Education and Research, SOA University, Bhubaneswar-751030, INDIA e-mail: (skkariter@rediffmail.com).

Manuscript received March 2011; revised

II. Orthogonal Functions and Their Properties

We consider two classes of orthogonal functions, namely BPFs and SLPs, and discuss their properties.

A. *BPFs and their properties [4], [7]*

A set of m BPFs, orthogonal over $t \in [t_0, t_f)$, is defined as

$$B_i(t) = \begin{cases} 1, & t_0 + iT \le t < t_0 + (i+1)T \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

for $i = 0, 1, 2, \dots, m-1,$ where

$$T = \frac{t_f - t_0}{m}, \text{ the block-pulse width} \tag{2}$$

A square integrable function $f(t)$ on $t_0 \le t \le t_f$ can be approximately represented in terms of BPFs as

$$f(t) \approx \sum_{i=0}^{m-1} f_i B_i(t) = \mathbf{f}^T \boldsymbol{B}(t) \tag{3}$$

where

$$\mathbf{f} = \left[\; f_0, \;\; f_1, \;\; \dots, \;\; f_{m-1} \; \right]^T \tag{4}$$

is an m - dimensional block-pulse spectrum of $f(t)$, and

$$\boldsymbol{B}(t) = \left[\; B_0(t), \;\; B_1(t), \;\; \dots, \;\; B_{m-1}(t) \; \right]^T \tag{5}$$

an m - dimensional BPF vector. f_i in Eq. (3) is given by

$$f_i = \frac{1}{T} \int_{t_0+iT}^{t_0+(i+1)T} f(t) dt \tag{6}$$

which is the average value of $f(t)$ over $t_0 + iT \le t \le t_0 + (i+1)T$. The product of two BPFs $B_i(t)$ and $B_j(t)$ can be expressed as

$$B_i(t)B_j(t) = \begin{cases} 0 & \text{if} \quad i \ne j \\ B_i(t) & \text{if} \quad i = j \end{cases} \tag{7}$$

Operational matrix of forward integration [4] :
Integrating $\boldsymbol{B}(t)$ from t_0 to t and expressing the result in m-set of BPFs, we have

$$\int_{t_0}^{t} \boldsymbol{B}(\tau) d\tau \approx P_f \boldsymbol{B}(t) \tag{8}$$

where

$$P_f = T \begin{bmatrix} \frac{1}{2} & 1 & 1 & \dots & 1 \\ 0 & \frac{1}{2} & 1 & \dots & 1 \\ 0 & 0 & \frac{1}{2} & \dots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2} \end{bmatrix} \tag{9}$$

is called the operational matrix of forward integration of BPFs and it is an $m \times m$ upper triangular matrix.

Operational matrix of backward integration [7] :

We integrate $\boldsymbol{B}(t)$ from t_f to t and express the result in m-set of BPFs to obtain

$$\int_{t_f}^{t} \boldsymbol{B}(\tau)d\tau \approx P_b\, \boldsymbol{B}(t) \tag{10}$$

where

$$P_b = -T \begin{bmatrix} \frac{1}{2} & 0 & 0 & \dots & 0 \\ 1 & \frac{1}{2} & 0 & \dots & 0 \\ 1 & 1 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & \frac{1}{2} \end{bmatrix} = -P_f^T \tag{11}$$

We call P_b the operational matrix of backward integration of BPFs. It is an $m \times m$ lower triangular matrix.

B. SLPs and their properties [8]

SLPs satisfy the recurrence relation

$$L_{i+1}(t) = \frac{(2i+1)}{(i+1)}\,\varphi(t)\, L_i(t) - \frac{i}{(i+1)}\, L_{i-1}(t) \tag{12}$$

for $i = 1, 2, 3, \dots\dots$ with

$$\varphi(t) = \frac{2(t-t_0)}{(t_f-t_0)} - 1 \tag{13}$$

$$L_0(t) = 1, \text{ and } L_1(t) = \varphi(t) \tag{14}$$

A function $f(t)$ that is square integrable on $t \in [t_0,\ t_f]$ can be represented in terms of SLPs as

$$f(t) \approx \sum_{i=0}^{m-1} f_i L_i(t) = \mathbf{f}^T \mathbf{L}(t) \tag{15}$$

Here $\mathbf{f}$ is called Legendre spectrum of $f(t)$, given in Eq. (4) and

$$\mathbf{L}(t) = \left[\ L_0(t),\ \ L_1(t),\ \ \dots,\ \ L_{m-1}(t)\ \right]^T \tag{16}$$

is called SLP vector. f_i in Eq. (15) is given by

$$f_i = \frac{(2i+1)}{(t_f-t_0)} \int_{t_0}^{t_f} f(t) L_i(t) dt \tag{17}$$

The product of two SLPs $L_i(t)$ and $L_j(t)$ can be expressed as

$$L_i(t) L_j(t) \simeq \sum_{k=0}^{m-1} \psi_{ijk} L_k(t) \tag{18}$$

where

$$\psi_{ijk} = \frac{(2k+1)}{(t_f-t_0)} \pi_{ijk} \tag{19}$$

$$\pi_{ijk} = \pi_{ikj} = \pi_{jik} = \pi_{jki} = \pi_{kji} = \pi_{kij} \tag{20}$$

$$\pi_{ijk} = \begin{cases} \frac{a_l a_{j-l} a_{i-j+l}}{a_{i+l}} \frac{(t_f-t_0)}{2(i+l)+1} & \text{if} \quad k = i-j+2l \\ 0 & \text{if} \quad k \neq i-j+2l \end{cases} \tag{21}$$

for $i \geq j$

$$a_0 = 1, \quad a_{l+1} = \frac{(2l+1)}{(l+1)} a_l \tag{22}$$

for $l = 0, 1, 2, \dots\dots$

Operational matrix of forward integration [8] :

Integrating $\mathbf{L}(t)$ from t_0 to t, and expressing the result in terms of the same set of SLPs, we have

$$\int_{t_0}^{t} \mathbf{L}(\tau)d\tau \approx P_f \mathbf{L}(t) \tag{23}$$

where

$$P_f = \frac{(t_f-t_0)}{2} \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \dots & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2m-3} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{24}$$

which is called the operational matrix of forward integration of SLPs.

Operational matrix of backward integration [5] :

If we integrate $\mathbf{L}(t)$ from t_f to t and express the result in terms of the same set of SLPs, we have

$$\int_{t_f}^{t} \mathbf{L}(\tau)d\tau \approx P_b \mathbf{L}(t) \tag{25}$$

where

$$P_b = \frac{(t_f-t_0)}{2} \begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \dots & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2m-3} \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \tag{26}$$

which is called the operational matrix of backward integration of SLPs.

III. The LQG Control Problem

Consider the linear dynamic system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + \mathbf{v}(t) \tag{27}$$

$$\mathbf{z}(t) = C\mathbf{x}(t) + \mathbf{w}(t) \tag{28}$$

where $\mathbf{x}(t)$ is n dimensional state vector, $\mathbf{u}(t)$ p dimensional control vector, and $\mathbf{z}(t)$ q dimensional output vector, and $\mathbf{v}(t)$ and $\mathbf{w}(t)$ the additive zero-mean white Gaussian system noise and measurement noise, respectively, i.e.

$$\mathrm{E}\left\{\mathbf{v}(t)\mathbf{v}^T(\tau)\right\} = Q_2\, \delta(t-\tau) \tag{29}$$

$$\mathrm{E}\left\{\mathbf{w}(t)\mathbf{w}^T(\tau)\right\} = R_2\, \delta(t-\tau) \tag{30}$$

where Q_2 is positive semi-definite and R_2 is positive definite symmetric matrices. Also $\mathbf{v}(t)$ is uncorrelated with $\mathbf{w}(t)$, i.e.

$$\mathrm{E}\left\{\mathbf{v}(t)\mathbf{w}^T(\tau)\right\} = 0 \tag{31}$$

Assume that the initial condition $\mathbf{x}(t_0)$ is Gaussian with mean $\bar{\mathbf{x}}(t_0)$ and covariance matrix

$$P = P_2(t_0) = \mathrm{E}\left\{[\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)]\,[\mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)]^T\right\}$$

which is symmetric positive semi-definite, and

$$\mathrm{E}\left\{\mathbf{v}(t)\,\mathbf{x}_0^T\right\} \;=\; \mathrm{E}\left\{\mathbf{w}(t)\,\mathbf{x}_0^T\right\} \;=\; 0 \quad \text{for } t \geq t_0 \quad (32)$$

Given this system, the objective is to find the control input $\mathbf{u}(t)$ which at every time t may depend only on the past measurements $\mathbf{z}(t_1),\ t_0 \leq t_1 < t$ such that the cost function

$$\begin{aligned} J &= \mathrm{E}\left\{\frac{1}{2}\mathbf{x}^T(t_f)S\mathbf{x}(t_f) \right.\\ &\left. + \frac{1}{2}\int_{t_0}^{t_f}\left[\mathbf{x}^T(t)Q_1\mathbf{x}(t) + \mathbf{u}^T(t)R_1\mathbf{u}(t)\right]dt\right\} \quad (33)\end{aligned}$$

is minimized, where the matrix R_1 is positive definite symmetric, and S and Q_1 are symmetric positive semi-definite matrices.

The LQG controller that solves the LQG control problem is specified by the equations

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + K_2(t)\left[\mathbf{z}(t) - C\hat{\mathbf{x}}(t)\right] \quad (34)\\ \hat{\mathbf{x}}(t_0) &= \mathrm{E}\left[\mathbf{x}(t_0)\right] \;=\; \bar{\mathbf{x}}_0 \\ \mathbf{u}(t) &= -K_1(t)\hat{\mathbf{x}}(t) \quad (35)\end{aligned}$$

The matrix $K_2(t)$ is called the Kalman gain of the associated Kalman filter represented by Eq. (34). At each time t this filter generates estimates $\hat{\mathbf{x}}(t)$ of the state $\mathbf{x}(t)$ using the past measurements and inputs. The Kalman gain is determined through the associated matrix Riccati differential equation

$$\begin{aligned} \dot{P}_2(t) &= AP_2(t) + P_2(t)A^T \\ &\quad -P_2(t)C^TR_2^{-1}CP_2(t) + Q_2 \quad (36)\\ P_2(t_0) &= P \end{aligned}$$

Given the solution $P_2(t),\ t_0 \leq t \leq t_f$ the Kalman gain equals

$$K_2(t) \;=\; P_2(t)C^TR_2^{-1} \quad (37)$$

The matrix $K_1(t)$ is called the feedback gain matrix which is determined through the associated matrix Riccati differential equation

$$\begin{aligned} -\dot{P}_1(t) &= A^TP_1(t) + P_1(t)A \\ &\quad -P_1(t)BR_1^{-1}B^TP_1(t) + Q_1 \quad (38)\\ P_1(t_f) &= S \end{aligned}$$

Given the solution $P_1(t),\ t_0 \leq t \leq t_f$ the feedback gain equals

$$K_1(t) \;=\; R_1^{-1}B^TP_1(t) \quad (39)$$

Observe the similarity of the two matrix Riccati differential Equations (36) and (38); the first one running forward in time, and the second one running backward in time. The first one solves the LQE problem and the second one solves LQR problem. So the LQG problem separates into LQE and LQR problems that can be solved independently.

The block diagram of the LQG problem is presented in Figure 1.

IV. Orthogonal Functions Approach

Integrating the Riccati equation (38) backward in time from t_f to t, we obtain

$$\begin{aligned} -\left[P_1(t) - S\right] &= \int_{t_f}^{t}\left[A^TP_1(\tau) + P_1(\tau)A \right.\\ &\quad \left. -P_1(\tau)FP_1(\tau) + Q_1\right]d\tau \quad (40)\end{aligned}$$

where $F = BR_1^{-1}B^T$. Expressing $P_1(t)$, $P_1(t)FP_1(t)$, Q_1 and S in terms of orthogonal functions $\{\phi_i(t)\}$, which may be BPFs $\{B_i(t)\}$ or SLPs $\{L_i(t)\}$, we have

$$P_1(t) \;\simeq\; \sum_{i=0}^{m-1}P_{1i}\phi_i(t) \;=\; \tilde{P}_1\left(\boldsymbol{\phi}(t)\otimes I_n\right) \quad (41)$$

where

$$\tilde{P}_1 \;=\; \left[\; P_{10},\;\; P_{11},\;\; \ldots,\;\; P_{1,m-1} \;\right] \quad (42)$$

Then

$$\begin{aligned} A^TP_1(t) &\simeq \bar{P}_1\left(\boldsymbol{\phi}(t)\otimes I_n\right) \quad (43)\\ P_1(t)A &\simeq \hat{P}_1\left(\boldsymbol{\phi}(t)\otimes I_n\right) \quad (44)\end{aligned}$$

where

$$\begin{aligned} \bar{P}_1 &= \left[\; A^TP_{10},\;\; A^TP_{11},\;\; \ldots,\;\; A^TP_{1,m-1} \;\right] \quad (45)\\ \hat{P}_1 &= \left[\; P_{10}A,\;\; P_{11}A,\;\; \ldots,\;\; P_{1,m-1}A \;\right] \quad (46)\end{aligned}$$

$$\begin{aligned} P_1(t)FP_1(t) &\simeq \sum_{i=0}^{m-1}P_{1i}FP_{1i}\,\phi_i(t) \quad (47)\\ &\quad \text{if BPFs are used} \\ &\simeq \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\sum_{k=0}^{m-1}P_{1i}FP_{1j}\,\psi_{ijk}\,\phi_k(t) \quad (48)\\ &\quad \text{if SLPs are used} \\ &\simeq \tilde{F}\left(\boldsymbol{\phi}(t)\otimes I_n\right) \quad (49)\end{aligned}$$

where

$$\begin{aligned} \tilde{F} &= \left[\; P_{10}FP_{10},\;\; P_{11}FP_{11},\;\; \ldots\ldots, \right.\\ &\quad \left. \ldots\ldots,\;\; P_{1,m-1}FP_{1,m-1} \;\right] \quad (50)\\ &\quad \text{if } \boldsymbol{\phi}(t) \text{ is } \mathbf{B}(t) \\ &= \left[\; \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\psi_{ij\,0}P_{1i}FP_{1j},\;\; \ldots\ldots \right.\\ &\quad \left. \ldots\ldots,\;\; \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\psi_{ij,m-1}P_{1i}FP_{1j} \;\right] \quad (51)\\ &\quad \text{if } \boldsymbol{\phi}(t) \text{ is } \mathbf{L}(t) \end{aligned}$$

$$Q_1 \;=\; \tilde{Q}_1\left(\boldsymbol{\phi}(t)\otimes I_n\right) \quad (52)$$

where

$$\begin{aligned} \tilde{Q}_1 &= \left[\; Q_1,\;\; Q_1,\;\; \ldots,\;\; Q_1 \;\right] \quad \text{if } \boldsymbol{\phi}(t) \text{ is } \mathbf{B}(t) \quad (53)\\ &= \left[\; Q_1,\;\; 0,\;\; \ldots\ldots,\;\; 0 \;\right] \quad \text{if } \boldsymbol{\phi}(t) \text{ is } \mathbf{L}(t) \quad (54)\end{aligned}$$

and

$$S \;=\; \tilde{S}\left(\boldsymbol{\phi}(t)\otimes I_n\right) \quad (55)$$

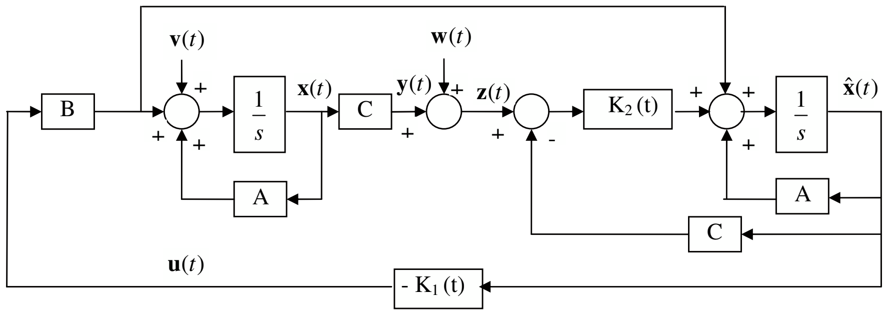

Fig. 1. Optimum linear combined estimation and control

where

$$\begin{aligned}\tilde{S} &= \left[\begin{array}{cccc} S, & S, & \ldots, & S \end{array}\right] \text{ if } \phi(t) \text{ is } \mathbf{B}(t) && (56)\\ &= \left[\begin{array}{cccc} S, & 0, & \ldots, & 0 \end{array}\right] \text{ if } \phi(t) \text{ is } \mathbf{L}(t) && (57)\end{aligned}$$

and $\otimes$ is the Kronecker product [3].

Substituting Eqs. (41), (43), (44), (49), (52) and (55) into Eq. (40) and making use of the backward integration operational property in Eq. (10) or (25), we have

$$-\tilde{P}_1 + \tilde{S} = \left[\bar{P}_1 + \hat{P}_1 - \tilde{F} + \tilde{Q}_1\right](P_b \otimes I_n)$$

$$\Rightarrow \tilde{P}_1 + \left[\bar{P}_1 + \hat{P}_1 - \tilde{F}\right](P_b \otimes I_n) = \tilde{S} - \tilde{Q}_1 (P_b \otimes I_n) \quad (58)$$

which is to be solved for the spectrum of $P_1(t)$. Similarly, the spectrum of $P_2(t)$ can also be found from the Riccati equation (36), and is given by

$$\tilde{P}_2 - \left[\bar{P}_2 + \hat{P}_2 - \tilde{G}\right](P_f \otimes I_n) = \tilde{P} + \tilde{Q}_2 (P_f \otimes I_n) \quad (59)$$

where

$$\begin{aligned}\bar{P}_2 &= \left[\begin{array}{cccc} AP_{20}, & AP_{21}, & \ldots, & AP_{2,m-1} \end{array}\right] && (60)\\ \hat{P}_2 &= \left[\begin{array}{cccc} P_{20}A^T, & P_{21}A^T, & \ldots, & P_{2,m-1}A^T \end{array}\right] && (61)\end{aligned}$$

and $G = C^T R_2^{-1} C$

Notice that both the Riccati equations are thus reduced to the non-linear algebraic equations (58) and (59), which can easily be solved using Newton-Raphson method.

A. Recursive algorithm via BPFs

For a scalar system it is possible to obtain a recursive algorithm if BPFs are used. This point is discussed here. Substituting the operational matrix of backward integration P_b in Eq. (11) into Eq. (58) and simplifying, we obtain the following recursive algorithm :

V. Numerical Example

Consider the linear system [2], [6]

$$\begin{aligned}\dot{x}(t) &= -0.5x(t) + u(t) + v(t)\\ \bar{x}(0) &= 10\end{aligned}$$

with the measurement

$$z(t) = x(t) + w(t)$$

and the cost function

$$J = \mathrm{E}\left\{0.5x^2(t_f)S + 0.5\int_0^{t_f}\left[2x^2(t) + u^2(t)\right]dt\right\}$$

where

$$\begin{aligned}\mathrm{E}\{v(t)v(\tau)\} &= 2\,\delta(t-\tau)\\ \mathrm{E}\{w(t)w(\tau)\} &= 0.25\,\delta(t-\tau)\\ \mathrm{E}\left\{[x(0)-\bar{x}(0)]^2\right\} &= 0\end{aligned}$$

If $S = 0$ and $t_f = 1$, the exact solutions of $P_1(t)$ and $P_2(t)$ are given by

$$\begin{aligned}P_1(t) &= -0.5 + 1.5\tanh(-1.5t + 1.8465736) \quad \text{and}\\ P_2(t) &= -0.125 + 0.125\sqrt{33}\tanh\left\{0.5\sqrt{33}\,t\right.\\ &\left. +\tan^{-1}\left(1/\sqrt{33}\right)\right\}\end{aligned}$$

So with $m = 4$ and 24 the above recursive algorithm via BPFs and with $m = 4$ the nonrecursive approach in Section 4 via SLPs are applied, and $P_1(t)$ and $P_2(t)$ are computed as shown in Figs. 2 and 3. The exact solutions are also shown in the same figures for comparison sake. The results are quite satisfactory even with four SLPs.

VI. Conclusion

A unique method to determine the filter gain and the regulator gain in LQG control problem is proposed. It is shown that the application of orthogonal functions (BPFs and SLPs) reduces differential calculus to algebra. A BPF based recursive algorithm is presented to solve the LQG control problem of linear time-invariant scalar systems. An illustrative example is

Recursive algorithm :

$$P_{1,m-1} = -\frac{1}{F}\left(\frac{1}{T}-A\right)+\sqrt{\left[\frac{1}{F}\left(\frac{1}{T}-A\right)\right]^2+\frac{Q_1}{F}} \quad (62)$$

$$P_{1,j} = -\frac{1}{F}\left(\frac{1}{T}-A\right)+ \sqrt{\left[\frac{1}{F}\left(\frac{1}{T}-A\right)\right]^2+\frac{2Q_1}{F}+\frac{2}{F}\left(\frac{1}{T}+A\right)P_{1,j+1}-P_{1,j+1}^2} \quad (63)$$

for $j=m-2,m-3,\dots,1,0$.
Similarly, substituting the operational matrix of forward integration P_f in Eq. (9) into Eq. (59) we have

$$P_{2,0} = -\frac{1}{G}\left(\frac{1}{T}-A\right)+\sqrt{\left[\frac{1}{G}\left(\frac{1}{T}-A\right)\right]^2+\frac{Q_2}{G}} \quad (64)$$

$$P_{2,j} = -\frac{1}{G}\left(\frac{1}{T}-A\right) +\sqrt{\left[\frac{1}{G}\left(\frac{1}{T}-A\right)\right]^2+\frac{2Q_2}{G}+\frac{2}{G}\left(\frac{1}{T}+A\right)P_{2,j-1}-P_{2,j-1}^2} \quad (65)$$

for $j=1,2,\dots,m-1$. Such a recursive algorithm is not possible with SLPs.

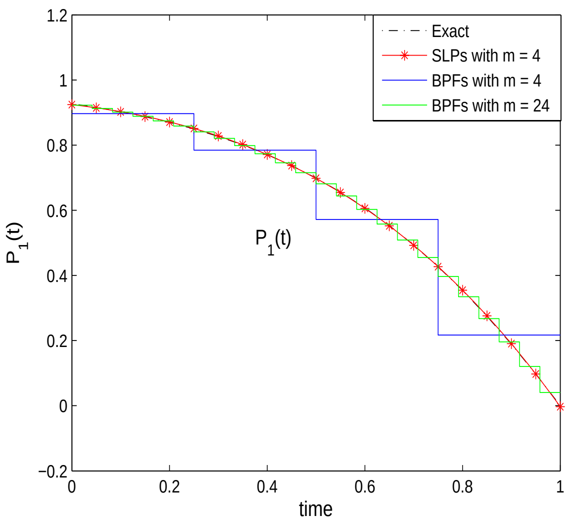

Fig. 2. Exact, SLP and BPF solutions of $P_1(t)$

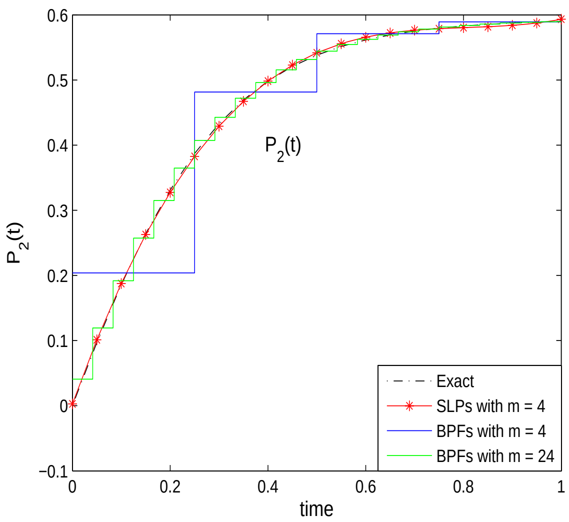

Fig. 3. Exact, SLP and BPF solutions of $P_2(t)$

included to demonstrate the usefulness of the unified approach via SLPs and the recursive algorithm via BPFs. As can be seen from Figs. 2 and 3, only four SLPs are good enough to obtain the result which is almost following the exact solution. One has to consider a large number of BPFs to improve upon the accuracy. This is because we are using piecewise constant functions (BPFs) to represent the smooth functions $P_1(t)$ and $P_2(t)$ in the present context.

Every approach (SLP or BPF) has its own advantage and disadvantage. SLP method does not require large number of polynomials in series expansion to represent smooth functions, but computationally it is not as much attractive as BPF method because SLPs are to be computed and used for signal representation while it is not so in BPF method as BPFs are all unity and disjoint.

References

[1] Athans, M., *The role and use of the stochastic linear-quadratic-Gaussian problem in control system design*, IEEE Trans. Automatic Control, vol. 16, no. 6, pp: 529-552, 1971.
[2] Sage, A. P. and White, C .C., *Optimum Systems Control*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1977.
[3] Brewer, J. W., *Kronecker products and matrix calculus in system theory*, IEEE Trans. Circuits and Systems, vol. 25, no. 9, pp: 772-781, 1978.
[4] Rao, G. P., *Piecewise Constant Orthogonal Functions and Their Application to Systems and Control*, LNCIS 55, Springer, Berlin, 1983.

[5] Hwang, C. and Chen, M. Y., *Analysis and optimal control of time-varying linear systems via shifted Legendre polynomials*, Int. J. Control, vol. 41, no. 5, pp: 1317-1330, 1985.
[6] Chang, Y. F. and Lee, T. T., *General orthogonal polynomials approximations of the linear-quadratic-Gaussian control design*, Int. J. Control, vol. 43, no. 6, pp: 1879-1895, 1986.
[7] Jiang, Z. H. and Schaufelberger, W., *Block-Pulse Functions and Their Applications in Control Systems*, LNCIS 179, Spinger, Berlin, 1992.
[8] Datta, K. B. and Mohan, B. M., *Orthogonal Functions in Systems and Control*, Advanced Series in Electrical and Computer Engineering, vol. 9, World Scientific, Singapore, 1995.
[9] Patra, A. and Rao, G. P., *General Hybrid Orthogonal Functions and Their Applications in Systems and Control*, LNCIS 213, Springer, London, 1996.
[10] Gupta, V., Hassibi, B. and Murray, R. M., *Optimal LQG control across packet-dropping links*, Systems & Control Letters, vol. 56, no. 6, pp: 439-446, 2007.
[11] Sinopoli, B., Schenato, L., Franceschetti, M., Poolla, K. and Sastry, S., *Optimal linear LQG control over lossy networks without packet acknowledgment*, Asian J. Control, vol. 10, no. 1, pp: 3-13, 2008.
[12] Kar, S. K., *Orthogonal functions approach to optimal control of linear time-invariant systems described by integro-differential equations*, KLEKTRIKA, vol. 11, no 1, pp: 15-18, 2009.
[13] Mohan, B. M. and Kar, S. K., *Optimal Control of Multi-Delay Systems via Orthogonal Functions*, Int. J. Advanced Research in Engineering and Technology, vol. 1, no. 1, pp: 1-24, 2010.
[14] Kar, S. K., *Optimal control of a linear distributed parameter system via shifted Legendre polynomials*, Int. J. Electrical and Computer Engineering (WASET), vol. 5, no. 5, pp: 292-297, 2010.
[15] Mohan, B. M. and Kar, S. K., *Orthogonal functions approach to optimal control of delay systems with reverse time terms*, J. The Franklin Institute, vol. 347, no. 9, pp: 1723-1739, 2010.
[16] Mohan, B. M. and Kar, S. K., *Optimal Control of Singular Systems via Orthogonal Functions*, Int. J. Control, Automation and Systems, vol. 9, no. 1, pp: 145-152, 2011.
[17] Mohan, B. M. and Kar, S. K., *Optimal control of multi-delay systems via shifted Legendre polynomials*, Int. Conf. on Energy, Automation and Signals (ICEAS), Bhubaneswar, INDIA, December 28-30, 2011.
[18] Mohan, B. M. and Kar, S. K., *Optimal control of nonlinear systems via orthogonal functions*, Int. Conf. on Energy, Automation and Signals (ICEAS), Bhubaneswar, INDIA, December 28-30, 2011.

B. M. Mohan was born in Tapeswaram, India in 1960. He received the Bachelors degree in Electrical Engineering from Osmania University in 1982, the Masters degree in Electrical Engineering (with Control Systems specialization) from Andhra University in 1985, and the Doctoral degree in Engineering from Indian Institute of Technology, Kharagpur, 1989. From July 1989 to October 1990 he was on the faculty of Electrical and Electronics Engineering Department, Regional Engineering College (now called National Institute of Technology), Tiruchirapalli. Since November 1990 he has been on the faculty of Electrical Engineering Department, Indian Institute of Technology, Kharagpur where he is currently a Professor.

He was a Visiting Professor at the Department of Electrical Engineering - Systems, University of Southern California, and Los Angeles, California in 2006. His research interests include Identification, Analysis, and Control of Dynamical Systems using Fuzzy Logic and Orthogonal Functions. He co-authored the research monograph 'Orthogonal Functions in Systems and Control (World Scientific, Singapore, 1995),' and several papers in international journals and conferences. He is a member of Asian Control Association, life member of Systems Society of India, senior member of the IEEE, and life fellow of Institution of Engineers (India). He is the Associate Editor of Int. J. Automation & Control (IJAAC), the Editorial Advisory Board member of Auto soft J. Intelligent Automation & Soft Computing, and the Editorial Board member of Int. J. Mathematics and Engineering with Computers (IJMAEC). He is the reviewer for Computing Reviews. He was the Chairman of Control Systems Chapter, IEEE Kharagpur Section. He was the chair of Control, Robotics & Motion Control track, ICIIS 2008. and Engineering with Computers

Sanjeeb Kumar Kar was born in Bankura, India in 1971. He received his B.Tech. in Electrical Engineering from College of Engineering & Technology, OUAT, Bhubaneswar in 1993, M. Tech in Control Systems from Indian Institute of Technology, Kharagpur in 2004 and Ph.D from Indian Institute of Technology, Kharagpur in 2011. From March 2004 to February 2005 he worked in Odisha State Electricity Board as Apprentice Engineer, from April 1995 to March 1997 worked in Pal Textile, Balasore as Assistant Manager. From August 1997 he joined CET, OUAT, Bhubaneswar as a faculty in Electrical Engineering and continued upto October 2008. Since November 2008 he has been on the faculty of Electrical Engineering Department, Institute of Technical Education & Research (ITER), Bhubaneswar. At present he is working as Associate Professor and Head of the Department of Electrical Engineering, ITER, SOA University, Bhubaneswar. He authored/co-authored several papers in international journals and conferences. He is a life member of Systems Society of India. His research interests include Estimation, Analysis, and Control of Dynamical Systems using Orthogonal Functions.