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On Suborbital Graphs of the Congruence Subgroup $\Gamma_0(N)$

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Abstract**—**In this paper we examine some properties of suborbital graphs for the congruence subgroup $\Gamma_0(N)$. Then we give necessary and sufficient conditions for graphs to have triangels.

Keywords**—**Congruence subgroup, Imprimitive action, Modular group, Suborbital graphs.

I. INTRODUCTION

LET Γ denote the inhomogeneous group $\mathrm{PSL}(2,\mathbb{Z})$ acting on the upper half plane $H := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ via:

$$A(z) = \frac{az+b}{cz+d},\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma .$$

Among the subgroups of Γ the congruence subgroups such as

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1 \bmod N ,b \equiv c \equiv 0 (\bmod N) \right\}$$

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \ (\bmod N) \right\}$$

have been the objects of detailed studies due to their signifiance in the arithmetic of elliptic curves, integral quadratic forms, elliptic modular forms in [5], [6]. In this paper, we define $\Gamma^*(N)$ as the group obtained by adding the stabilizer of ∞ to the congruence subroup $\Gamma(N)$, that is,

$$\Gamma^*(N) := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \Gamma(N) \right\rangle$$

which is easily seen that

$$\Gamma^*(N) = \left\{ \begin{pmatrix} 1+aN & b \\ cN & 1+dN \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ \det = 1 \right\}.$$

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II. THE ACTION OF $\Gamma_0(N)$ ON $\hat{\mathbb{Q}}$

Every element of $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ can be represented as a reduced fraction $\frac{x}{y}$, with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. Since $\frac{x}{y} = \frac{-x}{-y}$, this representation is not unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ on $\frac{x}{y}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \to \frac{ax+by}{cx+dy} .$$

It is easily seen that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\frac{x}{y} \in \hat{\mathbb{Q}}$ is a reduced fraction then, since $c(ax+by) - a(cx+dy) = -y$ and $d(ax+by) - b(cx+dy) = x$,

$$(ax+by, cx+dy) = 1 .$$

The action of a matrix on $\frac{x}{y}$ and on $\frac{-x}{-y}$ is identical.

Theorem 2.1. The action of $\Gamma_0(N)$ on $\hat{\mathbb{Q}}$ is not transitive.

Proof. From (1), for $\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$

$$\begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} 1 \\ N \end{pmatrix} = \frac{a+bN}{cN+dN}$$

is a reduced fraction, so $\frac{1}{N}$ is not sent to $\frac{1}{N+1}$ under the action of $\Gamma_0(N)$.

Without loss of generality, for making calculations easier, N will be a prime p throughout the paper.

Theorem 2.2. The orbits of $\Gamma_0(p)$ are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ p \end{pmatrix}$.

Proof. Using the corallaries from [2] we can write down the sets of orbits of $\Gamma_0(N)$ in general

$$\begin{pmatrix} a \\ b \end{pmatrix} = \left\{ \frac{x}{y} \in \hat{\mathbb{Q}} : (p, y) = b,\ x \equiv a \bmod \left(b, \frac{N}{b} \right) \right\} .$$

Then we have

$$\binom{1}{p} = \left\{ \frac{k}{yp} : k \in \mathbb{Z}, (k, yp) = 1 \right\}$$

and

$$\binom{1}{1} = \left\{ \frac{k}{\ell} : k, \ell \in \mathbb{Z}, (k, \ell) = 1 \right\}.$$

We now consider the imprimitivity of the action of $\Gamma_0(p)$ on $\hat{\mathbb{Q}}$.

Let (G,Ω) be transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called G–invariant if whenever $\alpha, \beta \in \Omega$ satisfy $\alpha \approx \beta$ then $g(\alpha) \approx g(\beta)$ for all g in G. The equivalence classes are called blocks.

We call (G,Ω) imprimitive if Ω admits some G – invariant equivalence relation different from

(i) the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$
(ii) the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Omega$.

Otherwise (G,Ω) is called primitive. We now give a lemma from [3].

Lemma 2.3. Let (G,Ω) be transitive. (G,Ω) imprimitive if and only if G_α, the stabilizer of a point $\alpha \in \Omega$, is a maximal subgroup of G for each $\alpha \in \Omega$.

What the lemma is saying is whenever $G_\alpha \lneq H \lneq G$, then Ω admits some G – invariant equivalence relation other than trivial cases. In fact, since G acts transitively, every element of Ω has the form $g(\alpha)$ for some $g \in G$. If we define the relation $\approx$ on Ω as

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH,$$

then it is easily seen that it is non-trivial G–invariant equivalence relation. That is (G,Ω) imprimitive.

From the above we see that the number of blocks is equal to the index $|G : H|$.

We now apply these ideas to the case where G is the $\Gamma_0(p)$ and Ω is $\hat{\mathbb{Q}}$. An obvious choice for H is $\Gamma^*(p)$. Clearly $\Gamma_\infty \lneq \Gamma^*(p) \lneq \Gamma_0(p)$. Then we have

Corollary 2.4. ($\Gamma_0(p)$, $\hat{\mathbb{Q}}$) is imprimitive permutation group.

$\Gamma_0(p)$ acts transitively and imprimitively on the set $\binom{1}{p}$.

Let $\approx$ denote the $\Gamma_0(p)$– invariant equivalence relation induced on $\binom{1}{p}$ by $\Gamma_0(p)$ as:

If $v = \frac{a_1}{pc_1}$ and $w = \frac{a_2}{pc_2}$ are elements of $\binom{1}{p}$, then $v = g(\infty)$ and $w = g'(\infty)$ for elements $g, g' \in \Gamma_0(p)$ of the form

$$g = \begin{pmatrix} a_1 & b_1 \\ pc_1 & d_1 \end{pmatrix} \quad , \quad g' = \begin{pmatrix} a_2 & b_2 \\ pc_2 & d_2 \end{pmatrix}.$$

Now $v \approx w$ if and only if $g^{-1}g' \in \Gamma^*(p)$, that is,

$$g^{-1}g' = \begin{pmatrix} d_1a_2 - p(c_2b_1) & d_1b_2 - b_1d_2 \\ p(a_1c_2 - c_1a_2) & a_1d_2 - p(c_1b_2) \end{pmatrix} \in \Gamma^*(p)$$

if and only if $d_1a_2 \equiv 1 \pmod p$ and $d_2a_1 \equiv 1 \pmod p$. Then $a_1d_1a_2 \equiv a_1 \pmod p$ and so $a_1 \equiv a_2 \pmod p$.

Hence we see that

$$v \approx w \text{ if and only if } a_1 \equiv a_2 \pmod p \tag{1}$$

By our general discussion of imprimitivity, the number $\psi(p)$ of equivalence class under $\approx$ is given by

$$\psi(p) = |\Gamma_0(p) : \Gamma^*(p)|.$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^p \in \Gamma(p)$, then $|\Gamma^*(p) : \Gamma(p)| = p$. From [6], we know that

$$|\Gamma : \Gamma(N)| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \text{ and } |\Gamma : \Gamma_0(N)| = N \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Calculating for $N = p$ and using the following equation

$$\underbrace{|\Gamma : \Gamma(p)|}_{p(p^2-1)} = \underbrace{|\Gamma : \Gamma_0(p)|}_{p+1} \cdot \underbrace{|\Gamma_0(p) : \Gamma^*(p)|}_{p-1} \cdot \underbrace{|\Gamma^*(p) : \Gamma(p)|}_{p},$$

we have that

$$\binom{1}{p} = \begin{bmatrix} 1 \\ p \end{bmatrix} \cup \begin{bmatrix} 2 \\ p \end{bmatrix} \cup \ldots \cup \begin{bmatrix} p-1 \\ p \end{bmatrix}.$$

From (1), it is clear that

$$\begin{bmatrix} 1 \\ p \end{bmatrix} = \left\{ \frac{1+xp}{yp} : x, y \in \mathbb{Z} \right\} \cong [\infty] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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III. SUBORBITAL GRAPHS

In 1967 Sims introduced the idea of suborbital graphs of a permutation group G acting on a set Ω : these are graphs with vertex set Ω, on which G induces automorphism in [7]. Also in [8] the applications are used in finite groups.

Let (G,Ω) be transitive permutation group. Then G acts on $\Omega\times\Omega$ by

$$g:(\alpha,\beta)\to(g(\alpha),g(\beta))\ ,\ g\in G \text{ and } \alpha,\beta\in\Omega .$$

The orbits of this action are called suborbitals of G, that containing (α,β) being denoted by $O(\alpha,\beta)$. From $O(\alpha,\beta)$ we can form a suborbital graph $G(\alpha,\beta)$: its vertices are the elements of Ω, and there is a directed edge from γ to δ, denoted by $\gamma\to\delta$, if $(\gamma,\delta)\in O(\alpha,\beta)$. We can draw this edge as a hyperbolic geodesic in the upper half–plane H.

In this final section, we determine the suborbital graphs for $\Gamma_0(p)$ on $\begin{pmatrix}1\\p\end{pmatrix}$. Since $\Gamma_0(p)$ acts transitively on $\begin{pmatrix}1\\p\end{pmatrix}$, each suborbital contains a pair (∞,v) for some $v\in\begin{pmatrix}1\\p\end{pmatrix}$; $v=\frac{u}{p}$, we denote this suborbital by $O_{u,p}$ and corresponding suborbital graph by $G_{u,p}$.

$G_{u,p}$ is a disjoint union of $\psi(p)$ subgraphs forming blocks with respect to $''\approx''$ $\Gamma_0(p)$–invariant equivalence relation. $\Gamma_0(p)$ permutes these blocks transitively and these subgraphs are all isomorphic [4].

Therefore, it is sufficient to do the calculations only for the block $[\infty]$. Let $F_{u,p}$ denote the subgraph of $G_{u,p}$ whose vertices form the block $[\infty]$.

Theorem 3.1. Let $\frac{r}{s}$ and $\frac{x}{y}$ be in the block $[\infty]$. Then there is an edge $\frac{r}{s}\to\frac{x}{y}$ in $F_{u,p}$ if and only if

$$x\equiv\pm ur \ (\mathrm{mod}\ p) \text{ and } r\equiv 1(\mathrm{mod}\ p),\ ry-sx=\pm p$$
$$y\equiv\pm su \ (\mathrm{mod}\ p) \text{ and } s\equiv 0(\mathrm{mod}\ p),\ ry-sx=\pm p.$$

Proof. Since $\frac{r}{s}\to\frac{x}{y}\in F_{u,p}$, then there exists some $T\in\Gamma^*(p)$ such that T sends the pair $\left(\frac{1}{0},\frac{u}{p}\right)$ to the pair $\left(\frac{r}{s},\frac{x}{y}\right)$, that is, for $T=\begin{pmatrix}1+ap & b\\ pc & 1+dp\end{pmatrix}\in\Gamma^*(p)$, det T = 1, $T\begin{pmatrix}1\\0\end{pmatrix}=\frac{r}{s}$ and $T\begin{pmatrix}u\\p\end{pmatrix}=\frac{x}{y}$. From these equations , it is clear that $x\equiv ur\ (\mathrm{mod}\ p)$ and $y\equiv su\ (\mathrm{mod}\ p)$.

Furthermore

$$\begin{pmatrix}1+ap & b\\ pc & 1+dp\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & p\end{pmatrix}=\begin{pmatrix}r & x\\ s & y\end{pmatrix},$$

so that $ry-sx=p$.

Conversely, let be $x\equiv ur\ (\mathrm{mod}\ p)$ and $y\equiv su\ (\mathrm{mod}\ p)$ and also $r\equiv 1\ (\mathrm{mod}\ p)$ and $s\equiv 0\ (\mathrm{mod}\ p)$.Then there are $b,d\in\mathbb{Z}$ such that $x=ur+bp$ and $y=su+dp$. If we put these equivalences in $ry-sx=p$, we obtain

$$r(us+dp)-s(ur+bp)=p .$$

Since

$$\begin{pmatrix}r & b\\ s & d\end{pmatrix}\begin{pmatrix}1 & u\\ 0 & p\end{pmatrix}=\begin{pmatrix}r & ur+bp\\ s & us+dp\end{pmatrix},$$

then $rd-bs=1$. As $rd-bs\equiv 1(\mathrm{mod}\ p)$ and $s\equiv 0\ (\mathrm{mod}\ p)$, then $rd\equiv 1(\mathrm{mod}\ p)$. Since $r\equiv 1\ (\mathrm{mod}\ p)$, we obtain $d\equiv 1\ (\mathrm{mod}\ p)$.

Consequently,

$$A=\begin{pmatrix}r & b\\ s & d\end{pmatrix},\ \det A=1 \text{ and } \begin{matrix}r\equiv d\equiv 1(\mathrm{mod}\ p)\\ s\equiv 0\ (\mathrm{mod}\ p)\end{matrix},$$

so $A\in\Gamma^*(p)$.

The proof for $(-)$ is similiar.

Theorem 3.2. $\Gamma^*(p)$ permutes the vertices and the edges of $F_{u,p}$ transitively.

Proof. Suppose that $u,v\in[\infty]$. As $\Gamma_0(p)$ acts on $\begin{pmatrix}1\\p\end{pmatrix}$ transitively, $g(u)=v$ for some $g\in\Gamma_0(p)$. Since $u\approx\infty$ and $''\approx''$ is $\Gamma_0(p)-$ invariant equivalence relation, then $g(u)\approx g(\infty)$, that is, $v\approx g(\infty)$. Thus, as $g(\infty)\in[\infty]$, $g\in\Gamma^*(p)$.

Assume that $v,w\in[\infty]$; $x,y\in[\infty]$ and $v\to w$, $x\to y\in F_{u,p}$. Then $(v,w)\in O_{u,p}$ and $(x,y)\in O_{u,p}$. Therefore, for some $S,T\in\Gamma_0(p)$

$$S(\infty)=v\ ,\ S\begin{pmatrix}u\\p\end{pmatrix}=w\ ;\ T(\infty)=x\ ,\ T(\infty)=y\ .$$

As $S(\infty),T(\infty)\in[\infty]$, then $S,T\in\Gamma^*(p)$. So this proof is completed.

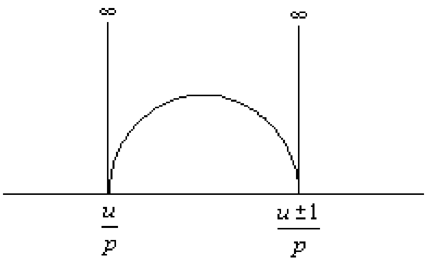

Fig. 1 $F_{u,p}$ – Suborbital Graph

Theorem 3.3. $F_{u,p}$ contains a triangle if and only if $u^2 \pm u + 1 \equiv 0 (\bmod p)$.

Proof. Since $\Gamma^*(p)$ permutes the vertices transitively $F_{u,p}$ and $\infty \to \frac{u}{p}$, then we may suppose that triangle has the form

$$\infty \to \frac{u}{p} \to v \to \infty .$$

Assume that $v = \frac{x}{yp}$, $y > 0$. Since $\frac{x}{yp} \to \frac{1}{0}$, then

$$0 \cdot x - yp = \pm p .$$

As $y > 0$, then $y = 1$. Therefore $v = \frac{x}{y}$. Since $\frac{u}{p} \to \frac{x}{y}$, then from Theorem 3.1 we obtain

$$u - x = 1 \quad \text{and} \quad x \equiv u^2 (\bmod p) \qquad (2)$$
$$u - x = -1 \quad \text{and} \quad x \equiv -u^2 (\bmod p) \qquad (3)$$

From (2) and (3), we have that

$$u^2 - u + 1 \equiv 0 (\bmod p) \quad \text{and} \quad u^2 + u + 1 \equiv 0 (\bmod p)$$

respectively.

Conversely, suppose that $u^2 \pm u + 1 \equiv 0 (\bmod p)$. Clearly, we have the triangle

$$\infty \to \frac{u}{p} \to \frac{u \pm 1}{p} \to \infty$$

from Theorem 3.1.

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