On the Efficiency of a Double-Cone Gravitational Motor and Generator

Barenten Suciu, Akio Miyamura

Abstract—In this paper, following the study-case of an inclined plane gravitational machine, efficiency of a double-cone gravitational motor and generator is evaluated. Two types of efficiency ratios, called *translational efficiency* and *rotational efficiency*, are defined relative to the intended duty of the gravitational machine, which can be either the production of translational kinetic energy, or rotational kinetic energy. One proved that, for pure rolling movement of the double-cone, in the absence of rolling friction, the total mechanical energy is conserved. In such circumstances, as the motion of the double-cone progresses along rails, the translational efficiency decreases and the rotational efficiency increases, in such way that sum of the rotational and translational efficiencies remains unchanged and equal to 1. Results obtained allow a comparison of the gravitational machine with other types of motor-generators, in terms of the achievable efficiency.

Keywords—Truncated double-cone, friction, rolling and sliding, efficiency, gravitational motor and generator.

I. Introduction

SINCE the mechanism consisted of a double-cone, self-propelled on straight V-shaped horizontal rails [1]-[3], is able to convert the potential energy gained in the terrestrial gravitational field into mechanical work or kinetic energy of rotation and translation, it was regarded as a gravitational motor [4], [5]. Recently, a truncated double-cone rolling on divergent-convergent rails, materialized by using either straight V-rails or eccentric circular rails, was used in the construction of a wave-powered electrical generator [6]. From a practical point of view, efficiency evaluation of such gravitational motor and generator is mandatory in order to properly decide the possible range of applications and the feasibility of industrial production.

In the present work, one firstly evaluates the efficiency of an inclined plane gravitational machine, in the case of the sliding of a flat body, and also in the case of the rolling of a revolution body, during their descending along the slope. Then, the efficiency of the double-cone motor and generator is defined according to the purpose of the mechanical system, and the influence of various geometrical parameters is clarified.

II. Geometrical Model of the Truncated Double-Cone Motor and Generator

Two identical cones having a height H, a radius R at the base circles, and a truncation radius R^* at the conical tips, are

fixedly joined together to achieve the so-called double-cone (see Fig. 1). Therefore, the apex angle can be calculated as [5]:

$$\Psi = \tan^{-1} \frac{R - R^*}{H} \tag{1}$$

On the other hand, the moment of inertia can be written as [5]:

$$I = 0.3mR^2 \frac{1 - (R^*/R)^5}{1 - (R^*/R)^3} \tag{2}$$

where m is the mass of the truncated double-cone. Note that, for small truncation radii ($R^* \ll R$), the moment of inertia can be quite accurately calculated as: $I \cong 0.3mR^2$ [5].

Straight rails, having a length L_0, an entrance span L_2 and an exit span L_1, are disposed on a horizontal table to form a V letter (see Fig. 1). For divergent rails, as considered in this work, the exit span exceeds the entrance span ($L_1 > L_2$). Degree of divergence can be quantified by defining an opening angle of the rails, as [5]:

$$\Phi = \sin^{-1} \frac{L_1 - L_2}{2L_0} \tag{3}$$

Start position of the double-cone on the rails is set at the distance L_S, measured from the entrance point. Static and dynamic sliding friction coefficients between the double-cone and the rails are denoted as μ_S and μ, respectively [7].
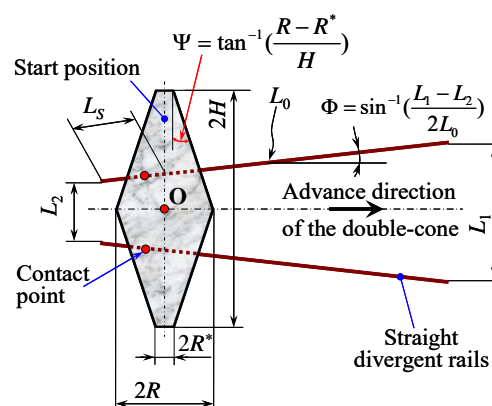

Fig. 1 Geometry of the truncated double-cone, self-propelling on V-shaped straight divergent rails, disposed on a horizontal table

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III. INCLINED PLANE GRAVITATIONAL MACHINE: CASE OF THE SLIDING OF A FLAT BODY

Fig. 2 shows the descending of a flat body of mass m on an inclined plane of angle θ, regarded here as a gravitational motor able to transform the potential energy of the block into the kinetic energy of translation. In one possible approach to find the *efficiency* of such gravitational motor, this mechanical system can be considered as a black-box, in which an amount of potential energy $\left| \Delta E_p \right|$ is inputted, as follows [8]-[10]:

$$INPUT = \left| \Delta E_p \right| = E_{p,i} - E_{p,f} = mg(H_i - H_f) \qquad (4)$$

where $E_{p,i}$ and $E_{p,f}$ are the potential energies at the initial and final positions of the block, corresponding to the heights H_i and H_f measured relative to the reference horizontal plane, and g is the gravitational acceleration. Initially halted block ($V_i = 0$) is accelerated during the descending, up to the final velocity $V_f = V$.
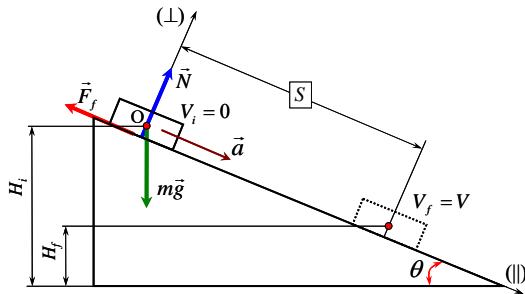

Fig. 2 Model for sliding of a flat body on an inclined plane

Since the *useful output* or the *duty* of the motor is to gain a certain amount of translational kinetic energy:

$$OUTPUT = \left| \Delta E_{k,tr} \right| = E_{k,tr,f} - E_{k,tr,i} = E_{k,tr,f} = \frac{m}{2}V^2 \qquad (5)$$

the translational efficiency $\eta_{ip,tr}$ of the incline, concerning the sliding movement of the flat block, can be defined as:

$$\eta_{ip,tr} = \frac{OUTPUT}{INPUT} = \frac{\left| \Delta E_{k,tr} \right|}{\left| \Delta E_p \right|} = \frac{V^2}{2g(H_i - H_f)} \qquad (6)$$

For such translation movement at constant acceleration a, the term V^2 of (6) can be written as:

$$V^2 = 2aS \qquad (7)$$

where the distance S traveled by the block along the inclined plane can be expressed as (see Fig. 2):

$$S = \frac{H_i - H_f}{\sin \theta} \qquad (8)$$

Thus, by substituting (8) in (7), and then, the obtained result in (6), the translational efficiency can be rewritten as:

$$\eta_{ip,tr} = \frac{a}{g \sin \theta} \qquad (9)$$

Next, in order to determine the acceleration a, Newton's 2nd Law of Dynamics, for the translation motion of the mass center O of the flat block, can be written as:

$$m\vec{a} = m\vec{g} + \vec{N} + \vec{F}_f \qquad (10)$$

which leads to the following scalar equations along the parallel and perpendicular axes attached to the inclined plane (Fig. 2):

$$\begin{cases} (\|): & mg \sin \theta - F_f = ma \\ \\ (\perp): & N - mg \cos \theta = 0 \end{cases} \qquad (11)$$

where N is the normal force, and $F_f = \mu N$ is the frictional force, in which μ is the dynamic sliding friction coefficient at the contact of the block with the inclined plane. From (11) one firstly finds the acceleration, as follows:

$$a = g(\sin \theta - \mu \cos \theta) \qquad (12)$$

and then, by substituting (12) in (9), the translational efficiency:

$$\eta_{ip,tr} = \frac{g(\sin \theta - \mu \cos \theta)}{g \sin \theta} = 1 - \frac{\mu}{\tan \theta} \le 1 \qquad (13)$$

One notes that the acceleration in sliding, in the presence of friction, given by (12), is always smaller than the acceleration in sliding without friction $g \sin \theta$, which can be obtained from (12) for nil friction coefficient ($\mu = 0$). Thus, friction is able to dissipate a part of the inputted potential energy into heat, which is un-recoverable to do useful work, i.e. to accelerate the block during the descending movement. As expected, for nil friction $\mu = 0$ between the block and the inclined plane, the efficiency maximizes to $\eta_{ip,tr} = 1$. On the other hand, when the angle θ equals the friction angle $\tan^{-1} \mu$, the efficiency minimizes to $\eta_{ip,tr} = 0$.

Obviously, the gravitational motor can operate only if the angle θ exceeds the friction angle ($\theta > \tan^{-1} \mu$), which is a condition well-known in the literature [7]-[11].

IV. INCLINED PLANE GRAVITATIONAL MACHINE: CASE OF THE ROLLING OF A REVOLUTION BODY

Fig. 3 shows the descending, on an inclined plane of angle

θ, of a revolution body, e.g. a cylinder, a sphere, or a double-cone, having a mass m, a radius R, and a moment of inertia I at rotation versus its axis of symmetry. This mechanical system can be also regarded as a gravitational machine able to transform the potential energy of the revolution body into the kinetic energy of rotation and translation. Since it has been previously proven that the truncated double-cone gravitational motor rolls without slip [5], one assumes here that the revolution body of Fig. 3 rolls also without slipping, down the slope. Such condition is easily satisfied for gentle slopes, but it may be violated for steep slopes, depending, of course, on the magnitude of the dynamic sliding friction coefficient [7].
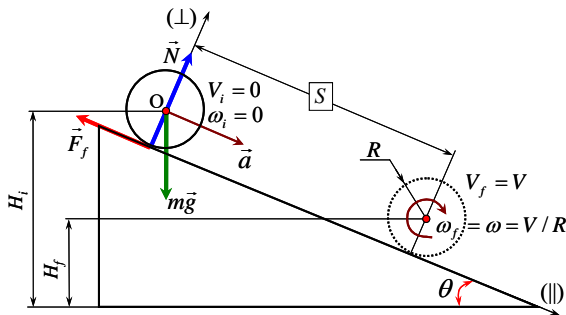

Fig. 3 Model for rolling of a revolution body on an inclined plane

Newton's 2nd Law of Dynamics, for the translation motion of the mass center O of the revolution body, displays the same vectorial form (10) and scalar form (11). However, the equivalent sliding friction coefficient, which can be attached to this ideal rolling movement, is unknown at this time. Therefore, instead of considering that the friction force is obtained by multiplying the normal force with a friction coefficient, the problem is solved by applying the Law of Dynamics for the rotation movement of the revolution body.

Note that the weight $m\vec{g}$ and the normal force $\bar{N}$ are vectors passing through the mass center O (see Fig. 3). Hence, they are unable to cause revolution of the body around its axis of symmetry. Only the friction force $\vec{F}_f$ is able to produce spinning of the revolution body, due to the clockwise traction torque $F_f R$ (see Fig. 3). Thus, equation describing the rolling movement on the inclined plan can be written as:

$$I\varepsilon = F_f R \qquad (14)$$

where ε is the angular acceleration, which in the case of pure rolling can be expressed as [5], [11]:

$$\varepsilon = a / r \qquad (15)$$

By substituting (15) in (14), and then the resulting friction force into the upper equation of (11), one obtains the acceleration, as:

$$a = g\sin\theta /(1+\bar{I}) \qquad (16)$$

where $\bar{I}$ is the dimensionless moment of inertia, given by:

$$\bar{I} = I /(mR^2) \qquad (17)$$

Note that $\bar{I}$ decreases from 1 for a cylinder shell, to 2/3 for a spherical shell, to 0.5 for a cylinder, to 0.4 for a sphere, and up to 0.3 for a cone or a double-cone. However, it is important to note that although for the majority of the revolution bodies, e.g. cylinders and spheres, the radius of the body equals the radius of contact, in the case of a double-cone, the radius R at the base circle is larger than the actual contact radius r (see Fig. 4). In fact, the double-cone has a variable contact radius, which decreases from a maximal value, at the initial position [5]:

$$r_i = \frac{R}{\cos^2\alpha}\{1-[\frac{L_2}{2H}+\frac{(L_1-L_2)L_S}{2HL_0}](1-\frac{R^*}{R})\} \qquad (18)$$

to a minimal value, at the final position [5]:

$$r_f = \frac{R}{\cos^2\alpha}[1-\frac{L_1}{2H}(1-\frac{R^*}{R})] \le r_i \qquad (19)$$

Here α is the angle between the rectilinear trajectory of the mass center and the horizontal line (see Fig. 4), given by [5]:

$$\alpha = \sin^{-1}[\frac{R-R^*}{H}\frac{L_1-L_2}{\sqrt{4L_0^2-(L_1-L_2)^2}}] \qquad (20)$$

For these reasons, dimensionless moment of inertia of a double-cone should be regarded as a function of the contact radius:

$$\bar{I}(r) = \frac{I}{mr^2} = \frac{0.3mR^2}{mr^2} = 0.3\frac{R^2}{r^2} \qquad (21)$$
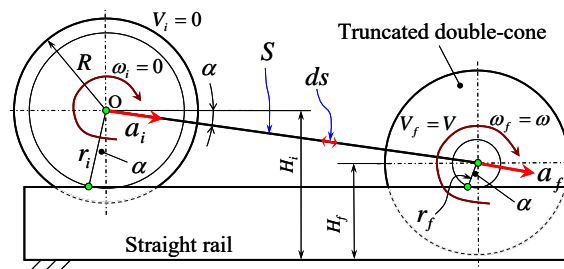

Fig. 4 Change in the contact radius, producing a continuous variation of the sliding acceleration of the double-cone rolling on straight rails

From (16), one concludes that the constant acceleration of a revolution body, rolling without slipping ($g\sin\theta$), is smaller than the value obtained when the same body slides down the same slope without friction. Largest reduction of acceleration ($0.5g\sin\theta$) is obtained for the cylindrical shell, and the smallest reduction of acceleration ($0.714g\sin\theta$) for the solid sphere. However, as it will be discussed in detail in the next

section, for the particular case of a double-cone, the sliding acceleration appears as variable (see (16), (21), and Fig. 4).

It is also useful to observe that an equivalent sliding friction coefficient μ_e can be defined, as follows (see (11), (16)):

$$\mu_e = \frac{F_f}{N} = \frac{m(g\sin\theta - a)}{mg\cos\theta} = \tan\theta \frac{\bar{I}}{1+\bar{I}} \qquad (22)$$

which, in order to satisfy the condition of pure rolling, should be smaller than the dynamic sliding friction coefficient μ.

In such conditions, (13) can be directly used to determine the translational efficiency $\eta_{ip,tr}$ of the inclined plane gravitational motor, in the case of a rolling body, relative to the translational kinetic output, as follows:

$$\eta_{ip,tr} = 1 - \frac{\mu_e}{\tan\theta} = \frac{1}{1+\bar{I}} < 1 \qquad (23)$$

Thus, (23) shows that during the rolling movement of the revolution body, only a part ($1/(1+\bar{I})$) of the potential energy is converted into the translational kinetic energy, the rest ($\bar{I}/(1+\bar{I})$) of the potential energy being transformed into the rotational kinetic energy. This can be verified by observing that the total kinetic energy $E_{k,total,f}$ of the revolution body at the final position on the inclined plane, can be written as the sum of the translational kinetic energy $E_{k,tr,f} = 0.5mV^2$ and the rotational kinetic energy $E_{k,rot,f} = 0.5I\omega^2$:

$$E_{k,total,f} = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mV^2[1 + \frac{I}{m}(\frac{\omega}{V})^2] \qquad (24)$$

Since during the pure rolling motion, the spinning speed ω varies directly proportionally to the sliding velocity of the mass center and inversely proportionally to the contact radius [11]:

$$\omega = V / R \qquad (25)$$

the total kinetic energy (24) can be rewritten as:

$$E_{k,total,f} = \frac{1}{2}mV^2(1+\bar{I}) = E_{k,tr,f}(1+\bar{I}) \qquad (26)$$

In conclusion, the machine efficiency defined relative to the rotational kinetic output can be written as:

$$\eta_{ip,rot} = \frac{E_{k,rot,f}}{|\Delta E_p|} = \eta_{ip,tr}\bar{I} = \frac{\bar{I}}{1+\bar{I}} < 1 \qquad (27)$$

Consequently, the efficiency defined relative to the total kinetic output becomes equal to 1:

$$\eta_{ip,total} = \frac{E_{k,total,f}}{|\Delta E_p|} = \eta_{ip,tr} + \eta_{ip,rot} = 1 \qquad (28)$$

Result (28) appears to be correct since the rolling friction of the revolution body, i.e. the small eccentricity of the normal force in relation with the center of mass, was neglected. If rolling friction is considered, a friction moment equal to the normal force multiplied by the above-mentioned eccentricity, opposes the clockwise traction torque $F_f R$ produced by the sliding friction force [7], [11].

Since for a revolution body, whose radius equals the contact radius, the dimensionless moment of inertia cannot exceed the unity ($\bar{I} \leq 1$), from (27) it appears that the rotational efficiency cannot exceed the translational efficiency:

$$\eta_{ip,rot} = \eta_{ip,tr}\bar{I} \leq \eta_{ip,tr} \qquad (29)$$

However, as it will be proven in the next section, in the case of a double-cone gravitational machine, the inequality (29) can be easily violated.

From (16) and (23), one concludes that the solid cylinder reaches the bottom of the slope faster than the hollow cylinder, since it has a smaller dimensionless moment of inertia $\bar{I}$, i.e. it has a larger acceleration and translational efficiency $\eta_{ip,tr}$. In fact, the most compact object, i.e. the object with the smallest $\bar{I}$, will achieve the fastest descending speed and the highest translational efficiency $\eta_{ip,tr}$, but the lowest efficiency of rotation $\eta_{ip,rot}$. Thus, depending on the prescribed duty of the gravitational machine, a solid sphere can be selected to achieve a high translational efficiency of 71.4 %, but a hollow cylinder might be preferred to gain the highest rotational efficiency of 50 %.

V. TRUNCATED DOUBLE-CONE GRAVITATIONAL MACHINE: CASE OF THE PERFECT HORIZONTAL RAILS

Fig. 1 shows the *no-slip* or *pure rolling* movement of a truncated double-cone along perfect horizontal rails. Condition of pure rolling has been previously checked, by demonstrating that the equivalent sliding friction coefficient μ_e is smaller than the traction coefficient μ_t, and further, by observing that the traction coefficient μ_t is about 10 to 100 times smaller than the dynamic sliding friction coefficient μ [5]:

$$\mu_e = \frac{0.3R^2\mu_t}{r^2\cos^2\alpha + 0.3R^2} < \mu_t \quad ; \quad \mu_t = \frac{\cos\Psi}{\cos\Phi}\sin\alpha < \mu \quad (30)$$

As reported by [1]-[5], the mass center of the double-cone moves on a descending straight line that displays an *inclination* or *contact angle* α relative to the horizontal line (see Fig. 4). Descending speed $\vec{V}$ and acceleration $\vec{a}$ are vectors normal to the instantaneous contact radius, and consequently, they move on the same straight line as the mass center of the revolution

body (see Fig. 4). Double-cone, which is initially halted ($V_i = 0$, $\omega_i = 0$) is then accelerated during the rolling along the straight divergent rails, up to the final translational velocity $V_f = V$, and up to the final rotational velocity $\omega_f = \omega$.

Substituting (21) in (16), and taking into account that the angle θ of the inclined plane should be replaced by the contact angle α (see Figs. 3 and 4), the variable acceleration of the mass center of the double-cone is obtained as a function of the contact radius, as follows:

$$a(r) = g \sin \alpha \frac{1}{1 + 0.3 R^2 / r^2} = g \sin \alpha \frac{r^2}{r^2 + 0.3 R^2} \qquad (31)$$

Note that (31) displays the same expression as that derived in [5], following a quite different approach for the kinematical analysis of the double-cone movement on the rails.

Fig. 4 shows that the gradual reduction of the contact radius leads to height decrease of the mass center of the double-cone, and hence the inputted potential energy can be calculated as:

$$\left| \Delta E_p \right| = mgS \sin \alpha = mg(H_i - H_f) = mg(r_i - r_f) \cos \alpha \qquad (32)$$

Translational kinetic energy output can be evaluated as the work of the inertial force $m\vec{a}$ along the descending straight line of length S, on which the mass center is moving:

$$\left| \Delta E_{k,tr} \right| = E_{k,tr,f} - E_{k,tr,i} = E_{k,tr,f} = \int_0^S ma(s) \cdot ds \qquad (33)$$

By performing a change of variables from the length s to the contact radius r:

$$ds = -dr / \tan \alpha \qquad (34)$$

and by substituting the acceleration (31) in (33), one arrives to the following translational output:

$$
\begin{aligned}
\left| \Delta E_{k,tr} \right| &= -\int_{r_i}^{r_f} mg \sin \alpha \frac{r^2}{r^2 + 0.3 R^2} \cdot \frac{dr}{\tan \alpha} = \\
&= mg \cos \alpha [(r_i - r_f) - \sqrt{0.3} R \tan^{-1} \frac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f} \sqrt{\bar{I}_i}}]
\end{aligned} \qquad (35)
$$

where the initial $\bar{I}_i$ and final $\bar{I}_f$ dimensionless moments of inertia of the double-cone can be calculated as:

$$\bar{I}_i = 0.3 \frac{R^2}{r_i^2} \quad ; \quad \bar{I}_f = 0.3 \frac{R^2}{r_f^2} \geq \bar{I}_i \qquad (36)$$

Based on (32) and (35), one determines the translational efficiency η_{tr} of the double-cone gravitational machine, as:

$$\eta_{tr} = \frac{\left| \Delta E_{k,tr} \right|}{\left| \Delta E_p \right|} = 1 - \frac{\sqrt{\bar{I}_f} \sqrt{\bar{I}_i}}{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}} \tan^{-1} \frac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f} \sqrt{\bar{I}_i}} < 1 \qquad (37)$$

On the other hand, rotational kinetic energy output can be calculated as the work of the inertial moment $I\varepsilon$ around the mass center, as follows:

$$\left| \Delta E_{k,rot} \right| = E_{k,rot,f} - E_{k,rot,i} = E_{k,rot,f} = \int_0^{2\pi n} I\varepsilon(\gamma) \cdot d\gamma \qquad (38)$$

where the instantaneous number n of rotations of the double-cone is given by [5]:

$$n = -\frac{1}{2\pi \tan \alpha} \ln \frac{r}{r_i} \qquad (39)$$

Based on (39), one performs a change of variables from the angular coordinate γ to the contact radius r:

$$\gamma = 2\pi n = -\frac{1}{\tan \alpha} \ln \frac{r}{r_i} \quad \Rightarrow \quad d\gamma = -\frac{1}{\tan \alpha} \frac{dr}{r} \qquad (40)$$

Then, by substituting the acceleration (31) in the expression (15) of the angular acceleration for pure rolling, and the result in (38), one finds the following rotational kinetic energy output:

$$
\begin{aligned}
\left| \Delta E_{k,rot} \right| &= -\int_{r_i}^{r_f} Ig \sin \alpha \frac{1}{r^2 + 0.3 R^2} \cdot \frac{dr}{\tan \alpha} = \\
&= \sqrt{0.3} mgR \cos \alpha \tan^{-1} \frac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f} \sqrt{\bar{I}_i}}
\end{aligned} \qquad (41)
$$

Thus, based on (32) and (41), one obtains the rotational efficiency η_{rot} of the double-cone gravitational machine, as:

$$\eta_{rot} = \frac{\left| \Delta E_{k,rot} \right|}{\left| \Delta E_p \right|} = \frac{\sqrt{\bar{I}_f} \sqrt{\bar{I}_i}}{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}} \tan^{-1} \frac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f} \sqrt{\bar{I}_i}} < 1 \qquad (42)$$

Similar to (28), one achieves the efficiency defined relative to the total kinetic output of the double-cone, as equal to 1:

$$\eta_{total} = \frac{\left| \Delta E_{k,tr} \right| + \left| \Delta E_{k,rot} \right|}{\left| \Delta E_p \right|} = \eta_{tr} + \eta_{rot} = 1 \qquad (43)$$

which appears as a correct result under the fair assumption that the rolling friction of the double-cone can be neglected, i.e. the total mechanical energy is conserved.

VI. TRUNCATED DOUBLE-CONE GRAVITATIONAL MACHINE: CASE OF THE SLIGHTLY INCLINED RAILS

In this section, the double-cone is supposed to roll on straight

divergent rails, when they are slightly inclined. For instance, Fig. 5 shows the movement of the double-cone on descending rails of inclination angle θ.
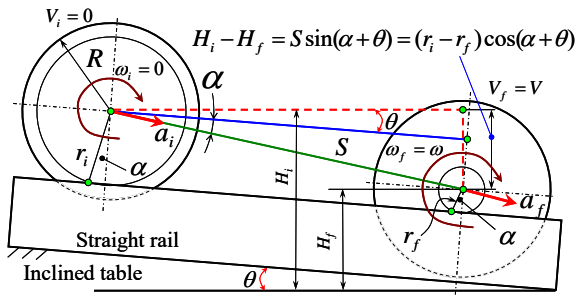

Fig. 5 Lateral view of the truncated double-cone, rolling on V-shaped straight divergent rails, disposed on a slightly descending table

In the case of inclined rails, according to Fig. 5, the inputted potential energy (32) of the double-cone gravitational machine should be rewritten as:

$$\left|\Delta E_p\right| = mg(H_i - H_f) = mgS\sin(\alpha \pm \theta) = \\ = mg(r_i - r_f)\cos(\alpha \pm \theta) \tag{44}$$

where the plus sign is related to descending rails (Fig. 5), and the minus sign is taken for ascending rails [2], of inclination angle θ. Here, *slightly inclined rails* means that the angle θ for ascending rails cannot exceed the contact angle α, i.e. $\theta < \alpha$. In the other words, if $\theta \geq \alpha$ the double-cone is unable to self-propel on the ascending rails, and in such circumstances, the gravitational machine is unable to perform its prescribed duty.

Additionally, acceleration (31) of the mass center of the double-cone should be revised as:

$$a(r) = g\sin(\alpha \pm \theta)\frac{r^2}{r^2 + 0.3R^2} \tag{45}$$

In such circumstances, the change of variables (34) can be rewritten as:

$$ds = -dr / \tan(\alpha \pm \theta) \tag{46}$$

and hence, the translational kinetic energy output (35) can be recalculated as:

$$\left|\Delta E_{k,tr}\right| = \\ = mg\cos(\alpha \pm \theta)[(r_i - r_f) - \sqrt{0.3}R\tan^{-1}\frac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f}\sqrt{\bar{I}_i}}] \tag{47}$$

On the other hand, since the change of variables (40) can be reconsidered as:

$$d\gamma = -\frac{1}{\tan(\alpha \pm \theta)}\frac{dr}{r} \tag{48}$$

the rotational kinetic energy output (41) can be recalculated as:

$$\left|\Delta E_{k,rot}\right| = \sqrt{0.3}mgR\cos(\alpha \pm \theta)\tan^{-1}\frac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f}\sqrt{\bar{I}_i}} \tag{49}$$

Based on (44), (47), and (49), one regains the translational, rotational, and total efficiencies, as follows:

$$\begin{cases} \eta_{tr} = 1 - \dfrac{\sqrt{\bar{I}_f}\sqrt{\bar{I}_i}}{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}\tan^{-1}\dfrac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f}\sqrt{\bar{I}_i}} \\[4ex] \eta_{rot} = \dfrac{\sqrt{\bar{I}_f}\sqrt{\bar{I}_i}}{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}\tan^{-1}\dfrac{\sqrt{\bar{I}_f} - \sqrt{\bar{I}_i}}{1 + \sqrt{\bar{I}_f}\sqrt{\bar{I}_i}} \\[4ex] \eta_{total} = \eta_{tr} + \eta_{rot} = 1 \end{cases} \tag{50}$$

which are identical with (37), (42), and (43). Thus, efficiency ratios (50), associated to the double-cone gravitational machine are not explicitly depending on the inclination angle of the rails.

VII. RESULTS AND DISCUSSIONS

From (50), one observes that the translational and rotational efficiencies vary during the movement of the double-cone on the rails, in such way that sum of the rotational and translational efficiencies remains invariant and equal to unity.

Concerning the efficiency at the initial position in Figs. 4-5, or at the start position in Fig. 1, by imposing $\bar{I}_f \to \bar{I}_i$ in (50), one gains the initial translational and rotational efficiencies as:

$$\eta_{tr,i} = \frac{1}{1 + \bar{I}_i} \quad ; \quad \eta_{rot,i} = \frac{\bar{I}_i}{1 + \bar{I}_i} \tag{51}$$

which are similar to (23) and (27), found for the inclined plane gravitational machine. Moreover, similar to (29) of the inclined plane, $\eta_{rot,i} \leq \eta_{tr,i}$ if $\bar{I}_i \leq 1$, i.e. if $r_i \geq \sqrt{0.3}R$. However, by selecting the distance L_S, i.e. the start position of the double-cone on the rails (see Fig. 1), in such a manner that $\bar{I}_i > 1$, i.e., $r_i < \sqrt{0.3}R$, the initial rotational efficiency can be altered to exceed the translational efficiency ($\eta_{rot,i} > \eta_{tr,i}$).

As the movement of the double-cone progresses on the rails, the final contact radius r_f decreases, and depending on the actual geometry of the mechanism, it might approach zero. In such conditions, the corresponding dimensionless moment of inertia $\bar{I}_f$ (see (36)) tends to infinity. For this reason, by imposing $\bar{I}_f \to \infty$ in (50), one obtains the corresponding translational and rotational efficiencies, as:

$$\eta_{tr,\infty} = 1 - \sqrt{\bar{I}_i}\,\tan^{-1}\frac{1}{\sqrt{\bar{I}_i}} \quad ; \quad \eta_{rot,\infty} = \sqrt{\bar{I}_i}\,\tan^{-1}\frac{1}{\sqrt{\bar{I}_i}} \qquad (52)$$

Based on (50)-(52), one presents in Fig. 6 the variation of the translational η_{tr}, rotational η_{rot}, and total η_{total} efficiency of the double-cone gravitational machine versus the square root of the final dimensionless moment of inertia $\sqrt{\bar{I}_f} \in [\sqrt{\bar{I}_i}, \infty)$, for various values of the initial dimensionless moment of inertia $\sqrt{\bar{I}_i} = \sqrt{0.3} = 0.548$ (red lines), 0.7 (violet lines), 0.8 (green lines), 1 (brown lines), 1.2 (blue lines), and 2 (black lines). From Fig. 6, one observes that the translational efficiency η_{tr} monotonically decreases from the initial value of $1/(1+\bar{I}_i)$ (see (51)) to the value of $1 - \sqrt{\bar{I}_i}\tan^{-1}(1/\sqrt{\bar{I}_i})$ (see (52). Moreover, the rotational efficiency η_{rot} monotonically increases from the initial value of $\bar{I}_i/(1+\bar{I}_i)$ (see (51)) to the value of $\sqrt{\bar{I}_i}\tan^{-1}(1/\sqrt{\bar{I}_i})$ (see (52). However, the total efficiency, i.e. the sum of rotational and translational efficiencies, invariably maintains the value of one. At augmentation of the square root of the initial dimensionless inertia moment $\sqrt{\bar{I}_i}$ (see Fig. 6), the translational efficiency decreases, but the rotational efficiency increases.
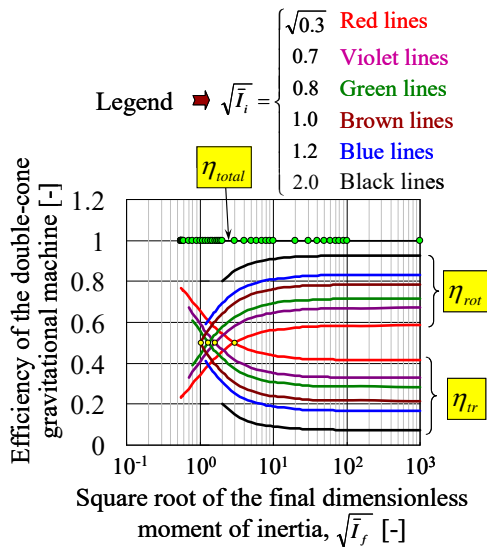

Fig. 6 Variation of the translational, rotational and total efficiency of the double-cone gravitational machine versus the square root of the final dimensionless moment of inertia, for various values of the square root of the initial dimensionless inertia moment

For $\bar{I}_i \le 1$, the graph of translational efficiency intersects the graph of rotational efficiency for a certain value of $\sqrt{\bar{I}_f}$, denoted as $\sqrt{\bar{I}_{f,eq}}$, and in such conditions $\eta_{tr} = \eta_{rot} = 0.5$ (see the yellow circular symbols on Fig. 6).

Accordingly, Fig. 7 shows the monotonical decreasing of the square root of the final dimensionless inertia moment $\sqrt{\bar{I}_{f,eq}}$, at which equality of the rotational and translational efficiencies is achieved, versus the square root of the initial dimensionless inertia moment $\sqrt{\bar{I}_i}$. On the other hand, one notes that, for $\bar{I}_i > 1$, regardless the value of $\sqrt{\bar{I}_f}$, rotational efficiency exceeds the translational efficiency ($\eta_{rot} > \eta_{tr}$), and this is a distinctive feature of the double-cone gravitational motor and generator, which cannot be achieved by the inclined plane gravitational machine.
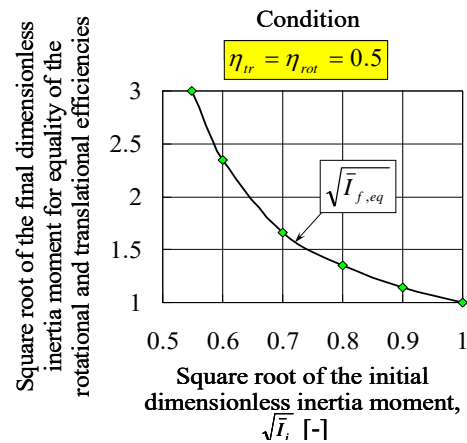

Fig. 7 Variation of the square root of the final dimensionless inertia moment, at which equality of the rotational and translational efficiencies is achieved, versus the square root of the initial dimensionless inertia moment

VIII. CONCLUSIONS

In this work, the efficiency of a double-cone gravitational motor and generator was theoretically investigated. Following conclusions can be drawn from the performed analysis:

1) According to the prescribed purpose of the double-cone gravitational machine, two types of efficiency ratios were defined, the translational and rotational efficiencies, in order to account for the production of kinetic energy, both at the movement of translation and rotation.

2) Efficiency ratios of the double-cone gravitational machine are not explicitly depending on the inclination angle of the straight divergent rails, but they appear to be functions of two geometrical parameters: the square roots of the initial and final dimensionless inertia moments.

3) Translational efficiency monotonically decreases, and the rotational efficiency monotonically increases against the square roots of the initial and final dimensionless inertia moments.

4) For an initial dimensionless inertia moment smaller than one, the rotational efficiency appears to be smaller than the translational efficiency, up to a certain value of the final dimensionless inertia moment, at which equality of the rotational and translational efficiency is obtained. Further

augmentation of the final dimensionless inertia moment leads to the reversed inequality between the rotational and translational efficiencies, and such effect is easier to be achieved for larger initial dimensionless inertia moments.

5) On the other hand, for an initial dimensionless inertia moment larger than one, the rotational efficiency exceeds translational efficiency, and such peculiar characteristic cannot be achieved in the case of the classical inclined plane gravitational machines.

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