

# On the Maximum Theorem: A Constructive Analysis

Yasuhito Tanaka

*Abstract*—We examine the maximum theorem by Berge from the point of view of Bishop style constructive mathematics. We will show an approximate version of the maximum theorem and the maximum theorem for functions with sequentially locally at most one maximum.

*Keywords*—Maximum theorem, Constructive mathematics, Sequentially locally at most one maximum.

## I. INTRODUCTION

WE examine the maximum theorem by Berge ([1]) from the point of view of constructive mathematics á la Bishop ([3], [5], [6]). In the next section we will show an approximate version of the maximum theorem, and in Section 3 we will prove the maximum theorem for functions with sequentially locally at most one maximum.

## II. APPROXIMATE MAXIMUM THEOREM

In classical mathematics the famous Berge's maximum theorem (see [1], [4], [7]) is expressed as follows;

Let  $X, Y$  be metric spaces, let  $f$  be a continuous function from  $X \times Y$  to the set of real numbers  $\mathbb{R}$ , and let  $F$  be a compact valued continuous (upper and lower hemi-continuous) multi-function (multi valued function or correspondence) from  $X$  to the set of nonempty subsets of  $Y$ . Consider a maximization problem;

$$\text{maximize } f(x, y) \text{ subject to } y \in F(x). \quad (1)$$

This has a solution, and

- 1) the function  $\varphi = \max_{x \in X, y \in F(x)} f(x, y)$  from  $X$  to  $\mathbb{R}$ , is continuous in  $X$ , and
- 2) the multi-function  $\Phi = \{y \in F(x) | f(x, y) = \varphi(x)\}$  from  $X$  to the set of nonempty subsets of  $Y$  is upper hemi-continuous.

In constructive mathematics, however, we can not prove that the maximization problem (1) has a solution in a compact set  $F(x)$  even if  $f$  is uniformly continuous with respect to  $y$  in  $F(x)$ . Instead we can prove that  $f$  has the supremum in  $F(x)$  (see Corollary 2.2.7 in [6]). The supremum  $\sup_{y \in F(x)} f(x, y)$  of  $f$  in  $F(x)$  satisfies

$$\sup_{y \in F(x)} f(x, y) \geq f(x, y') \text{ for all } y' \in F(x),$$

and

for any  $\varepsilon > 0$  there exists  $y' \in F(x)$  such that

$$\sup_{y \in F(x)} f(x, y) \leq f(x, y') + \varepsilon.$$

Y. Tanaka is with the Faculty of Economics, Doshisha University, Kyoto, 602-8580, Japan, e-mail: (yasuhito@mail.doshisha.ac.jp).

In constructive mathematics compactness of a set means *total boundedness with completeness*, and a nonempty set is called an *inhabited* set. A set  $S$  is totally bounded if for each  $\varepsilon > 0$  there exists a finitely enumerable  $\varepsilon$ -approximation to  $S$ . A set is finitely enumerable if there exist a natural number  $\mathbb{N}$  and a mapping of the set  $\{1, 2, \dots, N\}$  onto that set. An  $\varepsilon$ -approximation to  $S$  is a subset of  $S$  such that for each  $x \in S$  there exists  $y$  in that  $\varepsilon$ -approximation with  $|x - y| < \varepsilon$ . Completeness of a set, of course, means that every Cauchy sequence in the set converges. A set  $S$  is inhabited if there exists an element of  $S$ .

Note that in order to show that  $S$  is inhabited, we cannot just prove that it is impossible for  $S$  to be empty; we must actually construct an element of  $S$  (see page 12 of [6]).

We present some definitions. Let  $X$  and  $Y$  be metric spaces.

*Definition 1 (Continuity of functions):* A function  $f$  from  $X$  to  $Y$  is continuous at  $x \in X$  if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|f(x) - f(x')| < \varepsilon$  for all  $x' \in X$  with  $|x - x'| < \delta$ . If  $f$  is continuous at every  $x \in X$ , it is continuous in  $X$ .  $\delta$  depends on  $x$  and  $\varepsilon$ .

*Definition 2 (Uniform continuity of functions):* A function  $f$  from  $X$  to  $Y$  is uniformly continuous if for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $x, x' \in X$ , if  $|x - x'| < \delta$ , then  $|f(x) - f(x')| < \varepsilon$ .  $\delta$  depends on only  $\varepsilon$ .

*Definition 3: (Graph of multi-functions and uniformly closed graph)* A graph of a multi-function  $F$  from  $X$  to the set of inhabited subsets of  $Y$  is

$$G(F) = \cup_{x \in X} \{x\} \times F(x).$$

If  $G(F)$  is a closed set, we say that  $F$  has a closed graph. It implies the following fact.

Consider sequences  $(x_n)_{n \geq 1}$  and  $(y_n)_{n \geq 1}$  such that  $y_n \in F(x_n)$ . If  $x_n \rightarrow x$ , then for some  $y \in F(x)$  we have  $y_n \rightarrow y$ .

This means

For each  $\varepsilon > 0$  if there exists  $n_0$  such that  $|x_n - x| < \varepsilon$  when  $n \geq n_0$ , then there exists  $n'_0$  such that  $|y_n - F(x)| < \varepsilon$ , that is,  $|y_n - y| < \varepsilon$  for some  $y \in F(x)$  when  $n \geq n'_0$ .

$n_0$  and  $n'_0$  depend on  $x$  and  $\varepsilon$ . Further we consider a uniform version of this property for multi-functions, and call such a multi-function a *multi-function with uniformly closed graph*, or say that a *multi-function has a uniformly closed graph*. It means that  $n_0$  and  $n'_0$  depend on only  $\varepsilon$  not on  $x$ .

The closed graph property of multi-functions is closely related with upper hemi-continuity. But, we do not use such

a terminology. We define continuity of multi-functions as follows;

*Definition 4 (Continuity of multi-functions):* A multi-function  $F$  from  $X$  to the set of inhabited subsets of  $Y$  is continuous if

- 1) it has a uniformly closed graph, and
- 2) For every sequence  $(x_n)_{n \geq 1}$  such that  $x_n \rightarrow x$  and  $y \in F(x)$ , there exist a sequence  $(y_n)_{n \geq 1}$  such that  $y_n \in F(x_n)$  and  $y_n \rightarrow y$ .

This means

For each  $\varepsilon > 0$  if there exists  $n_0$  such that  $|x_n - x| < \varepsilon$  when  $n \geq n_0$ , then there exists  $n'_0$  such that  $|y_n - y| < \varepsilon$  when  $n \geq n'_0$ .

This definition is equivalent to the condition called the lower hemi-continuity. Again we do not use such a terminology.  $n_0$  and  $n'_0$  depend on  $x$  and  $\varepsilon$ . Further we consider a uniform version of this property. It means that  $n_0$  and  $n'_0$  depend on only  $\varepsilon$  not on  $x$ .

As stated above the supremum

$$\sup_{y \in F(x)} f(x, y). \tag{2}$$

of  $f$  exists. We define

$$\varphi(x) = \sup_{x \in X, y \in F(x)} f(x, y).$$

It is a function from  $X$  to  $\mathbf{R}$ , and define

$$\Phi(x) = \{y \in F(x) | \varphi(x) \geq f(x, y) \geq \varphi(x) - \varepsilon\}.$$

It is a multi-function from  $X$  to the set of inhabited subsets of  $Y$ .

We show the following theorem.

*Theorem 1:* Let  $X, Y$  be metric spaces, let  $f$  be a uniformly continuous function from  $X \times Y$  to  $\mathbf{R}$ , and let  $F$  be a compact valued continuous multi-function from  $X$  to the set of inhabited subsets of  $Y$ . Then,

- 1)  $\varphi$  is uniformly continuous in  $X$ , and
- 2)  $\Phi$  has a uniformly closed graph.

*Proof:* Consider sequences  $(x_n)_{n \geq 1}$  in  $X$  and  $(y_n)_{n \geq 1}$  in  $Y$  such that  $y_n \in \Phi(x_n)$ ,  $x_n \rightarrow x$  and  $y_n \rightarrow y$ .  $y_n \in \Phi(x_n)$  means  $y_n \in F(x_n)$  and  $f(x_n, y_n) \geq \varphi(x_n) - \varepsilon$ . Since  $F$  is a continuous multi-function, we have  $y \in F(x)$ , and for every  $y' \in F(x)$  there exist sequences  $(x_n)_{n \geq 1}$  and  $(y'_n)_{n \geq 1}$  such that  $y'_n \in \Phi(x_n)$ ,  $x_n \rightarrow x$  and  $y'_n \rightarrow y'$ . Assume  $f(x, y') > f(x, y) + \varepsilon$ . Then,  $f(x_n, y'_n) > f(x_n, y_n) + \varepsilon$  for sufficiently large  $n$ . But it contradicts  $y_n \in \Phi(x_n)$  because  $f(x_n, y_n) \geq \varphi(x_n) - \varepsilon$  and  $\varphi(x_n) \geq f(x_n, y'_n)$  imply  $f(x_n, y_n) \geq f(x_n, y'_n) - \varepsilon$ , and so  $f(x, y) \geq \varphi(x) - \varepsilon$  and  $y \in \Phi(x)$ . Since  $F$  has a uniformly closed graph,  $\Phi$  also has a uniformly closed graph.

Consider  $x, x' \in X$  and  $y \in \Phi(x)$ ,  $y' \in \Phi(x')$ . We have  $|\varphi(x) - f(x, y)| \leq \varepsilon$  and  $|\varphi(x') - f(x', y')| \leq \varepsilon$ . Therefore, if  $|f(x, y) - f(x', y')| < \varepsilon$ , we have  $|\varphi(x) - \varphi(x')| < 3\varepsilon$ . It means that  $\varphi$  is uniformly continuous because  $f$  is uniformly continuous. ■

### 3. The maximum theorem for functions with sequentially locally at most one maximum

In the previous section we have proved an approximate version of the maximum theorem. In this section by reference to the notion of *sequentially at most one maximum* in [2] we consider a property of functions which is called *sequentially locally at most one maximum*.

Let  $M$  be the supremum of a function  $f$  in a compact set  $X$ . The notion that  $f$  has at most one maximum in [2] is defined as follows;

*Definition 5 (At most one maximum):* For all  $x, y \in X$ , if  $x \neq y$ , then  $f(x) < M$  or  $f(y) < M$ .

On the other hand, *sequentially at most one maximum* also in [2] is defined as follows;

*Definition 6 (Sequentially at most one maximum):* All sequences  $(x_n)_{n \geq 1}, (y_n)_{n \geq 1}$  in  $X$  such that  $|f(x_n) - M| \rightarrow 0$  and  $|f(y_n) - M| \rightarrow 0$  are eventually close in the sense that  $|x_n - y_n| \rightarrow 0$ .

Next we define a notion *sequentially locally at most one maximum*. According to Corollary 2.2.12 of [6] about total boundedness of a set we have the following result.

*Lemma 1:* If a set  $X$  is totally bounded, for each  $\varepsilon > 0$  there exist totally bounded sets  $H_1, H_2, \dots, H_n$ , each of diameter less than or equal to  $\varepsilon$ , such that  $X = \cup_{i=1}^n H_i$ .

The definition of the notion *sequentially locally at most one maximum* is as follows;

*Definition 7: (Sequentially locally at most one maximum)* Let  $M = \sup f$  in  $X$ . There exists  $\bar{\varepsilon} > 0$  with the following property. For each  $\varepsilon > 0$  less than or equal to  $\bar{\varepsilon}$  there exist totally bounded sets  $H_1, H_2, \dots, H_m$ , each of diameter less than or equal to  $\varepsilon$ , such that  $X = \cup_{i=1}^m H_i$ , and if for all sequences  $(x_n)_{n \geq 1}, (y_n)_{n \geq 1}$  in each  $H_i$ ,  $|f(x_n) - M| \rightarrow 0$  and  $|f(y_n) - M| \rightarrow 0$ , then  $|x_n - y_n| \rightarrow 0$ .

Now we show the following lemma, which is based on Lemma 2 of [2].

*Lemma 2:* Let  $f$  be a uniformly continuous function from a compact set  $X$  to  $\mathbf{R}$ . Assume  $\sup_{x \in H_i} f(x) = M$  for some  $H_i \subset X$  defined above. If the following property holds:

For each  $\delta > 0$  there exists  $\varepsilon > 0$  such that if  $x, y \in H_i$ ,  $f(x) \geq M - \varepsilon$  and  $f(y) \geq M - \varepsilon$ , then  $|x - y| \leq \delta$ .

Then, there exists a point  $z \in H_i$  such that  $f(z) = M$ , that is,  $f$  has the maximum.

*Proof:* Choose a sequence  $(x_n)_{n \geq 1}$  in  $H_i$  such that  $f(x_n) \rightarrow M$ . Compute  $N$  such that  $f(x_n) \geq M - \varepsilon$  for all  $n \geq N$ . Then, for  $m, n \geq N$  we have  $|x_m - x_n| \leq \delta$ . Since  $\delta > 0$  is arbitrary,  $(x_n)_{n \geq 1}$  is a Cauchy sequence in  $H_i$ , and converges to a limit  $z \in H_i$ . The continuity of  $f$  yields  $f(z) = M$ . ■

Next we show the following lemma, which is based on Proposition 3 of [2].

*Lemma 3:* Each uniformly continuous function  $f$  from  $X$  to  $\mathbf{R}$ , which has sequentially locally at most one maximum, has the maximum.

*Proof:* Choose a sequence  $(z_n)_{n \geq 1}$  in  $H_i$  defined above such that  $f(z_n) \rightarrow M$ . In view of Lemma 2 it is enough to prove that the following condition holds.

For each  $\delta > 0$  there exists  $\varepsilon > 0$  such that if  $x, y \in H_i$ ,  $f(x) \geq M - \varepsilon$  and  $f(y) \geq M - \varepsilon$ , then  $|x - y| \leq \delta$ .

Assume that the set

$$K = \{(x, y) \in H_i \times H_i : |x - y| \geq \delta\}$$

is inhabited and compact (see Theorem 2.2.13 of [6]). Since the mapping  $(x, y) \rightarrow \min(f(x), f(y))$  is uniformly continuous, we can construct an increasing binary sequence  $(\lambda_n)_{n \geq 1}$  such that

$$\lambda_n = 0 \Rightarrow \sup_{(x, y) \in K} \min(f(x), f(y)) > M - 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \sup_{(x, y) \in K} \min(f(x), f(y)) < M - 2^{-n-1}.$$

It suffices to find  $n$  such that  $\lambda_n = 1$ . In that case, if  $f(x) > M - 2^{-n-1}$ ,  $f(y) > M - 2^{-n-1}$ , we have  $(x, y) \notin K$  and  $|x - y| \leq \delta$ . Assume  $\lambda_1 = 0$ . If  $\lambda_n = 0$ , choose  $(x_n, y_n) \in K$  such that  $\min(f(x_n), f(y_n)) > M - 2^{-n}$ , and if  $\lambda_n = 1$ , set  $x_n = y_n = z_n$ . Then,  $f(x_n) \rightarrow M$  and  $f(y_n) \rightarrow M$ , so  $|x_n - y_n| \rightarrow 0$ . Computing  $N$  such that  $|x_N - y_N| < \delta$ , we must have  $\lambda_N = 1$ . We have completed the proof. ■

This lemma means that  $f(x, y)$  has the maximum in  $F(x)$ , that is,  $\max_{y \in F(x)} f(x, y)$  exists. We define

$$\psi(x) = \max_{x \in X, y \in F(x)} f(x, y).$$

It is a function from  $X$  to  $\mathbf{R}$ , and define

$$\Psi(x) = \{y \in F(x) | f(x, y) = \psi(x)\}.$$

It is a multi-function from  $X$  to the set of inhabited subsets of  $Y$ .

Now we show the following theorem which is the maximum theorem for functions with sequentially locally at most one maximum.

*Theorem 2:* Let  $X, Y$  be metric spaces, let  $f$  be a uniformly continuous function with sequentially locally at most one maximum from  $X \times Y$  to  $\mathbf{R}$ , and let  $F$  be a compact valued continuous multi-function from  $X$  to the set of inhabited subsets of  $Y$ . Then,

- 1)  $\psi$  is uniformly continuous in  $X$ , and
- 2)  $\Psi$  has a uniformly closed graph.

*Proof:* Consider sequences  $(x_n)_{n \geq 1}$  in  $X$  and  $(y_n)_{n \geq 1}$  in  $Y$  such that  $y_n \in \Psi(x_n)$ ,  $x_n \rightarrow x$  and  $y_n \rightarrow y$ .  $y_n \in \Psi(x_n)$  means  $y_n \in F(x_n)$  and  $f(x_n, y_n) = \psi(x_n)$ . Since  $F$  is a continuous multi-function, we have  $y \in F(x)$ , and for every  $y' \in F(x)$  there exist sequences  $(x_n)_{n \geq 1}$  and  $(y'_n)_{n \geq 1}$  such that  $y'_n \in \Psi(x_n)$ ,  $x_n \rightarrow x$  and  $y'_n \rightarrow y'$ . Assume  $f(x, y') > f(x, y)$ . Then,  $f(x_n, y'_n) > f(x_n, y_n)$  for sufficiently large  $n$ . But it contradicts  $y_n \in \Psi(x_n)$ , and so  $f(x, y) = \psi(x)$  and  $y \in \Psi(x)$ . Since  $F$  has a uniformly closed graph,  $\Psi$  also has a uniformly closed graph.

Since  $\psi(x_n) = f(x_n, y_n) \rightarrow f(x, y) = \psi(x)$ ,  $\psi$  is uniformly continuous because  $f$  is uniformly continuous. ■

The maximum theorem is widely used in mathematical economics. If we consider  $f$  as a utility function of a consumer and  $F$  his budget constraint, then Theorem 2 implies the existence of the demand correspondence which has a uniformly closed graph.

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#### REFERENCES

- [1] C. Berge. *Topological Spaces*. Macmillan, 1963.
- [2] J. Berger, D. Bridges, and P. Schuster. The fan theorem and unique existence of maxima. *Journal of Symbolic Logic*, 71:713–720, 2006.
- [3] E. Bishop and D. Bridges. *Constructive Analysis*. Springer, 1985.
- [4] K. Border. *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge University Press, 1985.
- [5] D. Bridges and F. Richman. *Varieties of Constructive Mathematics*. Cambridge University Press, 1987.
- [6] D. Bridges and L. Vîță. *Techniques of Constructive Mathematics*. Springer, 2006.
- [7] T. Ichiishi. *Game Theory for Economic Analysis*. Academic Press, 1983.

**Yasuhito Tanaka** is Professor at the Faculty of Economics, Doshisha University, 602-8580, Japan.