

# On some subspaces of Entire sequence space of Fuzzy Numbers

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**Abstract**—In this paper we introduce some subspaces of fuzzy entire sequence space. Some general properties of these sequence spaces are discussed. Also some inclusion relation involving the spaces are obtained.

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## I. INTRODUCTION

The concepts of fuzzy set theory was introduced by Zadeh [1]. Later on sequence of fuzzy numbers have been discussed by Matloka [2] and developed by Mursaleen [3] Nanda [4] and Savas [5], Tripathy and Dutta [6] and many others. The sequence space  $\Gamma$  of entire sequences was introduced by Ganapathy Iyer [9]. The space  $\Gamma(F)$  of fuzzy entire sequences space was introduced by Kavikumar, Azme Bin Khamis and Kandasamy [8]. Also Orlicz space of Entire sequence of fuzzy numbers was introduced by Subramanian and Metin Basarir [9]. In this article we introduce some subspaces of fuzzy entire sequence space and some of their properties are discussed

## II. PRELIMINARIES AND DEFINITIONS

We begin with giving some required definitions and propositions and lemmas. A fuzzy number is a fuzzy set on the real axis i.e. a mapping  $u : \mathbb{R} \rightarrow [0, 1]$  which satisfies the following four conditions.

- (i)  $u$  is normal i.e. there exists an  $x_0 \in \mathbb{R}$  such that  $u(x_0) = 1$ .
- (ii)  $u$  is fuzzy convex i.e.  $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$  for all  $x, y \in \mathbb{R}$  and for all  $\lambda \in [0, 1]$ .
- (iii)  $u$  is upper semi-continuous.
- (iv) The set  $[u]_0 = \{x \in \mathbb{R} : u(x) > 0\}$  is compact (cf. Zadeh [1]) where  $\{x \in \mathbb{R} : u(x) > 0\}$  denotes the closure of the set  $\{x \in \mathbb{R} : u(x) > 0\}$  in the usual topology of  $\mathbb{R}$ . We denote the set of all fuzzy numbers on  $\mathbb{R}$  by  $E'$  and called it as the space of fuzzy numbers.  $\alpha$ -level set  $[u]_\alpha$  of  $u \in E'$  is defined by

$$[u]_\alpha = \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\}, & (0 < \lambda \leq 1) \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}}, & (\lambda = 0). \end{cases}$$

The set  $[u]_\alpha$  is a closed bounded and non-empty interval for each  $\alpha \in [0, 1]$  which is defined by  $[u]_\alpha = [u^-(\alpha), v^-(\alpha)]$ .

$\mathbb{R}$  can be embedded in  $E'$  since each  $r \in \mathbb{R}$  can be regarded as a fuzzy number  $\bar{r}$  defined by  $\bar{r} = \begin{cases} 1, & (x = r) \\ 0, & (x \neq r). \end{cases}$

Let  $u, v, w \in E'$  and  $k \in \mathbb{R}$ . Then the operations addition, scalar multiplication and product and division defined on  $E'$  by

$$u + v = w \Leftrightarrow [w]_\alpha = [u]_\alpha + [v]_\alpha \quad \text{for all } \alpha \in [0, 1]$$

$$\Leftrightarrow w^-(\alpha) = [u^-(\alpha), v^-(\alpha)] \quad \text{and}$$

$$w^+(\alpha) = [u^+(\alpha), v^+(\alpha)] \quad \text{for all } \alpha \in [0, 1]$$

$$[ku]_\alpha = k[u]_\alpha \quad \text{for all } \alpha \in [0, 1]$$

$$\text{and } uv = w \Leftrightarrow [w]_\alpha = [u]_\alpha [v]_\alpha \quad \text{for all } \alpha \in [0, 1]$$

where it is immediate that

$$w^-(\alpha) = \min\{u^-(\alpha), v^-(\alpha), u^-(\alpha), v^+(\alpha), u^+(\alpha), v^-(\alpha), u^+(\alpha), v^+(\alpha)\}$$

$$\text{and } w^+(\alpha) = \max\{u^-(\alpha), v^-(\alpha), u^-(\alpha), v^+(\alpha), u^+(\alpha), v^-(\alpha), u^+(\alpha), v^+(\alpha)\}$$

$$\frac{u}{v} = w = [w]_\alpha = [u]_\alpha / [v]_\alpha \quad \text{for all } \alpha \in [0, 1]$$

$$= [u^-(\alpha), v^+(\alpha)] \cdot \left[ \frac{1}{v^-(\alpha)}, \frac{1}{v^+(\alpha)} \right]$$

$$= \left[ \min \left\{ \frac{[u]_-(\alpha)}{[v]_+(\alpha)}, \frac{u^-(\alpha)}{v^-(\alpha)}, \frac{u^+(\alpha)}{v^+(\alpha)}, \frac{u^+(\alpha)}{v^-(\alpha)} \right\} \right]$$

Let  $W$  be the set of all closed and bounded intervals  $A$  of real numbers with endpoints  $\underline{A}$  and  $\bar{A}$  i.e.,  $A = [\underline{A}, \bar{A}]$ .

Define the relation  $d$  on  $W$  by  $d(A, B) = \max\{|\bar{A} - \bar{B}|, |\underline{A} - \underline{B}|\}$ . Then it can easily be observed that  $d$  is a metric on  $w$  (cf. Diamond and Kloeden [10]) and  $(W, d)$  is a complete metric space. Now we can define the metric  $D$  on  $E'$  by means of a Hausdroff metric  $d$  as

$$D(u, v) = \sup_{\alpha \in [0, 1]} d([u]_\alpha, [v]_\alpha) \\ = \sup_{\alpha \in [0, 1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\}$$

One can extend the natural order relation on the real line to intervals as follows:

$$A \leq B \quad \text{if and only if} \quad \underline{A} \leq \underline{B} \quad \text{and} \quad \bar{A} \leq \bar{B}$$

The partial order relation on  $E'$  is defined as follows.  $u \leq v$  if and only if  $[u]_\alpha \leq [v]_\alpha$  if and only if  $u^-(\alpha) \leq v^-(\alpha)$  and  $u^+(\alpha) \leq v^+(\alpha)$ .

An absolute value  $|u|$  of a fuzzy number  $u$  is defined by

$$|u|(t) = \begin{cases} \max\{u(t), u(-t)\}, & (t \geq 0) \\ 0, & (t > 0) \end{cases}$$

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$\alpha$ -level set  $[|u|]_\alpha$  of the absolute value of  $u \in E'$  is in the form

$[|u|]_\alpha = [u^-(\alpha), v^+(\alpha)]$  where

$$|u|^-(\alpha) = \max\{0, u^-(\alpha), -u^+(\alpha)\}$$

$$|u|^+(\alpha) = \max\{|u^-(\alpha)|, |u^+(\alpha)|\}$$

**Definition II.1.** A sequence  $u = (u_k)$  of fuzzy numbers is a function  $u$  from the set  $\mathbb{N}$  into  $E'$ .

The fuzzy number  $u_k$  denotes the value of the function at  $k \in \mathbb{N}$  and is called the  $k$ -th term of the sequence. The set of all fuzzy sequences is denoted by  $w(F)$ .

**Definition II.2.** A sequence  $u = (u_k) \in w(F)$  is called convergent with limit  $\ell \in E'$  if and only if for every  $\varepsilon > 0$  there exists an  $n_0 = n_0(\varepsilon) \in \mathbb{N}$  such that  $D(u_k, u) < \varepsilon$  for all  $k \geq n_0$ .

**Definition II.3.** A sequence  $u = (u_k) \in w(F)$  is called bounded if and only if the set of fuzzy numbers consisting of the terms of the sequence  $(u_k)$  is a bounded set. That is, a sequence  $(u_k) \in w(F)$  is said to be bounded if and only if there exist two fuzzy numbers  $m$  and  $M$  such that  $m \leq u_k \leq M$  for all  $k \in \mathbb{N}$ . In this article we define some subspaces  $\Gamma(F, \lambda)$ ,  $\chi(F)$  and  $\Gamma(F, 1, d)$  of fuzzy entire sequences space.

### III. MAIN RESULTS

For each fixed  $k$ , define the fuzzy metric

$$D(u_k, v_k) = \sup_{\alpha \in [0,1]} \max\{|u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, |u_k^+(\alpha) - v_k^+(\alpha)|^{1/k}\}$$

Clearly  $(E', D)$  is a complete metric space.

We define the space of all entire sequences of fuzzy numbers by

$$\Gamma(F) = \{u = (u(k)) \in w(F) : \lim_{k \rightarrow \infty} D(u_k, \bar{0}) = 0\}$$

**Theorem III.1.**  $\Gamma(F)$  is a complete metric space with respect to the metric

$$d(u, v) = \sup_k D(u_k, v_k)$$

*Proof:* Let  $\{u^i\}$  be a Cauchy sequence of fuzzy numbers in  $\Gamma(F)$ . Then for given  $\varepsilon > 0$  there exist  $n_0 \in \mathbb{N}$  such that

$$d(u^i, u^j) = \sup_k D[u_k^{(i)}, v_k^{(j)}] < \varepsilon \quad \text{for all } i, j \geq n_0 \quad (1)$$

Since this is true for all  $k$ , we have

$$D(u_k^{(i)}, v_k^{(j)}) < \varepsilon \quad \text{for all } i, j \geq n_0 \quad (2)$$

This leads to the fact that  $\{u_k^{(i)}\}$  is a Cauchy sequence of fuzzy number in  $E'$ . Since  $(E', D)$  is a complete metric space,  $\{u_k^{(i)}\} \rightarrow u_k$  as  $i \rightarrow \infty$ .

Therefore,  $D(u_k^{(i)}, u_k) < \varepsilon$ , since this is true for all  $k$ ,  $\sup_k D(u_k^{(i)}, u_k) < \varepsilon$  implies that  $u^{(i)} \rightarrow u$  in  $\Gamma(F)$ .

It is easy to see that  $u \in \Gamma(F)$ .

Hence  $\Gamma(F)$  is complete. ■

Now we proceed to define some subspaces of  $\Gamma(F)$ .

**Definition III.2.** Let  $\lambda = (\lambda_k)$  denote a fixed sequence of fuzzy numbers such that  $\lambda_k \neq 0$  for all  $k$ . We define the sequence space  $\Gamma(F, \lambda)$  as follows:

$$\Gamma(F, \lambda) = \{u \in \Gamma(F) : \lambda u \in \Gamma(F)\}.$$

**Theorem III.3.**  $\Gamma(F, \lambda) = \Gamma(F)$  if and only if  $\limsup\{D(\lambda_k, \bar{0})\} < \infty$ .

*Proof:* Suppose  $\Gamma(F, \lambda) = \Gamma(F)$ .

Let  $u \in \Gamma(F, \lambda)$ . Then  $\lambda u \in \Gamma(F)$ . Therefore for given  $\varepsilon > 0$  there exist  $n_0$  such that  $D(\lambda_k u_k, \bar{0}) < \varepsilon$  for all  $k \geq n_0$ . Suppose  $\limsup\{D(\lambda_k, \bar{0})\} = \infty$ . Then there exist a subsequence  $\{n_k\}$  such that  $D(\lambda_{n_k}, \bar{0}) > M$  for some  $M > 0$ . Therefore  $\sup_{\alpha \in [0,1]} \max\{|\lambda_{n_k}^+(\alpha)|^{1/k}, |\lambda_{n_k}^-(\alpha)|^{1/k}\} > M$ .

This implies that  $|\lambda_{n_k}^+(\alpha)|^{1/k} > M$  and  $|\lambda_{n_k}^-(\alpha)|^{1/k} > M$ .

Now, define a sequence of fuzzy numbers by

$$u_{n_k} = \begin{cases} \bar{1}, & \text{if } n = k, \\ 0, & \text{if } n \neq k. \end{cases}$$

Then  $u_{n_k}^-(\alpha) = 0$  and  $u_{n_k}^+(\alpha) = 1$ . Clearly  $(u_{n_k}) \in \Gamma(F)$ . But  $|u_{n_k}^+(\alpha) \lambda_{n_k}^-(\alpha)|^{1/k} > M$  and  $|u_{n_k}^+(\alpha) \lambda_{n_k}^+(\alpha)|^{1/k} > M$ , which contradicts the fact that  $D(\lambda_k u_k, \bar{0}) < \varepsilon$ .

Conversely, Suppose  $\limsup\{D(\lambda_k, \bar{0})\} < \infty$ . Then there exist  $M > 0$  such that  $D(\lambda_k, \bar{0}) < M$  for all  $k$ .

Obviously  $\Gamma(F, \lambda) \subseteq \Gamma(F)$ . Let  $u \in \Gamma(F)$ . Then  $D(u_k, \bar{0}) < \varepsilon/M$ . Now,  $D(\lambda_k u_k, \bar{0}) \leq D(\lambda_k, \bar{0}) \leq D(u_k, \bar{0}) < \varepsilon$  (see cf. Talo [11]).

Hence  $\lambda u \in \Gamma(F)$ . From this we get  $\Gamma(F) \subseteq \Gamma(F, \lambda)$ .

Consequently,  $\Gamma(F) = \Gamma(F, \lambda)$ . This completes the proof ■

**Theorem III.4.** If  $\lambda = (\lambda_k)$  and  $\mu = (\mu_k)$  are any two fixed sequences of fuzzy numbers and if  $\{D(\gamma_k, \bar{0})\} < M$  for all  $k$  and for some  $M > 0$ , where  $\gamma_k = \frac{\mu_k}{\lambda_k}$  then  $\Gamma(F, \lambda) \subset \Gamma(F, \mu)$ .

*Proof:* Suppose  $\{D(\gamma_k, \bar{0})\} < M$  for some  $M > 0$ .

Then  $\sup_{\alpha \in [0,1]} \max\{|\gamma_k^+(\alpha)|^{1/k}, |\gamma_k^-(\alpha)|^{1/k}\} < M$ .

This implies that  $\left|\left(\frac{\mu_k}{\lambda_k}\right)^-(\alpha)\right|^{1/k} < M$  and  $\left|\left(\frac{\mu_k}{\lambda_k}\right)^+(\alpha)\right|^{1/k} < M$ . From this we get

$$|\mu_k^-| < M |\lambda_k^-|, \quad |\mu_k^+| < M |\lambda_k^+| \quad (3a)$$

$$\text{and } |\mu_k^-| < M |\lambda_k^+|, \quad |\mu_k^+| < M |\lambda_k^-| \quad (3b)$$

Let  $u \in \Gamma(F, \lambda)$ . Then  $\lambda u \in \Gamma(F)$ . Therefore for given  $\varepsilon > 0$  there exist  $n_0 \in \mathbb{N}$  such that  $D(\lambda_k u_k, \bar{0}) < \varepsilon/M$  for all  $k \geq n_0$ .

This implies that

$\sup_{\alpha \in [0,1]} \max\{ |(\lambda_k u_k)^-(\alpha)|^{1/k}, |(\lambda_k u_k)^+(\alpha)|^{1/k} \} < \varepsilon/M$ . This means that  $|(\lambda_k u_k)^-(\alpha)|^{1/k} < \varepsilon/M$  and  $|(\lambda_k u_k)^+(\alpha)|^{1/k} < \varepsilon/M$ . From this we get

$$|\lambda_k^-(\alpha) u_k^-(\alpha)| < \varepsilon/M, \quad |\lambda_k^+(\alpha) u_k^+(\alpha)| < \varepsilon/M \quad (4)$$

Using (3) and (4) we get

$$\begin{aligned} |\mu_k^-(\alpha) u_k^-(\alpha)| &= |\mu_k^-(\alpha)| |u_k^-(\alpha)| < M, \\ |\lambda_k^-(\alpha) u_k^-(\alpha)| &< M \cdot \frac{\varepsilon}{M} = \varepsilon \end{aligned}$$

Similarly we have  $|\mu_k^-(\alpha) u_k^+(\alpha)| < \varepsilon$  and  $|\mu_k^+(\alpha) u_k^-(\alpha)| < \varepsilon$ ,  $|\mu_k^+(\alpha) u_k^+(\alpha)| < \varepsilon$ . Hence  $\sup_{\alpha \in [0,1]} \max\{ |(\mu_k u_k)^-(\alpha)|^{1/k}, |(\mu_k u_k)^+(\alpha)|^{1/k} \} < \varepsilon$ . This implies that  $D(\mu_k u_k, \bar{0}) < \varepsilon$ . Thus  $u \in \Gamma(F, \mu)$ . ■

**Remark:** The condition stated in Theorem III.5 is not necessary. Let us, now define a sequence  $(\lambda_k) = (\frac{1}{k!})$ , where  $k \in E'$  and  $\mu_k = (\bar{1})$  for all  $k$ .

Then  $\{D(\gamma_k, \bar{0})\}$  and  $\{D(\mu_k, \bar{0})\}$  are bounded sequences.

Thus by Theorem III.4,  $\Gamma(F, \lambda) = \Gamma(F, \mu) = \Gamma(F)$ .

Therefore  $\Gamma(F, \lambda) \subseteq \Gamma(F, \mu)$ .

But  $\left\{ D\left(\frac{\mu_k}{\lambda_k}, \bar{0}\right) \right\}$  is unbounded. □

$\Gamma(F, \lambda)$  is endowed with two topologies, one is the metric topology inherited from  $\Gamma(F)$ , its metric being

$$\begin{aligned} d(u, v) &= \sup_k \left\{ \sup_{\alpha \in [0,1]} \max\{ |u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, \right. \\ &\quad \left. |u_k^+(\alpha) - v_k^+(\alpha)|^{1/k} \} \right\} \end{aligned}$$

where  $u, v \in \Gamma(F, \lambda)$ . The other is the metric topology  $d_\lambda$  given by

$d_\lambda(u, v) = \sup_k D_\lambda(u_k, v_k)$ , where

$$\begin{aligned} D_\lambda(u_k, v_k) &= \sup_{\alpha \in [0,1]} \max\{ |\lambda_k^-(\alpha)|^{1/k} |u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, \\ &\quad |\lambda_k^-(\alpha)|^{1/k} |u_k^+(\alpha) - v_k^+(\alpha)|^{1/k} \} \end{aligned}$$

and  $E'$  is complete with respect to  $D_\lambda$ .

**Theorem III.5.** If  $\limsup\{D(\lambda_k, \bar{0})\} < \infty$  then  $d$  is finer than  $d_\lambda$ .

*Proof:* To prove the result, it is enough to prove that if  $\{u_k\}$  is a sequences of fuzzy numbers converging to  $u$  in  $[\Gamma(F, \lambda), d]$  then the sequence converges to  $u$  in  $[\Gamma(F, \lambda), d_\lambda]$ . Consider the identity mapping  $I$  from  $(\Gamma(F, \lambda), d)$  to  $(\Gamma(F, \lambda), d_\lambda)$  defined by  $u \rightarrow u$ . Take  $u = \bar{0}$ , where  $\bar{0}$  is the zero element of  $\Gamma(F)$ . Since  $(u_k)$  converges to  $u = \bar{0}$  in  $(\Gamma(F, \lambda), d)$ , given  $\varepsilon > 0$  there exists a positive integer  $n_0$  such that  $d(u, \bar{0}) = \sup D(u_k, \bar{0}) < \varepsilon$  for all

$k \geq n_0$ .

Let  $U = \limsup\{D(\lambda_k, \bar{0})\}$

$$\begin{aligned} d_\lambda(u, \bar{0}) &= \sup_k \left\{ \sup_{\alpha \in [0,1]} \max\{ |\lambda_k^-(\alpha)|^{1/k} |u_k^-(\alpha)|^{1/k}, \right. \\ &\quad \left. |\lambda_k^+(\alpha)|^{1/k} |u_k^+(\alpha)|^{1/k} \} \right\} \\ &\leq U \sup_k \left\{ \sup_{\alpha \in [0,1]} \max\{ |u_k^-(\alpha)|^{1/k}, |u_k^+(\alpha)|^{1/k} \} \right\} \\ &\leq U \sup_k D(u_k, \bar{0}) < U\varepsilon \end{aligned}$$

Hence  $(u_k)$  converges to  $\bar{0}$  in  $(\Gamma(F, \lambda), d_\lambda)$ . ■

**Theorem III.6.**  $(\Gamma(F, \lambda), d_\lambda)$  is a complete metric space if and only if

$\liminf\{D(\lambda_k, \bar{0})\} > 0$ .

*Proof:* Let  $\{u^i\}$  be a Cauchy sequence of fuzzy numbers in  $\Gamma(F, \lambda)$ . Therefore for given  $\varepsilon > 0$  there exist  $n_0 \in \mathbb{N}$  such that  $d_\lambda(u^i, u^j) = \sup D_\lambda(u_k^{(i)}, u_k^{(j)}) < \varepsilon$  for all  $i, j \geq n_0$ . Since this is true for all  $k$ ,  $D_\lambda(u_k^{(i)}, u_k^{(j)}) < \varepsilon$  for all  $i, j \geq n_0$ . This implies that

$$\begin{aligned} \sup_k \left\{ \sup_{\alpha \in [0,1]} \max\{ |\lambda_k^-(\alpha)|^{1/k} |u_k^{(i)-}(\alpha) - u_k^{(j)-}(\alpha)|^{1/k}, \right. \\ \left. |\lambda_k^+(\alpha)|^{1/k} |u_k^{(i)+}(\alpha) - u_k^{(j)+}(\alpha)|^{1/k} \} \right\} < \varepsilon \end{aligned} \quad (5)$$

Let  $L = \liminf\{D(\lambda_k, \bar{0})\}$

$$= \liminf \left\{ \sup_{\alpha \in [0,1]} \max\{ |\lambda_k^-(\alpha)|^{1/k}, |\lambda_k^+(\alpha)|^{1/k} \} \right\} \quad (6)$$

Using (5) and (6) we get

$$|u_k^{(i)-}(\alpha) - u_k^{(j)-}(\alpha)|^{1/k} < \frac{\varepsilon}{L} \quad \text{and} \quad (7)$$

$$|u_k^{(i)+}(\alpha) - u_k^{(j)+}(\alpha)|^{1/k} < \frac{\varepsilon}{L} \quad \text{for all } i, j \geq n_0 \quad (8)$$

Hence  $\{u_k^{(i)}\}$  is a Cauchy sequence in  $E'$  and since  $(E', D)$  is complete,

$$\{u_k^{(i)}\} \rightarrow u_k \quad \text{as } i \rightarrow \infty \quad (9)$$

Hence  $D(u_k^{(i)}, u_k) < \frac{\varepsilon}{L}$  for all  $i, j \geq n_0$ .

Letting  $j \rightarrow \infty$  in (7) we get

$$\begin{aligned} |u_k^{(i)-}(\alpha) - u_k^-(\alpha)|^{1/k} &< \frac{\varepsilon}{L} \quad \text{and} \\ |u_k^{(i)+}(\alpha) - u_k^+(\alpha)|^{1/k} &< \frac{\varepsilon}{L} \end{aligned}$$

Now

$$\begin{aligned} |\lambda_k^-(\alpha)|^{1/k} |u_k^{(i)-}(\alpha) - u_k^-(\alpha)|^{1/k} &< \varepsilon \quad \text{and} \\ |\lambda_k^+(\alpha)|^{1/k} |u_k^{(i)+}(\alpha) - u_k^+(\alpha)|^{1/k} &< \varepsilon \end{aligned}$$

Hence

$$\begin{aligned} \sup_k \sup_{\alpha \in [0,1]} \max\{ |\lambda_k^-(\alpha)|^{1/k} |u_k^{(i)-}(\alpha) - u_k^-(\alpha)|^{1/k}, \\ |\lambda_k^+(\alpha)|^{1/k} |u_k^{(i)+}(\alpha) - u_k^+(\alpha)|^{1/k} \} < \varepsilon \end{aligned}$$

Thus  $u_k^{(i)} \rightarrow u$  in  $(\Gamma(F, \lambda), d_\lambda)$ .

Since each  $(u^i)$  is in  $\Gamma(F)$  we have

$$D(u_k^{(i)}, \bar{0}) < \frac{\varepsilon}{L}. \quad (10)$$

Using (9) and (10),

$$\begin{aligned} D(\lambda_k u_k, \bar{0}) &\leq D(\lambda_k, \bar{0}) D(u_k, \bar{0}) \quad (\text{See [11]}) \\ &\leq D(\lambda_k, \bar{0}) \{D(u_k, \bar{0}) + D(u_k^{(i)}, \bar{0})\} \\ &\leq L \left( \frac{\varepsilon}{L}, \frac{\varepsilon}{L} \right) \end{aligned}$$

Hence  $u \in \Gamma(F, \lambda)$ . Thus  $\Gamma(F, \lambda)$  is complete.

Conversely suppose  $\liminf \{D(\lambda_k, \bar{0})\} = 0$ .

Then there exist a subsequence  $\{D(\lambda_k, \bar{0})\}$  which is steadily decreasing and tends to zero.

Consider a sequence  $\{P_n\}$  of polynomials where  $P_n(x) = 1 + x^{n_1} + x^{n_2} + \dots + x^{n_k}$ . Clearly this sequence is a Cauchy sequence in  $(\Gamma(F, \lambda), d_\lambda)$ .

But it fails to converge to a point in  $(\Gamma(F, \lambda), d_\lambda)$ .

This completes the proof. ■

We now define the subsequences  $\chi(F)$  and  $\Gamma(F, 1, d)$  and we show that they are complete.

**Definition III.7.** For each fixed  $k$ , we define a fuzzy metric

$$D_\chi(u_k, v_k) = \sup_{\alpha \in [0,1]} \max \{ |k!u_k^-(\alpha) - k!v_k^-(\alpha)|^{1/k}, |k!u_k^+(\alpha) - k!v_k^+(\alpha)|^{1/k} \}$$

where  $u = (u_k)$  and  $v = (v_k)$  are sequences of fuzzy numbers and we can easily see that  $(E', D_\chi)$  is complete.

**Definition III.8.** The subspace  $\chi(F)$  of  $\Gamma(F)$  is defined by

$$\chi(F) = \left\{ u = (u_k) \in \Gamma(F) : \lim_{k \rightarrow \infty} D_\chi(u_k, \bar{0}) = 0 \right\}.$$

In other words given  $\varepsilon > 0$  there exists a positive integer  $n_0 \in \mathbb{N}$  such that  $D_\chi(u_k, \bar{0}) < \varepsilon$  for all  $k \geq n_0$ .

**Theorem III.9.**  $\chi(F)$  is a complete metric space (See [10]).

**Definition III.10.** For each fixed  $k$  we define a fuzzy metric by

$$\begin{aligned} \bar{D}(u_k, v_k) &= \sup_{\alpha \in [0,1]} \max \{ |k!u_k^-(\alpha) - v_k^-(\alpha)|^{1/k}, \\ &\quad |k!u_k^+(\alpha) - v_k^+(\alpha)|^{1/k} \} \end{aligned}$$

where  $u = (u_k)$  and  $v = (v_k)$  are sequences of fuzzy numbers and it is clear that  $(E', \bar{D})$  is complete.

**Definition III.11.** We define the subsequence  $\Gamma(F, 1, d)$  of  $\Gamma(F)$  by

$$\Gamma(F, 1, d) = \left\{ u = (u_k) \in \Gamma(F) : \lim_{k \rightarrow \infty} \bar{D}(u_k, \bar{0}) = 0 \right\}$$

In other words given  $\varepsilon > 0$  there exist  $n_0 \in \mathbb{N}$  such that  $\bar{D}(u_k, \bar{0}) < \varepsilon$ .

**Theorem III.12.**  $\Gamma(F, 1, d)$  is a complete metric space with respect to the metric  $\bar{d}(u, v) = \sup \bar{D}(u_k, v_k)$ .

**Theorem III.13.**  $\chi(F)$  is a proper closed subspace of  $\Gamma(F, 1, d)$ .

*Proof:* Consider the sequence  $(u_k)$  defined by  $(u_k) = (\frac{1}{k!})$  where  $k \in E'$ .

Then  $(u_k) \in \Gamma(F, 1, d)$  but  $(u_k) \notin \chi(F)$ .

Therefore  $\chi(F)$  is a proper subspace of  $\Gamma(F, 1, d)$ .

Let  $u \in \Gamma(F, 1, d)$  be a limit point of  $\chi(F)$ .

Then there exist a sequence  $(u^i)$  in  $\chi(F)$  such that  $u^i \rightarrow u$ .

Therefore for given  $\varepsilon > 0$  there exist  $n_0$  such that

$$\bar{d}(u', u) = \sup_k (u_k^{(i)}, u) < \varepsilon \quad \text{for all } k \geq n_0.$$

This implies that  $k[|u_k^{(i)-}(\alpha) - u_k^-(\alpha)|]^{1/k} < \varepsilon$  and  $k[|u_k^{(i)+}(\alpha) - u_k^+(\alpha)|]^{1/k} < \varepsilon$  for all  $k \geq n_0$ .

Now our aim is to prove  $u \in \chi(F)$ .

$$\begin{aligned} [\angle k | u_k^-(\alpha) |]^{1/k} &\leq [\angle k | u_k^{(i)}(\alpha) |]^{1/k} \\ &\quad + [\angle k | u_k^{(i)-}(\alpha) - u_k^-(\alpha) |]^{1/k} \\ &\leq [\angle k | u_k^{(i)}(\alpha) |]^{1/k} + (\angle k)^{1/k} \frac{\varepsilon}{k} \end{aligned}$$

Therefore  $u \in \chi(F)$  and is closed. ■

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