

On Generalized New Class of Matrix Polynomial Set

Ghazi S. Kahmmash

Abstract—New generalization of the new class matrix polynomial set have been obtained. An explicit representation and an expansion of the matrix exponential in a series of these matrix are given for these matrix polynomials.

Keywords—Generating functions, Recurrences relation and Generalization of the new class matrix polynomial set.

I. INTRODUCTION

RECENTLY, Hermite matrix polynomials have been introduced and studied in [3]-[4] and generalization of Hermite matrix polynomials are given [6] for matrix in $C^{N \times N}$ whose eigenvalues are all situated in the right open half-plane. Moreover, some properties of the Hermite matrix polynomials have been presented in [1]-[2]. If D_0 is the complex plane cut along the negative, real axis and $\log(z)$ denotes the principal logarithm of z , then $z^{1/2}$

represents $\exp\left(\frac{1}{2}\log(z)\right)$. If A is a matrix

in $C^{N \times N}$. the set of all the eigenvalues of A is denoted by $\sigma(A)$. if $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z , which are defined in an open set Ω of the complex plane, and A is a matrix in $C^{N \times N}$ such that $\sigma(A) \subset \Omega$. Then from the properties of the matrix functional calculus [7]. It follows that

$f(A)g(A) = g(A)f(A)$. If A is a matrix with $\sigma(A) \subset D_0$, then $A^{1/2} = \sqrt{A}$ denotes the a image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A . We say that A is a positive stable matrix

$$\text{if } \operatorname{Re}(z) > 0 \text{ for all } z \in \sigma(A) \quad (1)$$

If $A(k, n)$ are matrix in $C^{N \times N}$ for $n \geq 0$ and $k \geq 0$ then it follows that [1].

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k) \quad (2)$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} A(k, n-2k) \quad (3)$$

Ghazi S. Kahmmash is with the Mathematics Department, Al-Aqsa University, Gaza -Palestine (e-mail: ghazikahmmash@yahoo.com).

For m is a positive, similarly to (2) one can find [6].

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} A(k, n-mk); \quad n > m \quad (4)$$

a new class of matrix polynomial $k_n(x, A)$ suggested by Hermite polynomials $H_n(x, A)$ have been introduced and discussed in " as discussed by Kahmmash [5]" as follows.

$$k_n(x, A) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{s=0}^{\lfloor \frac{n-2r}{3} \rfloor} \frac{(-1)^r (A/2)^{n-(3/2)r-3s} n! x^{2n-3r-6s} 3^{n-r-3s}}{(n-2r-3s)! r! s!} \quad (5)$$

Where $k_n(x, A)$ is a polynomial of degree precisely $2n$ in x and that

$$k_n(x, A) = (3(A/2))^n x^{2n} + \pi_{2n-3} \text{in } x.$$

The aim of this paper is to derive the generalization of new class of matrix polynomials set, an explicit representation, expand the matrix exponential in a series of the generalized new class of matrix polynomial set with some recurrence relations in particular the four terms recurrence relation for these matrix polynomials.

II. DEFINITION OF GENERALIZATION OF A NEW CLASS OF MATRIX POLYNOMIALS SET

Let A be a matrix in $C^{N \times N}$ such that $\operatorname{Re}(\mu) > 0$ for every eigenvalue $\mu \in \sigma(A)$. For

$n = 0, 1, 2, \dots, \lambda \in R^+$ and m is a positive integer, we define the generalized anew of matrix polynomials by

$$F(x, t) = e^{\left(\lambda(3(A/2)x^2 t - 3\sqrt{A/2} x t^{m-1} + t^m I)\right)} \quad (6)$$

$$= \sum_{n=0}^{\infty} k_{n,m}^{\lambda}(x, A) t^n$$

Since

$$\exp\left(\lambda\left(3(A/2)x^2 t - 3\sqrt{A/2} x t^{m-1} + t^m I\right)\right)$$

$$= e^{3\lambda(A/2)x^2 t} e^{-3\lambda\sqrt{A/2} x t^{m-1}} e^{\lambda t^m I}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(3(A/2)x^2)^n}{n! r! s!}$$

$$\cdot \lambda^{n+r+s} (-1)^r \left(3\sqrt{A/2x}\right)^r t^{n+mr+ms-r}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{s=0}^{\lfloor \frac{n-(m-1)r}{m} \rfloor} \frac{(3(A/2)x^2)^{n-2r-3s}}{(n-(m-1)r)!r!s!}$$

$$\cdot \lambda^{n-m+2r+s} (-1)^r \left(3\sqrt{A/2x}\right)^r t^{n+ms}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{s=0}^{\lfloor \frac{n-(m-1)r}{m} \rfloor} \frac{(3(A/2)x^2)^{n-(m-1)r-ms}}{(n-(m-1)r-ms)!r!s!}$$

$$\cdot \lambda^{n-m+2r-ms+s} (-1)^r \left(3\sqrt{A/2x}\right)^r t^n$$

$$k_{n,m}^{\lambda}(x,A)$$

$$= \sum_{r=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{s=0}^{\lfloor \frac{n-(m-1)r}{m} \rfloor} \frac{(-1)^r n! x^{2n-2(m-1)r-2ms+r} \lambda^{n-(m+2)r-(m-1)s}}{(n-(m-1)r-ms)!r!s!}$$

$$\cdot (A/2)^{n-(m-1)r-ms+(1/2)r} 3^{n-(m-1)r-ms+r}$$

$$= \lambda^n \sum_{r=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{s=0}^{\lfloor \frac{n-(m-1)r}{m} \rfloor} \frac{(-1)^r \lambda^{n-(m-2)r-ms} x^{2n-(2m-3)r-2ms}}{\lambda^{(m+2)r+(m-1)s} (n-(m-1)r-ms)!r!s!} \quad (7)$$

$$\cdot (A/2)^{n-(m-3/2)r-ms} 3^{n-(m-2)r-ms}$$

Where $k_{n,m}^{\lambda}(x,A)$ is a polynomial of degree precisely $2n$ in x and that

$$k_{n,m}^{\lambda}(x,A) = ((A/2))^n x^{2n} + \pi_{2n-m}(x)$$

Where $\pi_{2n-m}(x)$ is a matrix polynomial of degree $(2n-m)$ in x .

For simplicity we denote $k_{n,m}^{\lambda}(x,A)$ for the generalized new class of matrix polynomials when $\lambda=1$ it should be observed that, in view of the explicit representation (6), the generalized new

class of matrix polynomials $k_{n,3}^{\lambda}(x,A)$ reduces to the new class of matrix polynomials $k_n(x,A)/n!$ as given in (5)

Note that

$$\begin{aligned} & \left(3Ax - 3\sqrt{A/2} t^{m-2}\right)^{-1} \frac{d}{dx} e^{3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1}} \\ & = t e^{3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1}} \end{aligned}$$

This,

$$\exp\left(\left(3Ax - 3\sqrt{A/2} t^{m-2}\right)^{-m} \frac{d^m}{dx^m}\right)$$

$$\cdot \exp\left(3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(3Ax - 3\sqrt{A/2} t^{m-2}\right)^{-1}$$

$$\cdot \frac{d}{dx} \Big|^{mn} \exp\left(3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} t^{mn} \exp\left(3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1}\right)$$

$$= \exp\left(3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1} + t^m I\right)$$

There fore, we have

$$\exp\left(\left(3Ax - 3\sqrt{A/2} t^{m-2}\right)^{-m} \frac{d^m}{dx^m}\right) \exp\left(3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1}\right)$$

$$= \sum_{n=0}^{\infty} k_{n,m}(x,A) t^n$$

$$\exp\left(\left(3Ax - 3\sqrt{A/2} t^{m-2}\right)^{-m} \frac{d^m}{dx^m}\right)$$

$$\cdot \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(3(A/2)x - 3\sqrt{A/2} t^{m-2}\right)^n t^n$$

$$= \sum_{n=0}^{\infty} k_{n,m}(x,A) t^n$$

Identification of the coefficients of t^n in both sides gives a new representation for the generalized new class matrix polynomials for $\lambda=1$, in the form:

$$k_{n,m}(x,A) = \frac{1}{n!} \exp\left(\left(3Ax - 3\sqrt{A/2} t^{m-2}\right)^{-m} \frac{d^m}{dx^m}\right) \quad (8)$$

$$\cdot \left(3(A/2)x - 3\sqrt{A/2} t^{m-2}\right)^n x^n$$

For $m=2$, the expression (8) gives another representation for the new class matrix polynomials in the form .

$$k_{n,m}(x,A) = \frac{1}{n!} \exp\left(\left(3Ax - 3\sqrt{A/2}\right)^{-2} \frac{d^2}{dx^2}\right) \quad (9)$$

$$\cdot \left(3(A/2)x - 3\sqrt{A/2}\right)^2 x^n$$

Let B be a matrix in $C^{N \times N}$ satisfies the spectral property .

$$|\operatorname{Re}(\mu)| > |\operatorname{Im}(\mu)|, \forall \mu \in \sigma(B) \quad (10)$$

Suppose that $A = 2B^2$ in view of the spectral mapping theorem [7]. it is easy to find that

$$\sigma(A) = \{2b^2 : b \in \sigma(B)\} \text{ and by (10) we have}$$

$$\operatorname{Re}(2b^2) = 2\left[\left(\operatorname{Re}(b)\right)^2 - \left(\operatorname{Im}(b)\right)^2\right] > 0, b \in \sigma(B)$$

That is, A is appositve stable matrix .In (2.1), putting $t=1$ and $B = \sqrt{A/2}$ gives .

$$\exp\left(\lambda\left(3(AB)^2 - 3Bx + I\right)\right) = \sum_{n=0}^{\infty} k_{n,m}^{\lambda}(x, 2B^2)$$

Therefore for the matrix B satisfies (10),an expansion of $\exp\left(3\lambda x^2 B - 3x\right)B$ in a series of the new class matrix polynomials is obtained in the form:

$$\begin{aligned} &\exp\left(3\lambda x^2 B - 3x\right)B \\ &= \exp(\lambda) \sum_{n=0}^{\infty} k_{n,m}^{\lambda}(x, 2B^2), \quad -\infty < x < \infty. \end{aligned}$$

III. RECURRENCE RELATIONS

Now, since

$$\begin{aligned} F(x, t) &= e^{\left(\lambda(3(A/2)x^2 t - 3\sqrt{A/2}xt^{m-1} + t^m I)\right)} \\ &= \sum_{n=0}^{\infty} k_{n,m}^{\lambda}(x, A)t^n. \end{aligned} \tag{11}$$

Differentiating (3.1) with respect to x yields .

$$\begin{aligned} &\lambda\left(3Axt - 3\sqrt{A/2}t^2\right)e^{\left(\lambda(3(A/2)x^2 t - 3\sqrt{A/2}xt^{m-1} + t^m I)\right)} \\ &= \sum_{n=0}^{\infty} D k_{n-m,k,m}^{\lambda}(x, A)t^n. \end{aligned} \tag{12}$$

By (11) and (12),we have

$$\begin{aligned} &\lambda\left(3Axt - 3\sqrt{A/2}t^{m-1}\right) \\ &\cdot \sum_{n=0}^{\infty} k_{n,m}^{\lambda}(x, A)t^n = \sum_{n=0}^{\infty} D k_{n,m}^{\lambda}(x, A)t^n. \end{aligned}$$

$$\begin{aligned} &\lambda\left(3Ax - 3\sqrt{A/2}t^{m-2}\right) \\ &\cdot \sum_{n=0}^{\infty} k_{n,m}^{\lambda}(x, A)t^{n+1} = \sum_{n=0}^{\infty} D k_{n,m}^{\lambda}(x, A)t^n. \end{aligned}$$

Since $D k_{0,m}^{\lambda}(x, A) = 0$, then for $n \geq 1$ one obtains

$$\lambda\left(3Ax - 3\sqrt{A/2}t^{m-2}\right)k_{n-1,m}^{\lambda}(x, A) = D k_{n,m}^{\lambda}(x, A) \tag{13}$$

Iteration (13) ,for $0 \leq k \leq n$ gives.

$$\begin{aligned} &D^k k_{n,m}^{\lambda}(x, A) \\ &= \left[\lambda\left(3Ax - 3\sqrt{A/2}t^{m-2}\right)\right]^k k_{n-k,m}^{\lambda}(x, A) \end{aligned} \tag{14}$$

Differentiating (11) with respect to x and t , we find

$$\begin{aligned} &\partial f / \partial x = \lambda\left(3Axt - 3\sqrt{A/2}t^{m-1}\right) \\ &\cdot \exp\left(\lambda\left(3(A/2)x^2 t - 3\sqrt{A/2}xt^{m-1} + t^m I\right)\right). \end{aligned}$$

And

Therefore, $F(x, t)$ satisfies the partial matrix differential equation.

$$\begin{aligned} &\left(\left(A/2\right)x^2 I - (m-1)\sqrt{A/2}xt^{m-2} + \frac{m}{3}t^{m-1}I\right)\frac{\partial f}{\partial x} \\ &- \left(Axt - \sqrt{A/2}xt^{m-1}\right)\frac{\partial f}{\partial t} = 0. \end{aligned}$$

Which by using (11)

$$\begin{aligned} &\left(\frac{A}{2}x^2 I - (m-1)\sqrt{A/2}xt^{m-2} + \frac{m}{3}t^{m-1}I\right)\sum_{n=0}^{\infty} D k_{n,m}^{\lambda}(x, A)t^n \\ &- \left(Axt - \sqrt{A/2}xt^{m-1}\right)\sum_{n=1}^{\infty} n k_{n,m}^{\lambda}(x, A)t^{n-1} = 0. \end{aligned}$$

Or

$$\begin{aligned} &\sum_{n=1}^{\infty} A x n k_{n,m}^{\lambda}(x, A)t^n = \sum_{n=1}^{\infty} \sqrt{A/2} n k_{n,m}^{\lambda}(x, A)t^{n+m-2} \\ &+ \sum_{n=1}^{\infty} \frac{A}{2} x^2 D k_{n,m}^{\lambda}(x, A)t^n - \sum_{n=1}^{\infty} (m-1)\sqrt{A/2} x t^{n+m-2} D k_{n,m}^{\lambda}(x, A) \\ &+ \sum_{n=1}^{\infty} \frac{m}{3} D k_{n,m}^{\lambda}(x, A)t^{n+m-1}. \end{aligned}$$

Or

$$\begin{aligned} &\sum_{n=1}^{\infty} n k_{n,m}^{\lambda}(x, A)t^n = \sum_{n=1}^{\infty} \sqrt{1/2A} x n k_{n,m}^{\lambda}(x, A)t^{n+m-2} \\ &+ \sum_{n=1}^{\infty} \frac{x}{2} D k_{n,m}^{\lambda}(x, A)t^n - \sum_{n=1}^{\infty} (m-1)\sqrt{1/2A} (1/x)t^{n+m-2} \\ &\cdot D k_{n,m}^{\lambda}(x, A) + \sum_{n=1}^{\infty} \frac{m}{3Ax} D k_{n,m}^{\lambda}(x, A)t^{n+m-1}. \end{aligned}$$

Since $k_{n,m}^{\lambda}(x, A) = (\lambda x \sqrt{2A})^n / n!$, for $0 \leq n \leq m - 2$,then we get .

$$\begin{aligned} &n k_{n,m}^{\lambda}(x, A) = \frac{x}{2} D k_{n,m}^{\lambda}(x, A) \\ &+ \left(\left(n/x\right) - (m-1)\sqrt{1/2A}(1/x)\right) D k_{n-m+2,m}^{\lambda}(x, A) \\ &+ \frac{m}{3}(A/x) D k_{n-m+1,m}^{\lambda}(x, A). \end{aligned} \tag{15}$$

For $\lambda=1$, from (6),

$$\begin{aligned} &\exp\left(\left(3(A/2)x^2 t - 3\sqrt{A/2}xt^{m-1} + t^m I\right)\right) \\ &= e^{3\sqrt{A/2}xt^{m-1}} e^{-t^m I} \sum_{n=0}^{\infty} k_{n,m}(x, A)t^n. \end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(3\sqrt{A/2}x)^r (-1)^s}{r!s!} \cdot t^{n+(m-1)r+ms} k_{n,m}(x,A).$$

$$\sum_{n=0}^{\infty} \frac{(3(A/2))^n (x^2t)^n}{n!} = \sum_{n=0}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{s=0}^{\lfloor \frac{n-(m-1)r}{m} \rfloor} \frac{(-1)^s}{r!s!} \cdot (3\sqrt{A/2}x)^r t^n k_{n-(m-1)r-ms,m}(x,A).$$

By equating of the coefficients of t^n , one gets

$$\frac{x^{2n}}{n!} I = (3A/2)^{-n} \sum_{r=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{s=0}^{\lfloor \frac{n-(m-1)r}{m} \rfloor} \frac{(-1)^s}{r!s!} \cdot (3\sqrt{A/2}x)^r k_{n-(m-1)r-ms,m}(x,A). \tag{16}$$

Since

$$\begin{aligned} & \exp\left(\left(3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1} + t^m u^m I\right)\right) \\ &= \exp\left(3(A/2)x^2t - 3\sqrt{A/2}xt^{m-1} + t^m I\right) \cdot \exp(-t^m + t^m u^m I). \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} k_{n,m}(x,A)(tu)^n &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (1-u^m)^k}{k!} \cdot t^{mk} k_{n,m}(x,A)t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (1-u^m)^k}{k!} k_{n-mk,m}(x,A)t^n \end{aligned}$$

Which, by comparing the coefficients of t^n , we get

$$u^n k_{n,m}(x,A) = \sum_{n=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (1-u^m)^k}{k!} k_{n-mk,m}(x,A). \tag{17}$$

We thus arrive to the following result.

Theorem 3.1. The generalized new class of matrix polynomials set. Satisfy the following relations:

$$\begin{aligned} 1- D^k k_{n,m}^\lambda(x,A) &= \left[\lambda(3Ax - 3\sqrt{A/2}t^{m-2}) \right]^k k_{n-k,m}^\lambda(x,A) \\ 2- n k_{n,m}^\lambda(x,A) &= \frac{x}{2} D k_{n,m}^\lambda(x,A) + \left((n/x) - (m-1)\sqrt{1/2A}(1/x) \right) \cdot D k_{n-m+2,m}^\lambda(x,A) + \frac{m}{3} (A/x) D k_{n-m+1}^\lambda(x,A). \end{aligned}$$

$$3- \frac{x^{2n}}{n!} I = (3A/2)^{-n} \sum_{r=0}^{\lfloor \frac{n}{m-1} \rfloor} \sum_{s=0}^{\lfloor \frac{n-(m-1)r}{m} \rfloor} \frac{(-1)^s}{r!s!} \cdot (3\sqrt{A/2}x)^r k_{n-(m-1)r-ms,m}(x,A).$$

$$4- u^n k_{n,m}(x,A) = \sum_{n=0}^{\lfloor n/m \rfloor} \frac{(-1)^k (1-u^m)^k}{k!} k_{n-mk,m}(x,A).$$

Now, inserting (13) in (15), yields

$$\begin{aligned} n k_{n,m}^\lambda(x,A) &= \frac{x}{2} (3\lambda Ax - 3\lambda\sqrt{A/2}t^{m-2}) \cdot k_{n-1,m}^\lambda(x,A) + \left((n/x) - (m-1)\sqrt{1/2A}(1/x) \right) k_{n-m+2,m}^\lambda(x,A) \\ &+ \frac{m}{3} (A/x) D k_{n-m+1}^\lambda(x,A). \end{aligned} \tag{18}$$

Replacing n by $n - m + 1$ in (3.3), gives

$$D k_{n-m+1,m}^\lambda(x,A) = \lambda(3Ax - 3\sqrt{A/2}t^{m-2}) k_{n-m,m}^\lambda(x,A). \tag{19}$$

Substituting from (19) into (18), yields

The four terms recurrence relation as given in the following theorem:

Theorem 3.2. The generalized new class matrix polynomials $k_{n,m}^\lambda(x,A)$, satisfy the

four terms recurrence relation:

$$\begin{aligned} n k_{n,m}^\lambda(x,A) &= \frac{3\lambda x}{2} (Ax - \sqrt{A/2}t^{m-2}) k_{n-1,m}^\lambda(x,A) \\ &+ \left((n/x) - (m-1)\sqrt{1/2A}(1/x) \right) k_{n-m+2,m}^\lambda(x,A) \\ &+ \frac{\lambda m}{Ax} (Ax - \sqrt{A/2}t^{m-2}) k_{n-m}^\lambda(x,A), n \geq m. \end{aligned} \tag{20}$$

With initial values

$$k_{n,m}^\lambda(x,A) = (\lambda x \sqrt{2A}) / n!, 0 \leq n \leq m - 2.$$

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