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On the Efficiency of Five Step Approximation Method for the Solution of General Third Order Ordinary Differential Equations

N. M. Kamoh, M. C. Soomiyol

Abstract**—**In this work, a five step continuous method for the solution of third order ordinary differential equations was developed in block form using collocation and interpolation techniques of the shifted Legendre polynomial basis function. The method was found to be zero-stable, consistent and convergent. The application of the method in solving third order initial value problem of ordinary differential equations revealed that the method compared favorably with existing methods.

Keywords**—**Shifted Legendre polynomials, third order block method, discrete method, convergent.

I. Introduction

DIFFERENTIAL equations are mathematical equations that involve one or several variables of an unknown function that relate the function and its derivatives. These equations play very important role in engineering, physics, economics and many other disciplines. Differential equations arise in many areas of social, science and technology. Differential equations can predict the world around us; it describes the population growth of animals (Malthusian Law of populatpion growth) or the change in investment return over time [16]

Differential equations are studied for several reasons. It is mostly found in the field of medicine for modeling cancer growth and in the spread of diseases, the movement of electricity in engineering and the modeling of chemical reactions in chemistry. Other areas that ODEs play very important roles are economics in finding optimum investment strategies and describing the motion of waves, pendulums or chaotic systems in physics. It is also used in physics in Newton's second law of motion and the law of cooling, in Hooke's law for modeling the motion of a spring or in representing models for population growth and money flow or circulation. Only the simplest differential equations are readily solvable by explicit formulas. If an explicit formula for the solution is not available, the solution may be approximated numerically (see [16] and [17]).

While many numerical methods have been developed to determine the solution of initial value problems of ordinary differential equations with a given degree of accuracy by different researchers, [13] introduced a new derivation of continuous multistep method using power series as basis function. A continuous implicit hybrid one-step method for the solution of second order ordinary differential equations was investigated using power series as basis function by [1]. A family of implicit uniformly accurate order block integrators for the solution of second order differential equations using power series as basis function was investigated in [2]. Reference [9] constructed numerical solution of third order ordinary differential equations using a seven-step block method. A robust optimal order formula for direct integration of second order orbital problems was developed by [14]. Collocation techniques were employed in this work to derive the block discrete formulae which shall be used in a block-mode to solve third order initial value problems of ordinary differential equations directly without the need of starting values as discussed by researchers such as [1], [2], [8], [10] [13]. Similarly, work on third order methods were investigated using the block approach by many researchers. Reference [4] developed a class of hybrid collocation methods for third order of ordinary differential equations. Reference [7] used linear multistep method for the numerical integration of third order boundary value problems. Reference [5] introduced some multi-derivative hybrid block methods for the solution of general third order ordinary differential equations. A four-point fully implicit method for the numerical integration of third-order ordinary differential equations was developed in [3]. Reference [15] investigated the approximate solution with continuous coefficients for solving third order ordinary differential equations. There is therefore the need to develop a method which is self-starting thereby eliminating the use of predictors with better accuracy and efficiency. This paper seeks to present a block method which is based on the idea of multistep collocation using the shifted Legendre polynomials as basis function for the direct solution of third order initial value problems of ordinary differential equations.

II. Derivation of the Method

Consider the initial value problems of ordinary differential equations of the third order of the form;

$$y'''(x) = f\left(x, y(x), y'(x), y''(x)\right), y(0) = \alpha_1, y'(0) = \alpha_2, y''(0) = \alpha_3 \quad (1)$$

The idea here is to approximate the exact solution $y(x)$ of

N. M. Kamoh is with the Department of Mathematics, University of Jos, Nigeria (phone: 08056612785; e-mail: mahwash1477@gmail.com).

M. C. Soomiyol is with the Department of Mathematics and Computer Science, Benue State University, Nigeria (e-mail: mrumunsoomiyol@yahoo.com).

(1) in the partition $I_n := a = x_0 < x_1 < x_2 < \cdots < x_n = b$ of the integration interval $[a, b]$ with a constant step size $h = x_{n+1} - x_n, n = 0,1,2, \ldots, N, Nh = b$ given by a shifted Legendre polynomial of the form

$$y(x) = \sum_{i=0}^{m+s-1} c_i P_i(t) \tag{2}$$

where $c_i \in \mathbb{R}$, $y \epsilon C^3(a, b)$, m and s are interpolation and collocation points respectively.

The third derivative of (2) gives

$$y'''(x) = \sum_{i=0}^{m+s-1} c_i P'''_i(t) \tag{3}$$

Substituting (3) into (1) gives a differential system of the form

$$y'''(x) = \sum_{i=0}^{m+s-1} c_i P'''_i(t) = f\left(x, y(x), y'(x), y''(x)\right) \tag{4}$$

The interpolation and collocation equations (2) and (4) are evaluated at $x_{n+r}, r = 0,1$ and $k-1$ and $x_{n+r}, r = 0(1)k$ respectively to give a system of algebraic equations of the form

$$\boldsymbol{AX} = \boldsymbol{B} \tag{5}$$

where

$$A = \begin{bmatrix} P_0(0) & P_1(0) & & P_{m+s-2}(0) & P_{m+s-1}(0) \\ P_0(h) & P_1(h) & & P_{m+s-2}(h) & P_{m+s-1}(h) \\ P_0((k-1)h) & P_1((k-1)h) & & P_{m+s-2}((k-1)h) & P_{m+s-1}((k-1)h) \\ P_0'''(0) & P_1'''(0) & & P_{m+s-2}'''(0) & P_{m+s-1}'''(0) \\ P_0'''(h) & P_1'''(h) & & P_{m+s-2}'''(h) & P_{m+s-1}'''(h) \\ . & . & & . & . \\ . & . & & . & . \\ . & . & & . & . \\ P_0'''((k-2)h) & P_1'''((k-2)h) & & P_{m+s-2}'''((k-2)h) & P_{m+s-1}'''((k-2)h) \\ P_0'''((k-1)h) & P_1'''((k-1)h) & & P_{m+s-2}'''((k-1)h) & P_{m+s-1}'''((k-1)h) \\ P_0'''(kh) & P_1'''(kh) & & P_{m+s-2}'''(kh) & P_{m+s-1}'''(kh) \end{bmatrix}$$

$$X = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ . \\ . \\ c_{m+s-2} \\ c_{m+s-1} \end{bmatrix} \quad B = \begin{bmatrix} y_n \\ y_n \\ y_{n+k-1} \\ f_n \\ . \\ . \\ . \\ f_{n+k-1} \\ f_{n+k} \end{bmatrix}$$

Solving (5) for $c_i's$, $i = 0(1)m + s - 1$ by inverse of a matrix method and substituting the result into (2) produce a continuous implicit method of the form

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{k} \beta_j(x) f(x_{n+j}, y_{n+j}) \tag{6}$$

where $\alpha_j(x)$ and $\beta_j(x)$ are coefficients to be determined.

Evaluating (6) at the non-interpolation points and its first and second derivatives evaluated at the points $x_{n+r}, r = 0(1)k$ give the block discrete scheme of the form

$$A^{(0)} w_m = B^{(0)} q_m + h^3 D^{(0)} F_m \tag{7}$$

where $A^{(0)}$, $B^{(0)}$ and $D^{(0)}$ are square matrices

$$w_m = [y_{n+1}, \ldots, y_{n+k}, hU_{n+1}, \ldots, hU_{n+k}, h^2 V_{n+1}, \ldots, h^2 V_{n+k}]^T$$

$$q_m = [y_n, hU_n, h^2 V_n]^T, F_m = [f_n, f_{n+1}, f_{n+2}, \ldots, f_{n+k}]^T$$

Specification of the Method

Considering $k = 5$, the interpolation of (2) at $x_{n+r}, r = 0, 1, 4$ and collocating (4) at $x_{n+r}, r = 0, 1, 2, 3, 4, 5$ and solving for the $c_i's$ and substituting in (2), leads to the continuous linear multistep method of the form

$$y(x) = \sum_{i=0}^{4} \alpha_i(t) y_{n+i} + h^3 \sum_{i=0}^{5} \beta_i(t) f_{n+i} \tag{8}$$

where

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$$\alpha_0(t)=1-\frac{5}{4h}t+\frac{1}{4h^2}t^2,\alpha_1(t)=0,\alpha_2(t)=0,\alpha_3(t)=0,\alpha_4(t)=\frac{1}{12h^2}t^2-\frac{1}{12h}t$$

$$\beta_0(t)=\frac{1}{6}t^3-\frac{419}{2688}ht^2+\frac{589}{10080}h^2t-\frac{137}{1440h}t^4+\frac{1}{32h^2}t^5-\frac{1}{2880h^3}t^6+\frac{1}{1680h^4}t^7-\frac{1}{40320h^5}t^8$$

$$\beta_1(t)=-\frac{7753}{13440}ht^2+\frac{653}{1440}h^2t+\frac{5}{24h}t^4-\frac{77}{720h^2}t^5+\frac{71}{2880h^3}t^6-\frac{1}{360h^4}t^7+\frac{1}{8064h^5}t^8$$

$$\beta_2(t)=\frac{551}{20160}ht^2+\frac{23}{336}h^2t-\frac{5}{24h}t^4+\frac{107}{720h^2}t^5-\frac{59}{1440h^3}t^6+\frac{13}{2520h^4}t^7-\frac{1}{4032h^5}t^8$$

$$\beta_3(t)=-\frac{1117}{6720}ht^2+\frac{107}{1008}h^2t+\frac{5}{36h}t^4-\frac{13}{120h^2}t^5+\frac{49}{1440h^3}t^6-\frac{1}{210h^4}t^7+\frac{1}{4032h^5}t^8$$

$$\beta_4(t)=\frac{613}{13440}ht^2-\frac{239}{10080}h^2t-\frac{5}{96h}t^4+\frac{61}{1440h^2}t^5+\frac{11}{5040h^3}t^6-\frac{1}{8064h^5}t^8 \tag{9}$$

$$\beta_5(t)=-\frac{59}{8064}ht^2+\frac{13}{3360}h^2t+\frac{1}{120h}t^4-\frac{1}{144h^2}t^5-\frac{41}{2880h^3}t^6-\frac{1}{2520h^4}t^7+\frac{1}{40320h^5}t^8$$

Evaluating (9) at $x_{n+r}, r = 2, 3,5$ and its first and second derivatives evaluated at $x_{n+r}, r = 0, 1, 2, 3, 4, 5$, the following block discrete scheme is obtained

$$\begin{bmatrix}
1 & 0 & 0 & -\frac{840}{13440} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4}{3} & 1 & 0 & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{13440}{3360} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{3} & 0 & 0 & -\frac{5}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4480}{6720h} & 0 & 0 & -\frac{560}{6720h} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{315}{1260h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{13440}{20160h} & 0 & 0 & -\frac{8400}{20160h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1960}{3360h} & 0 & 0 & -\frac{1960}{3360h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{40320}{20160h} & 0 & 0 & -\frac{15120}{20160h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{13440}{20160h^2} & 0 & 0 & -\frac{3360}{20160h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{13440}{20160h^2} & 0 & 0 & -\frac{3360}{20160h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{13440}{20160h^2} & 0 & 0 & -\frac{3360}{20160h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\frac{13440}{20160h^2} & 0 & 0 & -\frac{3360}{20160h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{13440}{20160h^2} & 0 & 0 & -\frac{3360}{20160h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ U_{n+1} \\ U_{n+2} \\ U_{n+3} \\ U_{n+4} \\ U_{n+5} \\ V_{n+1} \\ V_{n+2} \\ V_{n+3} \\ V_{n+4} \\ V_{n+5} \end{bmatrix}
=
\begin{bmatrix}
\frac{12600}{13440} & 0 & \frac{10080h}{13440} \\
-\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 0 & 0 \\
-\frac{10080}{3360} & 0 & \frac{20160h^2}{3360} \\
1 & 0 & 0 \\
-\frac{5040}{6720h} & 0 & 0 \\
-\frac{312}{1260h} & 0 & 0 \\
\frac{5040}{20160h} & 0 & 0 \\
\frac{2520}{3360h} & 0 & 0 \\
\frac{25200}{20160h} & 0 & 0 \\
\frac{10080}{20160h^2} & 0 & 0 \\
\frac{10080}{20160h^2} & 0 & 0 \\
\frac{10080}{20160h^2} & 0 & 0 \\
\frac{10080}{20160h^2} & 0 & 0 \\
\frac{10080}{20160h^2} & 0 & 0
\end{bmatrix}
\begin{bmatrix} y_n \\ U_n \\ V_n \end{bmatrix}
+
\begin{bmatrix}
-\frac{589h^3}{13440} & -\frac{4571h^3}{13440} & -\frac{690h^3}{13440} & -\frac{1070h^3}{13440} & \frac{239h^3}{13440} & -\frac{39h^3}{13440} \\
-\frac{h^3}{288} & -\frac{113h^3}{480} & -\frac{257h^3}{720} & -\frac{47h^3}{720} & -\frac{h^3}{160} & \frac{h^3}{1440} \\
-\frac{h^3}{240} & -\frac{7h^3}{30} & -\frac{21h^3}{40} & -\frac{7h^3}{30} & -\frac{h^3}{240} & 0 \\
-\frac{6285h^3}{3360} & -\frac{23259h^3}{3360} & -\frac{1102h^3}{3360} & \frac{6702h^3}{3360} & -\frac{1839h^3}{3360} & \frac{295h^3}{3360} \\
\frac{h^3}{144} & \frac{23h^3}{48} & \frac{85h^3}{72} & \frac{85h^3}{72} & \frac{23h^3}{48} & \frac{h^3}{144} \\
-\frac{61h^2}{6720} & -\frac{1829h^2}{6720} & -\frac{1202h^2}{6720} & -\frac{266h^2}{6720} & -\frac{h^2}{6720} & -\frac{h^2}{6720} \\
-\frac{2h^2}{1260} & -\frac{160h^2}{1260} & -\frac{516h^2}{1260} & -\frac{160h^2}{1260} & -\frac{2h^2}{1260} & -\frac{315h^2}{1260} \\
\frac{35h^2}{20160} & \frac{2383h^2}{20160} & \frac{3390h^2}{20160} & -\frac{2306h^2}{20160} & -\frac{145h^2}{20160} & \frac{3h^2}{20160} \\
\frac{17h^2}{3360} & \frac{1207h^2}{3360} & \frac{2842h^2}{3360} & \frac{2374h^2}{3360} & \frac{293h^2}{3360} & -\frac{13h^2}{3360} \\
\frac{253h^2}{20160} & \frac{11597h^2}{20160} & \frac{31890h^2}{20160} & \frac{31130h^2}{20160} & \frac{21217h^2}{20160} & \frac{1353h^2}{20160} \\
\frac{365h}{20160} & -\frac{3281h}{20160} & -\frac{10070h}{20160} & \frac{46h}{20160} & -\frac{583h}{20160} & \frac{83h}{20160} \\
-\frac{13h}{20160} & \frac{5637h}{20160} & \frac{4238h}{20160} & -\frac{3566h}{20160} & \frac{495h}{20160} & -\frac{71h}{20160} \\
\frac{141h}{20160} & \frac{4335h}{20160} & \frac{15466h}{20160} & \frac{7662h}{20160} & -\frac{807h}{20160} & \frac{83h}{20160} \\
-\frac{13h}{20160} & \frac{5413h}{20160} & \frac{11854h}{20160} & \frac{21970h}{20160} & \frac{8111h}{20160} & -\frac{295h}{20160} \\
\frac{365h}{20160} & \frac{2991h}{20160} & \frac{18602h}{20160} & \frac{10798h}{20160} & \frac{28089h}{20160} & \frac{6355h}{20160}
\end{bmatrix}
\begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix}$$

Note: $U_{n+r} = y'(x_{n+r})$ and $V_{n+r} = y''(x_{n+r}), r = 0, 1, 2, 3, 4, 5$.

III. Analysis of the Method

Order and Error Constant

Expanding the block method (10) in Taylor's series and collecting like terms in powers of h, we obtain the following:

$$\check{C}_0 = \check{C}_1 = \check{C}_2 = \cdots = \check{C}_7 = \check{C}_8 = (0,0,0,0,0)^T,$$
$$\check{C}_9 = \left(\frac{2789}{1209600}, -\frac{13}{36288}, -\frac{1}{60480}, -\frac{599}{8640}, \frac{17}{22680}\right)^T.$$

Hence the block discrete scheme has order $\check{p} = 6$ with error constant $\check{C}_9 = \left(\frac{2789}{1209600}, -\frac{13}{36288}, -\frac{1}{60480}, -\frac{599}{8640}, \frac{17}{22680}\right)^T$.

Consistency

Following [6] the block discrete scheme is consistent since it has order $\breve{p} = 6 > 1$.

Zero Stability

The block method (10) is said to be zero stable if the roots z_r; $r = 1, \dots, n$ of the first characteristic polynomial $p(z)$, defined by $p(z) = det|zQ - T|$ satisfy $|z_r| \leq 1$ and every root with $|z_r| = 1$ has multiplicity not exceeding three in the limit as $h \to 0$

From the block method (10), substituting for Q and T in $p(z) = det|zQ - T|$ leads to

$$z^{15} - 2z^{14} + z^{13} = 0$$
$$z = (0, 0, 0, 0, 0, 0, 0, 0,0,0,0,0,0,1, 1)$$

This shows that the block method is zero stable since all roots with modulus one do not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$

Convergence

The block method is convergent since it is both consistent and zero stable following [6].

IV. Numerical Illustrations and Results

To achieve the validity, the accuracy and support the theoretical discussion of the proposed method; the computations associated with the examples are performed using MAPLE 18. Furthermore, the performance of the method is tested on some examples in the literature. The developed block method is used in evaluating each of the test problems to determine its performance. For each example, absolute errors of the approximate solutions are compared with existing methods.

Example 1. Consider the differential equation problem of the third order given by

$$y'''(x) = e^x, y(0) = 3, y'(0) = 1, y''(0) = 5, 0 \leq x \leq 1, h = 0.1$$

Exact solution $y(x) = 2 + 2x^2 + e^x$, the results are shown in Table I.

Example 2. Consider the differential equation of the third order given by

$$y'''(x) = -y, y(0) = 1, y'(0) = -1, y''(0) = 1, 0 \leq x \leq 1, h = 0.1$$

Exact solution $y(x) = e^x$, the results are shown in Table II.

Example 3. Consider the differential equation of the third order given by

$$y'''(x) = 3\mathrm{S}inx, y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$$

Exact solution $y(x) = 3cosx + \frac{x^2}{2} - 2$, the results are shown in Table III.

TABLE I
COMPARING PROPOSED RESULTS WITH [18]

x	Exact solution	Result of Proposed Method	Error in Proposed Method $K = 5, P = 6$	Error in [18] $K = 7, P = 9$
0.1	3.12517091807565	3.12517091807901	3.36×10^{-12}	0
0.2	3.30140275816017	3.30140275818176	2.16×10^{-11}	2.84×10^{-13}
0.3	3.52985880757600	3.52985880762931	5.33×10^{-11}	1.67×10^{-12}
0.4	3.81182469764127	3.81182469774112	9.99×10^{-11}	3.00×10^{-11}
0.5	4.14872127070013	4.14872127085997	1.60×10^{-11}	3.18×10^{-11}
0.6	4.54211880039051	4.54211880064155	2.51×10^{-10}	9.19×10^{-11}
0.7	4.99375270747048	4.99375270786647	3.96×10^{-10}	8.95×10^{-11}
0.8	5.50554092849247	5.50554092908497	5.93×10^{-10}	1.92×10^{-10}
0.9	6.07960311115695	6.07960311199962	8.27×10^{-10}	2.11×10^{-10}
1.0	6.71828182845905	6.71828182960333	1.14×10^{-9}	4.94×10^{-10}

TABLE II
COMPARING PROPOSED RESULTS WITH [11]

x	Exact solution	Result of Proposed Method	Error in Proposed Method $P = 6$	Error in [11], $P = 9$
0.1	0.904837418035960	0.904837418033781	2.18×10^{-12}	2.18×10^{-12}
0.2	0.818730753077982	0.818730753064047	1.39×10^{-11}	1.39×10^{-11}
0.3	0.740818220681718	0.740818220647252	3.45×10^{-11}	3.44×10^{-11}
0.4	0.670320046035639	0.670320045971130	6.45×10^{-11}	6.45×10^{-11}
0.5	0.606530659712633	0.606530659609410	1.03×10^{-10}	1.03×10^{-10}
0.6	0.548811636094026	0.548811635935252	1.59×10^{-10}	1.50×10^{-10}
0.7	0.496585303791410	0.496585303553820	2.38×10^{-10}	2.05×10^{-10}
0.8	0.449328964117222	0.449328963778233	3.39×10^{-10}	2.68×10^{-10}
0.9	0.406569659740599	0.406569659277454	4.63×10^{-10}	6.94×10^{-10}
1.0	0.367879441171442	0.367879440562268	6.09×10^{-10}	1.42×10^{-10}

TABLE III
COMPARING PROPOSED RESULTS WITH [12]

x	Exact solution	Result of Proposed Method	Error in Proposed Method P = 6	Error in [12], P = 7
0.1	0.990012495834077	0.9900124958323234	1.75×10^{-12}	3.41×10^{-11}
0.2	0.96019973352373	0.960199733512464	1.13×10^{-11}	1.24×10^{-10}
0.3	0.911009467376818	0.9110094673490696	2.77×10^{-11}	1.77×10^{-10}
0.4	0.843182982008655	0.8431829819566858	5.20×10^{-11}	4.09×10^{-10}
0.5	0.757747685671118	0.757747685587981	8.31×10^{-11}	3.71×10^{-10}
0.6	0.656006844729035	0.6560068445948196	1.34×10^{-10}	7.10×10^{-10}
0.7	0.539526561853465	0.5395265616278406	2.26×10^{-10}	7.47×10^{-10}
0.8	0.410120128041496	0.4101201276862894	3.55×10^{-10}	1.96×10^{-10}
0.9	0.269829904811994	0.2698299042869376	5.25×10^{-10}	3.89×10^{-10}
1.0	0.120906917604419	0.1209069168714127	7.33×10^{-10}	6.40×10^{-10}

V. Discussion of Results

In this work, a block method of order six is developed for the direct solution of general third order ordinary differential equations. The method was found to be consistent, zero stable and convergent. Furthermore, the results obtained revealed that the proposed method compares favorably with some existing methods despite its low order and step number.

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