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On the Strong Solutions of the Nonlinear Viscous Rotating Stratified Fluid

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***Abstract*—**A nonlinear model of the mathematical fluid dynamics which describes the motion of an incompressible viscous rotating fluid in a homogeneous gravitational field is considered. The model is a generalization of the known Navier-Stokes system with the addition of the Coriolis parameter and the equations for changeable density. An explicit algorithm for the solution is constructed, and the proof of the existence and uniqueness theorems for the strong solution of the nonlinear problem is given. For the linear case, the localization and the structure of the spectrum of inner waves are also investigated.

***Keywords*—**Galerkin method, Navier-Stokes equations, nonlinear partial differential equations, Sobolev spaces, stratified fluid.

I. Introduction

We consider a bounded domain $\Omega \in R^3$ with a smooth boundary, and the following nonlinear system of fluid dynamics

$$\begin{cases} \dfrac{\partial v_1}{\partial t} - \omega v_2 - \nu_1 \Delta v_1 + v' \cdot \nabla v_1 + \dfrac{\partial p}{\partial x_1} = f_1 \\ \dfrac{\partial v_2}{\partial t} + \omega v_1 - \nu_1 \Delta v_2 + v' \cdot \nabla v_2 + \dfrac{\partial p}{\partial x_2} = f_2 \\ \dfrac{\partial v_3}{\partial t} - \nu_1 \Delta v_3 + v' \cdot \nabla v_3 + g v_5 + \dfrac{\partial p}{\partial x_3} = f_3 \\ \dfrac{\partial v_4}{\partial t} - \nu_2 \Delta v_4 + v' \cdot \nabla v_4 + \gamma v_3 = f_4 \\ \dfrac{\partial v_5}{\partial t} - \dfrac{N^2}{g} v_3 = f_5 \\ \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0\ . \end{cases} \tag{1}$$

$x = (x_1, x_2, x_3)$ is the space variable, $v(x,t) = (v', v_4, v_5)$, $v'(x,t) = (v_1, v_2, v_3)$ is the velocity field, v_4 is the temperature, v_5 is the density, $p(x,t)$ is the scalar field of the pressure, $f(x,t) = (f_1, f_2, f_3, f_4, f_5)$ is a known function from $L_2(Q_T)$, $Q_T = \Omega \times [0,T)$, $\nu_1 > 0$ is the kinematic viscosity parameter, $\nu_2 > 0$ is the heat conductivity coefficient, and ω, N, g, γ are positive constants. The system (1) describes the nonlinear motions of three-dimensional incompressible viscous fluid which is rotating over the vertical axis with the angular velocity $\vec{\omega} = [0,0,\omega]$, also with consideration of heat transfer and non-homogeneous stationary distribution of density which is described by the function e^{-Nx_3}. For the linear non-viscous case, (1) are deduced, for example, in [1], [2]. For the non-linear case without the stratification and density, (1) appear, for example, in [3], where the considered model was used for numerical calculations. There exists a considerable amount of bibliography dedicated to classical Navier-Stokes equations, some results may be found in [4]-[6]. For the simplified case of linearized compressible fluid without rotation, the system (1) was studied in [7], where the structure and the localization of the essential spectrum of normal vibrations were established. Due to the presence of the fourth and fifth equations for the unknown functions of density and temperature, and also due to the presence of the rotation parameter, (1) represent a novelty with respect to classical Navier-Stokes equations. In [8], the system (1) was considered without rotation and heat transfer, and there were established the properties of the existence and uniqueness of the weak solution.

Our aim is to prove the theorems of the existence and uniqueness of the strong solution for (1), as well as to construct an explicit algorithm for that solution.

If we introduce the following notations; $\tilde{v} = (v', v_4)$,

$$Mv = \begin{bmatrix} -\omega v_2 \\ \omega v_1 \\ g v_5 \\ \gamma v_3 \\ -\dfrac{N^2}{g} v_3 \end{bmatrix}, \qquad \nu \Delta v = \begin{bmatrix} \nu_1 \Delta v_1 \\ \nu_1 \Delta v_2 \\ \nu_1 \Delta v_3 \\ \nu_2 \Delta v_4 \\ 0 \end{bmatrix},$$

then we can write the system (1) as:

$$\begin{cases} \frac{\partial v}{\partial t} + (v' \cdot \nabla) \tilde{v} - \nu \Delta v + Mv + \nabla p = f \\ \operatorname{div} v' = 0, \quad x \in \Omega, \ t \geq 0. \end{cases} \tag{2}$$

We associate the system (2) with the following conditions

$$\begin{cases} v\big|_{t=0} = 0 \\ v\big|_{\partial\Omega} = 0 \end{cases} \tag{3}$$

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in the bounded domain $Q_T = \Omega \times [0,T]$.

Let us choose an orthonormal complete set of functions $\{u_k\}$ in the Hilbert functional space

$$H = \left\{ \varphi(x) = (\varphi_1, ..., \varphi_5) \ : \ \varphi \in \overset{0}{W}{}_2^1(\Omega), \operatorname{div} \varphi' = 0 \right\}.$$

Finally, let us introduce the notations

$$\beta = \max\left\{ \omega, g, \gamma, \frac{N^2}{g} \right\}, \ \nu_0 = \min\{\nu_1, \nu_2\}.$$

If we multiply the system (2) by $2v$ in $L_2(\Omega)$ and integrate the obtained result by parts and also with respect to $\tau \in [0,T]$, then we can easily obtain the estimate for the weak solution:

$$\|v\|^2_{L_2(\Omega)} + \nu_0 \|v_x\|^2_{L_2(Q_t)} \le C(\beta, T, \Omega) \|f\|^2_{L_2(Q_t)}. \quad (4)$$

We observe that, due to the vector representation of $\nu\Delta v$, the component v_{5_x} is equal to zero in (4). This property will be also valid for all further estimates which involve the term $\nu_0 \|v_x\|^2_{L_2(Q_t)}$. Now, let $\Phi(x,t) = (\Phi_1, ..., \Phi_5)$ be test functions from $L_2(Q_T)$, which for every $0 \le t \le T$ belong to the Sobolev space $\overset{0}{W}{}_2^1(\Omega)$, and which also satisfy the conditions:

$$\operatorname{div}\Phi' = 0, \quad \Phi\big|_{t=T} = 0, \quad \Phi\big|_{\partial\Omega} = 0.$$

For the weak solution v we require the same conditions as for the functions Φ. We will call $v(x,t)$ a *weak solution* of the problem (2), (3), if v satisfies the integral identity

$$\int_{Q_T} \Big[-(v, \Phi_t) + \nu_1 \sum_{i=1}^{3} (\nabla v_i, \nabla \Phi_i) + \nu_2 (\nabla v_4, \nabla \Phi_4) + (\tilde{v}, (v' \cdot \nabla)\tilde{\Phi}) + (Mv, \Phi) \Big] dxdt = \int_{Q_T} (f, \Phi) dxdt, \quad (5)$$

for all the functions Φ. To construct the weak solution, we will use the Galerkin method. We find the approximate solutions of the problem (2), (3) in the following form

$$v^N(x,t) = \sum_{k=1}^{N} C_k^N(t) u_k(x). \quad (6)$$

In the system (2), we put $v = v^N$, multiply by u_k in sense of $L_2(\Omega)$ and integrate by parts in Ω. In this way, we obtain a Cauchy problem for the system of ordinary differential equations of the type

$$\frac{d}{dt} C_k^N(t) + \sum_{j=1}^{N} C_j^N \{u_j, u_k\} - \int_\Omega \left(\tilde{v}^N, (v'^N \cdot \nabla) \tilde{u}_k \right) dx - {} + \int_\Omega (Mv^N, u_k) dx = F_k(t), \ k = 1, ..., N, \quad (7)$$

where

$$\{\vec{u}, \vec{v}\} = \int_\Omega \left[\nu_1 \sum_{i=1}^{3} (\nabla v_i, \nabla \Phi_i) + \nu_2 (\nabla v_4, \nabla \Phi_4) \right] dx,$$

$$F_k(t) = \int_\Omega (f, u_k) dx \ , \ k \ge 1.$$

To prove that (7) is solvable uniquely, we have to verify the “a priori” boundedness of the functions $C_k^N(t)$, $t \in [0,T]$, in the norm $L_2(\Omega)$. Evidently, the required property follows from the inequalities:

$$\begin{aligned} &\|v^N\|^2_{L_2(\Omega)} + \nu_0 \|v_x{}^N\|^2_{L_2(Q_T)} \le C \|f\|^2_{L_2(Q_T)} \\ &\|v^N\|^2_{L_2(Q_T)} \le C \|f\|^2_{L_2(Q_T)}. \end{aligned} \quad (8)$$

The relations (8) are obtained by the same reasoning as in (4). It follows from (8) that the Galerkin approximations (6) are “a priori” bounded. From the sequence $\{v^N\}_{N=1}^{\infty}$, keeping in mind the estimates (8), we can choose the subsequence $\{v^{N_k}\}$ which is weakly convergent to some function $v(x,t)$ in $L_2(Q_T)$, together with its first derivatives with respect to x_k, $k = 1,2,3$. The last fact follows from the weak compactness of bounded sets in the Hilbert space $L_2(Q_T)$. It is easy to see that the subsequence $\{v^{N_k}\}$ also tends strongly to v in sense of $L_2(Q_T)$, which follows from the generalized Friedrichs lemma ([9]):

$$\|v^{N_k} - v^{N_m}\|^2_{L_2(Q_T)} \le \sum_{l=1}^{N^\varepsilon} \int_0^T \left(v^{N_k} - v^{N_m}, u_l \right)^2 + {} + \varepsilon \|v_x{}^{N_k} - v_x{}^{N_m}\|^2_{L_2(Q_T)}.$$

Therefore, the sequence $\{v^{N_k}\}$ tends strongly to v. It can be easily verified that the mentioned function $v(x,t)$ satisfies (5); i.e., it is a weak solution of (2). In this way, we have outlined the general idea of the proof of the following theorem.

Theorem 1. There exists at least one weak solution for the problem (2), (3), which can be found as the limit of the approximations (6).

We will need the following auxiliary result as well.

Theorem 2. Let Ω be a bounded domain in $R^3, \partial\Omega \in C^2$,

J_0^∞ - a space of smooth solenoidal functions with compact support, and J_2^1 - the closure of J_0^∞ in norm of $W_2^1(\Omega)$. Suppose that v' is a weak solution of the Stokes system; $v' \in J_2^1$ and the integral identity holds:

$$\sum_{j=1}^{3}\int_{\Omega}\left(\nu_1\nabla v_j,\nabla\Phi_j\right)dx=\int_{\Omega}\left(f,\Phi\right)dx \quad \forall\Phi\in J_0^\infty .$$

If $f \in L_2(\Omega)$, then there exist $v' \in W_2^2(\Omega)$ and $p \in W_2^1(\Omega)$ such that $\{v', p\}$ is the strong solution of the Stokes system $\begin{cases} -\Delta v' + \nabla p = f \\ \operatorname{div} v' = 0,\ v'_{\partial\Omega} = 0 \end{cases}$, and the following estimate is valid:

$$\|v'\|_{W_2^2(\Omega)} + \|\nabla p\|_{L_2(\Omega)} \le C\|f\|_{L_2(\Omega)} .$$

Proof. From the Cattabriga theorem [10], we have that $\{v', p\}$ is the strong solution of the Stokes system, $v' \in W_2^2(\Omega)$, $p \in W_2^1(\Omega)$. From the conditions of the theorem, we have

$$\sum_{j=1}^{3}\int_{\Omega}\left(\nu_1\nabla v_j,\nabla\Phi_j\right)dx=\int_{\Omega}\left(f,\Phi\right)dx \quad \forall\Phi\in J_0^\infty .$$

Thus, v' is a weak solution. To prove that it is unique, we have to verify that $v' = 0$ if $\{v', \Phi\} = 0 \ \ \forall \Phi \in J_0^\infty$. Indeed, let $\Phi_k \to v'$ in $\overset{0}{W}{}_2^1$. Then, we will have that $\|v'\|_{\overset{0}{W}{}_2^1(\Omega)} = 0$, where

$$\|v'\|_{\overset{0}{W}{}_2^1(\Omega)} = \sqrt{\{v', v'\}_{\overset{0}{W}{}_2^1(\Omega)}} = \left(\sum_{j=1}^{3}\int_{\Omega}\left|\nabla v_j\right|^2 dx\right)^{\frac{1}{2}} .$$

In this way, the theorem is proved.

II. Problem Formulation

We introduce a real parameter λ and consider the problem (2), (3) in Q_T as:

$$\begin{cases} v_t + (v'\cdot\nabla)\tilde{v} - \nu\Delta v + Mv + \nabla p = \lambda f(t,x) \\ \operatorname{div} v' = 0,\ v\big|_{t=0} = 0,\ v\big|_{\partial\Omega} = 0 \ . \end{cases} \tag{9}$$

We will obtain the needed solution later by putting $\lambda = 1$.

Let us assume that the solution $\{v, \nabla p\}$ of the problem (9) is analytic with respect to λ near the point $\lambda = 0$. (Later we will prove it).

For $m \ge 0$, we introduce the following notations.

$$v(t,x,\lambda) = \sum_{m=0}^{\infty}\frac{v^{(m)}(t,x)}{m!}\lambda^m,$$

$$v^{(m)}(t,x) = \left.\frac{\partial^m v(t,x,\lambda)}{\partial\lambda^m}\right|_{\lambda=0},$$

$$\nabla p(t,x,\lambda) = \sum_{m=0}^{\infty}\frac{\nabla p^{(m)}(t,x)}{m!}\lambda^m,$$

$$\nabla p^{(m)}(t,x) = \left.\frac{\partial^m v(t,x,\lambda)}{\partial\lambda^m}\right|_{\lambda=0} .$$

We will define the functions $v^{(m)}(t,x)$ as solutions of the following iterative linear problems.

First, let $m = 0$ and let us consider the problem

$$\begin{cases} v_t^{(0)} + \left(v'^{(0)}\cdot\nabla\right)\tilde{v}^{(0)} - \nu\Delta v^{(0)} + Mv^{(0)} + \nabla p^{(0)} = 0 \\ \operatorname{div} v'^{(0)} = 0,\ v^{(0)}\big|_{t=0} = 0,\ v^{(0)}\big|_{\partial\Omega} = 0 \ . \end{cases} \tag{10}$$

We choose $v^{(0)} = 0$, $v^{(0)} = 0$, $\nabla p^{(0)} = 0$ as the solution of the homogeneous problem (10).

For $m = 1$, the pair $\{v^{(1)}(t,x), \nabla p^{(1)}(t,x)\}$ will be the solution of the linear problem

$$\begin{cases} v_t^{(1)} - \nu\Delta v^{(1)} + Mv^{(1)} + \nabla p^{(1)} = f(t,x) \\ \operatorname{div} v'^{(1)} = 0,\ v^{(1)}\big|_{t=0} = 0,\ v^{(1)}\big|_{\partial\Omega} = 0 \ . \end{cases} \tag{11}$$

The problem (11) is obtained from (9) by differentiating once with respect to λ and taking into account the values $v^{(0)} = 0$, $\nabla p^{(0)} = 0$. Using the assumption of smoothness with respect to λ, we differentiate (9) $m \ge 1$ times and thus obtain

$$\begin{cases} v_t^{(m)} - \nu\Delta v^{(m)} + Mv^{(m)} + \nabla p^{(m)} = f^{(m-1)}(t,x) \\ \operatorname{div} v'^{(m)} = 0,\ v^{(m)}\big|_{t=0} = 0,\ v^{(m)}\big|_{\partial\Omega} = 0 \ , \end{cases} \tag{12}$$

where

$$f^{(m-1)}(t,x) = -\sum_{j=1}^{m-1}\frac{m!}{j!(m-j)!}\left(v'^{(j)}\cdot\nabla\right)\tilde{v}^{(m-j)},\ \ m \ge 2.$$

Let us consider the auxiliary problem (12) in the linearized form

$$\begin{cases} v_t - \nu\Delta v + Mv + \nabla p = f(t,x) \\ \operatorname{div} v' = 0, \ \ v\big|_{t=0} = 0, \ \ v\big|_{\partial\Omega} = 0 \ . \end{cases} \tag{13}$$

To establish the properties of weak solutions for the problem (13), we repeat the reasoning for the proof of Theorem 1 and thus we can easily prove the following result.

Theorem 3. There exists a unique weak solution $v(t,x)$ of the problem (13) such that $v_t \in L_2(Q_T)$, $D_x^\alpha v \in L_2(Q_T), |\alpha| = 1$, $v' \in J_2^1$, $v'_t \in J_2$ for almost every $t \in (0,T]$; and the following estimate holds

$$\|v_t\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|=1} \|D_x^\alpha v\|_{L_2(Q_T)} \le C\|f\|_{L_2(Q_T)},$$

where the positive constant C depends only on $\beta, T, \partial\Omega$.

Now, we need to establish the unique strong solvability of the problem (13).

Theorem 4. There exists a unique strong solution $\{v(t,x), \nabla p(t,x)\}$ of (13) such that $v_t \in L_2(Q_T)$, $D_x^\alpha v \in L_2(Q_T), |\alpha| \le 2$, $v' \in J_2^1$ for almost every $t \in (0,T]$, $\nabla p \in L_2(Q_T)$; and the following estimate holds

$$\|v_t\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\le 2} \|D_x^\alpha v\|_{L_2(Q_T)} + \|\nabla p\|_{L_2(Q_T)} \le C\|f\|_{L_2(Q_T)},$$

where the positive constant C depends only on $\beta, T, \partial\Omega$.

Proof. From Theorem 3, there exists a weak solution $v \in W_2^1(Q_T)$ of (13) such that $v' \in J_2^1$ for almost every $t \in (0,T]$. Since $v_t \in L_2(Q_T)$, then, by virtue of Fubini theorem, the norm $\|v_t\|_{L_2(\Omega)}$ is finite for almost every $t \in (0,T]$. For the problem (13), we will introduce the notations:

$$\begin{aligned} & v = (v', v''), \ \ v' = (v_1, v_2, v_3), \ \ v'' = (v_4, v_5), \\ & \Phi = (\Phi', \Phi''), \ \ \Phi' = (\Phi_1, \Phi_2, \Phi_3), \ \ \Phi'' = (\Phi_4, \Phi_5), \\ & F' = (F_1, F_2, F_3), \ \ F'' = (F_4, F_5), \ \text{where} \\ & F = -v_t - Mv + f. \end{aligned}$$

In this way, we will split the problem (13) into two following problems:

a) a Dirichlet problem for the linearized Stokes system

$$\begin{cases} -\nu_1\Delta v' + \nabla p = F'(t,x) \\ \operatorname{div} v' = 0, \ \ v'\big|_{\partial\Omega} = 0 \end{cases}, \text{ and}$$

b) a Dirichlet problem for Poisson equation

$$\begin{cases} -\nu_2\Delta v_4 = F_4(t,x), \ F_5(t,x) = 0, \\ v''\big|_{\partial\Omega} = 0 \end{cases}.$$

From Theorem 2 and the estimates

$$\|v\|_{L_2(\Omega)}^2 + \nu_0\|v_x\|_{L_2(Q_T)}^2 \le C\|f\|_{L_2(Q_T)}^2,$$

$$\|v_t\|_{L_2(Q_T)}^2 + \nu_0\|v_x\|_{L_2(\Omega)}^2 \le C\|f\|_{L_2(Q_T)}^2.$$

It follows that there exist the derivatives $D_x^\alpha v' \in L_2(Q_T)$, $|\alpha| \le 2$, and also that there exists the function $\nabla p \in L_2(Q_T)$, such that the estimate holds:

$$\|v'_t\|_{L_2(Q_T)} + \nu_1 \sum_{|\alpha|\le 2} \|D_x^\alpha v'\|_{L_2(Q_T)} + \|\nabla p\|_{L_2(Q_T)} \le C\|F'\|_{L_2(Q_T)},$$

where the positive constant C depends only on $\beta, T, \partial\Omega$.

For the test functions $\Phi' = 0$, we define the weak solution as:

$$\int_{Q_T} \{(\nu_2\nabla v_4, \nabla\Phi_4) - (F'', \Phi'')\}\, dxdt = 0,$$
$$\forall \Phi'' \in W_{2,x,t}^{1,0}(Q_T) \ : \ \Phi''\big|_{\partial\Omega} = \Phi''\big|_{t=T} = 0.$$

It is easy to see that $v \in W_2^2(\Omega)$ for almost every $t \in (0,T)$, and that the estimate is valid:

$$\nu_2 \sum_{|\alpha|\le 2} \|D_x^\alpha v''\|_{L_2(Q_T)} \le C\|F''\|_{L_2(Q_T)},$$

in the other terms, we have that every weak solution $v'' \in \overset{0}{W}{}_2^1(\Omega)$ of the problem $\int_\Omega \{(\nu_2\nabla v_4, \nabla u_4)\}\, dx = \int_\Omega (F'', u'')\, dx$ with $F'' \in L_2(\Omega)$, will be a solution from $W_2^2(\Omega)$ for fixed values of $t \in (0,T]$.

In this way, we have obtained that the solution $v = (v', v'')$ has all the derivatives of the second order such that $D_x^\alpha v \in L_2(Q_T)$, $|\alpha| \le 2$, $v_t \in L_2(Q_T), \nabla p \in L_2(Q_T)$, and the required estimate holds. Thus, the strong solution of the linearized problem (13) that exists and is unique, concludes the proof.

III. Problem Solution

To construct the strong solution of the main nonlinear problem, we will use two following helpful results from [11].

Lemma 1. Let Ω be a bounded domain in R^3, $\partial\Omega \in C^2$, $n \geq 2$, $\frac{(n+3)}{3} \leq p < \infty$, $\nu > 0, T > 0$. Let, additionally, E be a Banach space with the norm

$$\|f\|_E = \|f_t\|_{L_p(Q_T)} + \nu \sum_{|\alpha|\leq 2} \|D_x^\alpha f\|_{L_p(Q_T)}.$$

Then, there exists a constant $C_0 > 0$ which depends on n, p, Ω, such that the estimate holds:

$$\left(\int_{Q_T} |u|^p |v_x|^p \, dxdt\right)^{\frac{1}{p}} \leq C_0 \nu^{-\frac{n+p}{2p}} T^{\frac{3}{2}-\frac{n+2}{2p}} \|u\|_E \|v\|_E$$

for all $u, v \in E : u(x,0) = v(x,0) = 0, \; x \in \Omega$.

Lemma 2. Let $\{\beta_m\}_{m=1}^\infty$ be a sequence of non-negative real numbers such that $\beta_1 = \beta \geq 0$ and $\beta_m \leq \gamma \sum_{j=1}^{m-1} \beta_j \beta_{m-j}$ for $m \geq 2$, $\gamma > 0$. Then, the series

$$\sum_{m=1}^\infty \beta_m t^m, \qquad \sum_{m=2}^\infty t^m \left(\sum_{j=1}^{m-1} \beta_j \beta_{m-j}\right)$$

converge for $4\beta\gamma|t| < 1$.

Now, we shall prove the main result of this paper.

Theorem 5. Let Ω be a bounded domain in R^3, $\partial\Omega \in C^2$, $f \in L_2(Q_T)$ and suppose that the condition holds:

$$8C^2 C_0 \nu_0^{-\frac{5}{4}} T^{\frac{1}{4}} \|f\|_{L_2(Q_T)} \leq 1,$$

where C_0 is the constant from Lemma 1 and C is the constant from theorem 4. Then, there exists a strong solution $\{v(t,x), \nabla p(t,x)\}$ of the nonlinear problem (2), (3) such that $v' \in J_2^1$ for almost all $t \in (0,T)$, $v_t \in L_2(Q_T)$, $D_x^\alpha v \in L_2(Q_T), |\alpha| \leq 2$, $\nabla p \in L_2(Q_T)$. This solution is unique and is defined by $\lambda = 1$ in (9).

Proof. We apply Theorem 4 to the problem (11) and thus conclude that there exists a unique strong solution $\{v^{(1)}, \nabla p^{(1)}\}$ of (11) such that $v'^{(1)} \in J_2^1$ for almost all $t \in (0,T)$, $v_t^{(1)} \in L_2(Q_T)$, $D_x^\alpha v^{(1)} \in L_2(Q_T), |\alpha| \leq 2$; $\nabla p^{(1)} \in L_2(Q_T)$. Additionally, by virtue of Theorem 4, we have the estimate for $m = 1$:

$$\|v_t^{(1)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(1)}\|_{L_2(Q_T)} + \|\nabla p^{(1)}\|_{L_2(Q_T)} \leq C\|f\|_{L_2(Q_T)}.$$

Now, we apply theorem 4 to the problem (12) and obtain that there exists a unique strong solution $\{v^{(m)}, \nabla p^{(m)}\}$ of (12) such that $v'^{(m)} \in J_2^1$ for almost all $t \in (0,T)$, $v_t^{(m)} \in L_2(Q_T), D_x^\alpha v^{(m)} \in L_2(Q_T), |\alpha| \leq 2$, $\nabla p^{(m)} \in L_2(Q_T)$. From Theorem 4, we also have that for $m \geq 2$ the estimate holds:

$$\|v_t^{(m)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(m)}\|_{L_2(Q_T)} + \|\nabla p^{(m)}\|_{L_2(Q_T)} \leq \leq C\|f^{(m-1)}\|_{L_2(Q_T)}.$$

Therefore, from (12) and Lemma 1 we have

$$\|v_t^{(m)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(m)}\|_{L_2(Q_T)} \leq$$
$$\leq C_1(\nu_0, T)\left(\sum_{j=1}^{m-1} \frac{m!}{j!(m-j)!}\right)\left(\|v_t^{(j)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(j)}\|_{L_2(Q_T)}\right) \times$$
$$\times \left(\|v_t^{(m-j)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(m-j)}\|_{L_2(Q_T)}\right),$$

where $C_1(\nu_0, T) = CC_0 \nu_0^{-\frac{5}{4}} T^{\frac{1}{4}}$.

If we introduce the notations

$$\beta_1 = \|v_t^{(1)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(1)}\|_{L_2(Q_T)},$$
$$\beta_m = \frac{1}{m!}\left(\|v_t^{(m)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(m)}\|_{L_2(Q_T)}\right),$$

then we can write the last inequality in the following way:

$$\beta_m \leq C_1(\nu_0, T) \sum_{j=1}^{m-1} \beta_j \beta_{m-j}.$$

therefore, by Lemma 2, the series

$$\sum_{m=0}^\infty \frac{\lambda^m}{m!}\left\{\|v_t^{(m)}\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\leq 2} \|D_x^\alpha v^{(m)}\|_{L_2(Q_T)}\right\} \tag{14}$$

is convergent, if the following condition holds:

$$\left\{\left\|v_t^{(1)}\right\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\le 2}\left\|D_x^\alpha v^{(1)}\right\|_{L_2(Q_T)}\right\} < \frac{1}{4|\lambda| C_1(\nu_0,T)}.$$

Thus, we obtain

$$4|\lambda| C_1(\nu_0,T)\left\{\left\|v_t^{(1)}\right\|_{L_2(Q_T)} + \nu_0 \sum_{|\alpha|\le 2}\left\|D_x^\alpha v^{(1)}\right\|_{L_2(Q_T)}\right\} \le$$
$$\le 4|\lambda| C_1(\nu_0,T) C\|f\|_{L_2(Q_T)}.$$

It follows from the assumption of this theorem that

$$4|\lambda| C^2 C_0 \nu_0^{-\frac{5}{4}} T^{\frac{1}{4}} \|f\|_{L_2(Q_T)} \le \frac{|\lambda|}{2}.$$

In this way, the series (14) will be convergent for $|\lambda| < 2$. Again, let us denote

$$v(t,x,\lambda) = \sum_{m=0}^{\infty} \frac{v^{(m)}(t,x)}{m!}\lambda^m,\ \nabla p(t,x,\lambda) = \sum_{m=0}^{\infty} \frac{\nabla p^{(m)}(t,x)}{m!}\lambda^m.$$

Since the series (14) is convergent, it is evident that $v' \in J_2^1$ for almost all $t \in (0,T), D_x^\alpha v \in L_2(Q_T), |\alpha| \le 2$ for $|\lambda| < 2$.

From (12) and Lemma 1, we have

$$\frac{1}{m!}\left\|f^{(m-1)}\right\|_{L_2(Q_T)} \le \frac{C_1(\nu_0,T)}{C}\sum_{j=1}^{m-1}\beta_j\beta_{m-j},\quad m \ge 2.$$

Therefore, it follows from Lemma 2 that the series

$$\sum_{m=2}^{\infty}\frac{\lambda^m}{m!}\left\|f^{(m-1)}\right\|_{L_2(Q_T)}$$

Converges for $|\lambda| < 2$. As a corollary, we have the convergence of the series $\sum_{m=2}^{\infty}\frac{\lambda^m}{m!}\left\|\nabla p^{(m)}\right\|_{L_2(Q_T)}$, which implies $\nabla p(t,x,\lambda) \in L_2(Q_T)$ for $|\lambda| < 2$. Due to the above construction, we have proved that for $|\lambda| < 2$ there exists a strong solution $\{v(t,x), \nabla p(t,x)\}$ of the problem (9) such that $v'(t,x,\lambda) \in J_2^1$ for almost all $t \in (0,T)$, $v_t \in L_2(Q_T), D_x^\alpha v(t,x,\lambda) \in L_2(Q_T), |\alpha| \le 2, \nabla p(t,x,\lambda) \in L_2(Q_T)$. This solution is analytic with respect to λ for $|\lambda| < 2$. Considering $\lambda = 1$ in the problem (9), we obtain the statement of Theorem 5 for the problem (2), (3). Using standard techniques, it can be easily shown that the strong solution $\{v, \nabla p\}$ of the problem (2), (3), is unique in the interval $(0,\tau)$. After repeating the same considerations k times, where $k = \left[\frac{T}{\tau}\right] + 1$, we will obtain the uniqueness of the strong solution for the whole interval $(0,T)$; in other terms, for the domain Q_T. Therefore, the theorem is proved.

We would like to investigate now some spectral properties of the differential operator which is generated by linearized system (2). Additionally, without loss of generality, we put $\gamma = 0$. In this way, we consider the system

$$\begin{cases}\frac{\partial v}{\partial t} - \nu\Delta v + Mv + \nabla p = 0\\ \operatorname{div} v' = 0,\quad x \in \Omega,\ t \ge 0,\quad v\big|_{\partial\Omega} = 0,\end{cases} \tag{15}$$

where

$$Mv = \begin{bmatrix} -\omega v_2 \\ \omega v_1 \\ g v_5 \\ 0 \\ -\frac{N^2}{g} v_3 \end{bmatrix},\qquad \nu\Delta v = \begin{bmatrix} \nu_1\Delta v_1 \\ \nu_1\Delta v_2 \\ \nu_1\Delta v_3 \\ \nu_2\Delta v_4 \\ 0 \end{bmatrix}.$$

Let us consider the following problem of normal vibrations

$$\tilde{v}(x,t) = \tilde{u}(x)e^{-\lambda t}$$
$$v_5(x,t) = \frac{N}{g}u_5(x)e^{-\lambda t}$$
$$p(x,t) = u_6(x)e^{-\lambda t},\quad \lambda \in C.$$

We denote $u = (\tilde{u}, u_5, u_6)$ and write the system (15) in the matrix form $Lu = 0$, where $L = A - \lambda I_5$, and

$$A = \begin{pmatrix} -\nu_1\Delta & -\omega & 0 & 0 & 0 & \frac{\partial}{\partial x_1} \\ \omega & -\nu_1\Delta & 0 & 0 & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & -\nu_1\Delta & 0 & N & \frac{\partial}{\partial x_3} \\ 0 & 0 & 0 & -\nu_2\Delta & 0 & 0 \\ 0 & 0 & -N & 0 & 0 & 0 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 & 0 & 0 \end{pmatrix}.$$

We define the domain of the differential operator A associated with the considered boundary condition as follows.

$$D(A)=\left\{(\tilde{u},u_5)\in\left(\overset{0}{W}{}_2^1(\Omega)\right)^5, u_6\in L_2(\Omega):\ Au\in\left(L_2(\Omega)\right)^6\right\}.$$

Evidently, the operator A is a closed operator, and its domain is dense in $\left(L_2(\Omega)\right)^6$. We recall that the essential spectrum

$$\sigma_{ess}(A)=\left\{\lambda\in C:(A-\lambda I)^{-1}\text{ is not of Fredholm type}\right\},$$

is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity [12].

Theorem 6. The essential spectrum of the operator A is composed of one real point $\sigma_{ess}(A)=\left\{\frac{1}{\nu_1}\right\}$.

The proof is based on the concept of ellipticity in sense of Douglis-Nirenberg [13] and is analogous to the proof of the Theorem 2 in [14].

IV. Conclusion

The results of this paper, particularly the explicit algorithm for construction of the strong solution, may be applied in the theoretical and computational study of the Atmosphere and the Ocean. The essential spectrum depends only on the kinematic viscosity, which corresponds to the results of [8] for a different model of barotropic stratified fluid.

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