

Optimal Policy for a Deteriorating Inventory Model with Finite Replenishment Rate and with Price Dependant Demand Rate and Cycle Length Dependant Price

Hamed Sabahno

Abstract—In this paper, an inventory model with finite and constant replenishment rate, price dependant demand rate, time value of money and inflation, finite time horizon, lead time and exponential deterioration rate and with the objective of maximizing the present worth of the total system profit is developed. Using a dynamic programming based solution algorithm, the optimal sequence of the cycles can be found and also different optimal selling prices, optimal order quantities and optimal maximum inventories can be obtained for the cycles with unequal lengths, which have never been done before for this model. Also, a numerical example is used to show accuracy of the solution procedure.

Keywords—Deteriorating items, Dynamic programming, Finite replenishment rate, Inventory control, Operation Research.

I. INTRODUCTION

INVENTORIES generally can be classified into three meta-categories based on: 1-obsolescence 2-deterioration 3-no obsolescence/deterioration.

Obsolescence refers to inventories that become obsolete at a certain time, for instance because of rapid changes in technology or introduction of a new product. Deterioration refers to damage, spoilage, dryness, vaporization, etc. The products like fresh food (meat, fish, fruits, and vegetables), photographic films, batteries, human blood, photographic films, etc having a maximum usable lifetime are known as perishable products and the products like alcohol, gasoline, radioactive substances, lubricants, glues, paints, chemical ingredients, etc having no shelf-life at all are known as decaying products. No obsolescence/deterioration refers to inventories that their shelf life can be indefinite and hence they would fall under no obsolescence/deterioration category. In this paper the deteriorating inventories are considered. There are many classifications for the deterioration. The deteriorating properties of inventory are classified by Ghare and Schrader [3] into three categories: 1-direct spoilage, e.g., vegetable, fruit and fresh food etc.; 2-physical depletion, e.g., gasoline and alcohol etc.; 3-deterioration such as radiation change, negative spoiling and loss of efficacy in inventory, e.g., electronic components and medicine. From another point of view, deterioration can

also be classified by the time-value or the items' life of inventory. Deterioration is categorized by Raafat [6] by the time-value of inventory: 1-utility constant: its utility does not change significantly as time passes within its valid usage period, e.g., liquid medicine; 2-utility increasing: its utility increases as time passes, e.g., some alcoholic drinks; 3- utility decreasing: its utility decreases as time passes, e.g., vegetables, fruits and fresh foods, etc. On the other hand, deterioration is classified by Nahmias [5] by the items' life of inventory: 1-fixed lifetime: items' lifetime is pre-specified and its lifetime is independent of the deteriorating factors; therefore, it is called time independent deterioration. In fact, the utility of these items decreases during their lifetime, and when passing their lifetime, the item will perish completely and become of no value, e.g., milk, inventory in a blood bank, and food, etc. 2-random lifetime: there is no specified lifetime for these items. The lifetime for these items is assumed as a random variable, and its probability distribution could be gamma, weibull, exponential, etc. These Items are called time dependent deteriorating items, e.g., electronic components, chemicals, and medicine, etc. The scope of this study covers those deteriorating items which are classified as utility decreasing (as regards as their time-value) and also as random lifetime (as regard as their lifetime).

Research in this area started with the work of Whitin [10] who considered fashion goods deteriorating at the end of prescribed storage period. An exponentially decaying inventory was first developed by Ghare and Schrader [3]. The certain commodities were observed to shrink with time by a proportion which can be approximated by a negative exponential function of time.

The concept of inflation and time value of money was employed by Wee and Law [7] into a model where the demand is price dependent and shortage is allowed. A production environment with a finite replenishment rate was considered in the model. An optimization framework was presented to derive optimal production and pricing policies when the total net present value is maximized.

A production-inventory system for deteriorating items with time varying demands and completely backlogged shortages was modeled by Wee and Wang [9]. The production schedules for this system were constructed using the traditional scheduling strategy in which each cycle starts with replenishment and ends with shortages.

A deteriorating inventory model under time value of money and a deterministic inventory system with the price-dependent demand was considered by Wee and Law

Hamed Sabahno is with the Industrial Engineering Department, National Iranian Oil Refining and Distribution Company (NIORDC), Tehran, Iran (phone:00989126216681; e-mail:hamed_sabahno@yahoo.com).

[8]. The discounted cash flows (DCF) approach was applied to the problem analysis. A heuristic approach was presented to derive the near optimal replenishment and pricing policy that tries to maximize the total net present value profit.

The work of Wee and Wang [9] was extended by Zhou et al [11] by permitting part of the backlogged shortages to turn into lost sales, which is assumed to be a function of currently backlogged amount. Also, an alternative scheduling strategy was considered in which each cycle of a schedule starts with a period of shortages and then followed by continuous replenishment. By examining the profit performance of the two scheduling strategies (i.e., "start with replenishment" versus "start with shortages"), they showed that their alternative strategy produces schedules with superior cost and profit values.

The dynamic programming method was used by Chen [1], [2] in order to find the optimal sequence of the cycles for his inventory models with the objective of minimizing the present worth of the total system cost and with time dependant demand rate, complete backlogging shortages and without considering the selling price and the revenue. His procedure is changed and modified by us in order to find the optimal sequence of the cycles for the current model.

Most of the researchers who worked in this area, assumed that selling price, order quantity, maximum inventory and cycle length are constant over the planning horizon (finite or infinite). In the present research, a deteriorating inventory model with constant replenishment rate, price dependant demand rate, time value of money and inflation, finite time horizon, lead time and exponential deterioration rate with the objective of maximizing the present worth of the total system profit is presented. Using a dynamic programming based solution algorithm, the optimal sequence of the cycles can be found and also different optimal selling prices, optimal order quantities and optimal maximum inventories can be obtained for the cycles with unequal lengths, which have never been done before for this model. Also, a numerical example is used to show accuracy of the solution procedure.

II. MODEL DEVELOPMENT

The assumptions and the notations used in this model are:

- 1-The replenishment starts with zero inventory level at time i . The replenishment stops at time j , where the inventory attains its maximum level. During the period $[i, j]$, the inventory level is affected by replenishment, demand and deterioration of items. Due to reasons of demand and deterioration of items, the inventory level gradually diminishes during the period $[j, k]$ and ultimately falls to zero at time k .
- 2-The demand rate, $D(s)$ is a linear, non-negative, continuous, convex, decreasing function of the selling price, s . $D(s) = a - bs$, where a and b are both positive constants.
- 3-A single item is considered, whose time to deterioration is distributed as exponential. $f(t) = \lambda e^{-\lambda t}$, where λ is a positive constant (deterioration rate) and $t > 0$.
- 4-The inflation rate (denoted by m) and the discount rate (denoted by r) representing the time value of money are constant. Therefore, the net discount rate of inflation is constant: $R = r - m$. The present worth of X_t is:

$X_t e^{-Rt}$, $t \geq 0$. where X_t is the value of X at time t .

Notice that both the internal (company) inflation rate and the external (general economy) inflation rate can be estimated by a unique inflation rate.

5-The replenishment rate is finite and constant, P , where $P > D(s)$.

6-Lead time (L) is constant but it is limited to $0 \leq L \leq (k - i)$, because of the inequality of the cycle lengths.

7-Holding cost applies to good units only.

8-Deterioration of units occurs only when the item is effectively in stock, and there is no repair or replacement of deteriorated units during planning horizon.

9-Shortages are not allowed.

In addition, the following notations are used in this paper:

$I_1(t)$: inventory level at time t , where $i \leq t \leq j$.

$I_2(t)$: inventory level at time t , where $j \leq t \leq k$.

Q : ordering quantity per cycle.

A : ordering cost per cycle at time zero.

C : unit purchasing cost at time zero, where $C < S$.

h : unit holding cost per unit of time at time zero.

H : planning horizon.

The inventory level can be described by the following differential equations:

$$i \leq t \leq j \quad \frac{\delta I_1(t)}{\delta t} + \lambda I_1(t) = P - D(s) \quad (1)$$

$$\frac{\delta I_2(t)}{\delta t} + \lambda I_2(t) = -D(s) \quad j \leq t \leq k \quad (2)$$

By applying the boundary conditions, multiplying $e^{\lambda t}$ on the both sides of (1), (2) and integrating by part, we have:

$$I_1(t) = (P - D(s))e^{-\lambda(t-i)} \int_i^t e^{\lambda(u-i)} du =$$

$$(P - D(s))e^{-\lambda t} \int_i^t e^{\lambda u} du \quad i \leq t \leq j$$

$$I_2(t) = D(s)e^{-\lambda(t-i)} \int_i^t e^{\lambda(u-i)} du = D(s)e^{-\lambda t} \int_j^k e^{\lambda u} du \quad j \leq t \leq k$$

Because the maximum inventory takes place at time j , we have:

$$I_{\max} = (P - D(s))e^{-\lambda j} \int_i^j e^{\lambda u} du = D(s)e^{-\lambda j} \int_j^k e^{\lambda u} du \quad (3)$$

After simplification, we have:

$$P.e^{-\lambda j} \int_i^j e^{\lambda u} du = D(s)e^{-\lambda j} \int_j^k e^{\lambda u} du$$

By solving the above equation with respect to j , we have:

$$j = \frac{\ln[\lambda \frac{D(s)}{P} \int_i^k e^{\lambda u} du + e^{\lambda i}]}{\lambda} = \frac{\ln[\frac{D(s)}{P} (e^{\lambda k} - e^{\lambda i}) + e^{\lambda i}]}{\lambda} \quad (4)$$

Ordering quantity for each cycle can be obtained as follows:

$$Q = D(s) \int_i^k du + \int_i^j \lambda I_1(t) dt + \int_j^k \lambda I_2(t) dt =$$

$$D(s)(k - i) + (P - D(s)) \lambda \int_i^j \int_i^t e^{\lambda(u-i)} du dt + D(s) \lambda \int_j^k \int_t^k e^{\lambda(u-t)} du dt \quad (5)$$

Reorder point in each cycle is (obviously, the order quantity would be the next cycle's need):

$$\text{If } 0 \leq Lt \leq (k-j) \text{ then } RoP = \int_{k-Lt}^k D(s) e^{\lambda(u-(k-Lt))} du \quad (6)$$

$$\text{If } (k-i) \geq Lt \geq (k-j) \text{ then} \quad (7)$$

$$RoP = \int_i^{k-Lt} (P-D(s)) e^{\lambda(u-(k-Lt))} du$$

Notice that if $Lt = (k-j)$ then either one is usable.

The present worth of the relevant costs and the revenue for each cycle are as follows:

The Present worth of the holding cost:

$$Hc = h \int_i^j e^{-Rt} I_1(t) dt + h \int_j^k e^{-Rt} I_2(t) dt =$$

$$h.(P-D(s)) \int_i^j e^{-Rt} e^{-\lambda t} \int_i^t e^{\lambda u} du dt + h.D(s) \int_j^k e^{-Rt} e^{-\lambda t} \int_t^k e^{\lambda u} du dt$$

$$\text{The present worth of the purchasing cost: } Pc = \int_i^j P.c.e^{-Ru} du$$

$$\text{The present worth of the ordering cost: } Oc = A.e^{-R(i-Lt)}$$

$$\text{The present worth of the revenue: } R = s \int_i^k D(s) e^{-Ru} du$$

Now we can compute the present worth of the total system profit as follows:

$$TP = R - HC - PC - OC$$

$$TP = s.D(s) \int_i^k e^{-Ru} du - h.(P-D(s)) \int_i^j e^{-Rt} e^{-\lambda t} \int_i^t e^{\lambda u} du dt -$$

$$h.D(s) \int_j^k e^{-Rt} e^{-\lambda t} \int_t^k e^{\lambda u} du dt - c.P \int_i^j e^{-Ru} du - A.e^{-R(i-Lt)} \quad (8)$$

In order to determine the optimal selling price for each cycle we differentiate (8) with respect to s and we set the result equal to zero. Then, we have:

$$\frac{\partial TP}{\partial s} = (a - 2b.s) \int_i^k e^{-Ru} du + b[-h \int_i^j e^{-Rt} e^{-\lambda t} \int_i^t e^{\lambda u} du dt +$$

$$h \int_j^k e^{-Rt} e^{-\lambda t} \int_t^k e^{\lambda u} du dt] = 0$$

$$\Rightarrow s = (a \int_i^k e^{-Ru} du + b[-h \int_i^j e^{-Rt} e^{-\lambda t} \int_i^t e^{\lambda u} du dt +$$

$$h \int_j^k e^{-Rt} e^{-\lambda t} \int_t^k e^{\lambda u} du dt] / 2b \int_i^k e^{-Ru} du) \quad (9)$$

To satisfy the optimally condition we must have $\frac{\partial^2 TP}{\partial^2 s} < 0$.

$$\frac{\partial^2 TP}{\partial^2 s} = -2b \int_i^k e^{-Ru} du < 0.$$

Then, the optimally condition is strongly satisfied, and the obtained s will maximize the present worth of the total system profit and also TP is concave with respect to s .

III. SOLUTION PROCEDURE

For obtaining the optimal sequence of the cycles, as mentioned in the introduction section, dynamic programming model and algorithm used by Chen [1], [2] are modifies and changed for current model, as follows:

$$TP_k = \text{Max}_{0 \leq i < j < k \leq H} \{TP_i + W(i, s^*, j^*, k)\}$$

Vol.2, No.8-2008

This recursive procedure works in a forward fashion to determine the maximal present worth of the total profit over the time horizon, H . Where, $0 \leq i < j < k \leq H$, TP_k is the optimal present worth of the profit until time k so that TP_H is the optimal present worth of the total system profit and $W(i, s^*, j^*, k)$ is the optimal present worth of the profit in cycle $[i, k]$ knowing s^* which is the optimal s in cycle $[i, k]$ and j^* which is the optimal point where the replenishment ends in cycle $[i, k]$.

The dynamic programming based solution algorithm to solve this model is:

Step 1: Enter parameters: $P, c, H, h, \lambda, a, b, R, lt, A$.

Step 2: Let $i = 0, TP_0 = 0$

For $K = 1: H$

Obtain j, S from (4), (9); (solve together)

Obtain $W(i, s^*, j^*, k)$ from (8)

Let $TP_k = W(i, s^*, j^*, k), s_{i,k}^* = s, j_{i,k}^* = j, T_k = i$

End

Step 3: For $k = 2: H$

For $i = 1: k-1$

Obtain j, S from (4), (9); (solve together)

Obtain $W(i, s^*, j^*, k)$ from (8)

If $TP_k < TP_i + W(i, s^*, j^*, k)$

Then $TP_k = TP_i + W(i, s^*, j^*, k), s_{i,k}^* = s, j_{i,k}^* = j, T_k = i$

End

End

Step 4: Let $k = H$

While $k > 0$

Cycle time $= [T_k, k]$

Optimal price $= s_{T_k,k}^*$

Replenishment ending time $= j_{T_k,k}^*$

Optimal maximum inventory $= I_{\max T_k,k}^*$, (from (3))

Optimal order quantity $= Q^*(T_k, s^*, j^*, k)$, (from (5))

$0 \leq Lt \leq (k-j)$ (if, $RoP_{T_k,k}$ Optimal reorder point:

from (7) and $(k-i) \geq Lt > (k-j)$ from (6), if

if $Lt > (k-i)$ then 'Lead time not acceptable')

Cumulated total profit $= TP_k$

Let $k = T_k$

End

End

Where, T_k is the starting point of the last replenishment cycle from time zero to time k (i.e., $[T_k, k]$) $T_k = 0, 1, 2, \dots, k-1$ and $k = 1, 2, \dots, H$, $s_{T_k,k}^*$ is the optimal selling price in cycle $[T_k, k]$, $j_{T_k,k}^*$ is the time in cycle $[T_k, k]$ where the replenishment ends, $Q^*(T_k, s^*, j^*, k)$ is the optimal order quantity in cycle $[T_k, k]$ knowing $s_{T_k,k}^*$ and $j_{T_k,k}^*$, and $RoP_{T_k,k}$ is the optimal reorder point in cycle $[T_k, k]$.

IV. NUMERICAL EXAMPLE

In this section, a numerical example is tested to show accuracy of the model and the solution procedure.

The parameters' quantities are:

$$a = 50, b = 0.9, R = 0.1, \lambda = 0.05, h = 0.5, c = 8, A = 100, H = 10, P = 95, Lt = 0.5.$$

The model is solved by MATLAB 7.1 to obtain the necessary results and they can be seen in Table I. Then, the proposed algorithm is used to obtain the optimal sequence of the cycles. After running the algorithm by MATLAB 7.1, there is two cycles with lengths of 2 and two cycles with

Vol:2, No:120083, with the present worth of the total system profit equal to 2565.4, as follows:

$$TP_{10} = W(8, s^*, j^*, 10) + W(6, s^*, j^*, 8) + W(3, s^*, j^*, 6) + W(0, s^*, j^*, 3) = 2565.4$$

The optimal price and the optimal order quantity for the optimal cycles are (for optimal maximum inventory, optimal reorder point, optimal replenishment ending time and more details refer to Table I):

$$s_{6,8}^* = s_{8,10}^* = 27.8936, s_{0,3}^* = s_{3,6}^* = 27.9491$$

$$Q_{6,8}^* = Q_{8,10}^* = 51.6576, Q_{0,3}^* = Q_{3,6}^* = 78.7628$$

TABLE I
SOFTWARE OUTPUT FOR THE NUMERICAL EXAMPLE

i	k	$j_{i,k}^*$	$s_{i,k}^*$	$W(i, s^*, j^*, k)$	I_{\max}^*	Q^*	RoP^*
0	1	0.2675	27.8364	350.6848	18.6131	25.4108	12.6309
0	2	0.5438	27.8936	734.2736	37.6066	51.6576	12.6048
1	2	1.2675	27.8364	317.3127	18.6131	25.4108	12.6309
0	3	0.8291	27.9491	1.0527e+003	56.9745	78.7628	12.5795
1	3	1.5438	27.8936	664.3982	37.6066	51.6576	12.6048
2	3	2.2675	27.8364	287.1164	18.6131	25.4108	12.6309
0	4	1.1237	28.0024	1.3123e+003	76.7101	106.7502	12.5552
1	4	1.8291	27.9491	952.5110	56.9745	78.7628	12.5795
2	4	2.5438	27.8936	601.1724	37.6066	51.6576	12.6048
3	4	3.2675	27.8364	259.7937	18.6131	25.4108	12.6309
0	5	1.4278	28.0534	1.5191e+003	96.8060	135.6444	12.5320
1	5	2.1237	28.0024	1.1875e+003	76.7101	106.7502	12.5552
2	5	2.8291	27.9491	861.8676	56.9745	78.7628	12.5795
3	5	3.5438	27.8936	543.9632	37.6066	51.6576	12.6048
4	5	4.2675	27.8364	235.0710	18.6131	25.4108	12.6309
0	6	1.7418	28.1015	1.6782e+003	117.2538	165.4713	12.5100
1	6	2.4278	28.0534	1.3745e+003	96.8060	135.6444	12.5320
2	6	3.1237	28.0024	1.0745e+003	76.7101	106.7502	12.5552
3	6	3.8291	27.9491	779.8500	56.9745	78.7628	12.5795
4	6	4.5438	27.8936	492.1983	37.6066	51.6576	12.6048
5	6	5.2675	27.8364	212.7011	18.6131	25.4108	12.6309
0	7	2.0659	28.1465	1.7946e+003	138.0444	196.2578	12.4895
1	7	2.7418	28.1015	1.5185e+003	117.2538	165.4713	12.5100
2	7	3.4278	28.0534	1.2437e+003	96.8060	135.6444	12.5320
3	7	4.1237	28.0024	972.2056	76.7101	106.7502	12.5552
4	7	4.8291	27.9491	705.6375	56.9745	78.7628	12.5795
5	7	5.5438	27.8936	445.3594	37.6066	51.6576	12.6048
6	7	6.2675	27.8364	192.4599	18.6131	25.4108	12.6309
0	8	2.4003	28.1881	1.8727e+003	159.1675	228.0320	12.4706
1	8	3.0659	28.1465	1.6238e+003	138.0444	196.2578	12.4895
2	8	3.7418	28.1015	1.3740e+003	117.2538	165.4713	12.5100
3	8	4.4278	28.0534	1.1254e+003	96.8060	135.6444	12.5320
4	8	5.1237	28.0024	879.6880	76.7101	106.7502	12.5552
5	8	5.8291	27.9491	638.4872	56.9745	78.7628	12.5795
6	8	6.5438	27.8936	402.9779	37.6066	51.6576	12.6048
7	8	7.2675	27.8364	174.1449	18.6131	25.4108	12.6309
0	9	2.7455	28.2258	1.9165e+003	180.6120	260.8229	12.4534
1	9	3.4003	28.1881	1.6945e+003	159.1675	228.0320	12.4706
2	9	4.0659	28.1465	1.4693e+003	138.0444	196.2578	12.4895
3	9	4.7418	28.1015	1.2432e+003	117.2538	165.4713	12.5100
4	9	5.4278	28.0534	1.0183e+003	96.8060	135.6444	12.5320
5	9	6.1237	28.0024	795.9746	76.7101	106.7502	12.5552
6	9	6.8291	27.9491	577.7271	56.9745	78.7628	12.5795
7	9	7.5438	27.8936	364.6295	37.6066	51.6576	12.6048
8	9	8.2675	27.8364	157.5728	18.6131	25.4108	12.6309
0	10	3.1017	28.2593	1.9297e+003	202.3652	294.6604	---
1	10	3.7455	28.2258	1.7341e+003	180.6120	260.8229	---
2	10	4.4003	28.1881	1.5332e+003	159.1675	228.0320	---
3	10	5.0659	28.1465	1.3295e+003	138.0444	196.2578	---
4	10	5.7418	28.1015	1.1249e+003	117.2538	165.4713	---
5	10	6.4278	28.0534	921.3642	96.8060	135.6444	---
6	10	7.1237	28.0024	720.2276	76.7101	106.7502	---
7	10	7.8291	27.9491	522.7491	56.9745	78.7628	---
8	10	8.5438	27.8936	329.9304	37.6066	51.6576	---
9	10	9.2675	27.8364	142.5778	18.6131	25.4108	---

A single item inventory model with constant replenishment rate, exponential deteriorating rate, price dependant demand rate, time value of money and inflation, lead time, finite time horizon and with the objective of maximizing the present worth of the total system profit, was developed in this paper. A numerical example was used to solve the model and different optimal selling prices, optimal order quantities, optimal maximum inventories and optimal time fractions of replenishment were concluded for the cycles with unequal lengths. Then, a computerized dynamic programming based algorithm was performed to obtain the optimal sequence of the cycles and the results showed that the optimal system has two cycles with lengths of 2 and two cycles with lengths of 3.

In addition, according to the numerical analysis, can be concluded that with increasing/decreasing of the cycle length; present worth of cycle profit, optimal selling price, optimal order quantity, optimal maximum inventory and optimal time fraction of replenishment will increase /decrease. Even with reasonable changes in the parameters' quantities, the above results will be still satisfied (this was tested by changing the parameters' quantities).

REFERENCES

- [1] J.M.Chen. An inventory model for deteriorating items with time-proportional demand and shortages under inflation and time discounting. *Int. J. Production Economics* 55 (1998) 21-30.
- [2] J.M. Chen and C.S. Lin. An optimal replenishment model for inventory items with normally distributed deterioration. *Production planning and control*, 2002, VOL. 13, NO. 5, 470-480.
- [3] P.N. Ghare and G.F. Schrader. A model for exponentially decaying inventories. *Journal of Industrial Engineering* 15 (1963), pp. 238-243.
- [4] S.K. Goyal, B.C.Giri. Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research* 134 (2001), 1-16.
- [5] S. Nahmias. Perishable inventory theory: A review. *Operations Research* 30 3 (1982), pp. 680-708.
- [6] F. Raafat. Survey of literature on continuously deteriorating inventory model. *Journal of the Operational Research Society* 42 (1991), pp. 27-37.
- [7] H.M. Wee and S.T. Law. Economic production lot size for deteriorating items taking account of the time value of money. *Computers & Operations Research* 26 (1999), pp. 545-558.
- [8] H.M. Wee, S.T. Law. Replenishment and pricing policy for deteriorating items taking into account the time-value of money. *Int. J. Production Economics* 71 (2001) 213-220.
- [9] H.M. Wee, W.T.Wang. A variable production scheduling policy for deteriorating items with time-varying demand. *Computers and Operations Research* 1999;26:237-54.
- [10] T.M. Whitin. *Theory of Inventory Management*, Princeton University Press, Princeton, NJ (1957).
- [11] Yong-Wu Zhou, Hon-Shiang Lau, Shan-Lin Yang. A new variable production scheduling strategy for deteriorating items with time-varying demand and partial lost sale. *Computers & Operations Research* 30 (2003) 1753-1776.