

On Cross-Ratio in some Moufang-Klingenberg Planes

Atilla Akpinar and Basri Celik

Abstract—In this paper we are interested in Moufang-Klingenberg planes $M(\mathcal{A})$ defined over a local alternative ring \mathcal{A} of dual numbers. We show that a collineation of $M(\mathcal{A})$ preserve cross-ratio. Also, we obtain some results about harmonic points.

Keywords—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio, harmonic points.

I. INTRODUCTION

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection [4, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by $M(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

(an alternative field \mathbf{A} , $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [8]. We will show that a collineation of $M(\mathcal{A})$ given in [2] preserves cross-ratio. Moreover, we will obtain some results related to harmonic points. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $M(\mathcal{A})$, respectively, it can be seen the papers of [10], [5], [9] or [8], [1].

The paper is organized as follows: Section 2 includes some basic definitions and results from the literature. In Section 3 we will give a collineation of $M(\mathcal{A})$ from [2] and we show that this collineation preserves cross-ratio. Finally, we obtain some results on harmonic points.

II. PRELIMINARIES

Let $M = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation ' \sim ' (neighbour relation) on \mathbf{P} and on \mathbf{L} , respectively. Then M is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q .

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h .

(PK3) There is a projective plane $M^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : M \rightarrow M^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

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hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in \mathbf{L}$, $C \in \mathbf{P}$, C is not symmetric to h and k . Then the well-defined bijection

$$\sigma := \sigma_C(k, h) : \begin{cases} h \rightarrow k \\ X \rightarrow XC \cap k \end{cases}$$

mapping h to k is called a *perspectivity* from h to k with center C . A product of a finite number of perspectivities is called a *projectivity*.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of M .

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane M that generalizes a Moufang plane, and for which M^* is a Moufang plane (for the exact definition see [3]).

An *alternative ring (field)* \mathbf{R} is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2b, (ba)a = ba^2, \forall a, b \in \mathbf{R}.$$

An alternative ring \mathbf{R} with identity element 1 is called *local* if the set \mathbf{I} of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [12, Theorem 3.1]).

Lemma 2.2: The identities

$$\begin{aligned} x(y(xz)) &= (xyx)z \\ ((yx)z)x &= y(xzx) \\ (xy)(zx) &= x(yz)x \end{aligned}$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [11, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [3].

Let \mathbf{R} be a local alternative ring. Then $M(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined

as follows:

$$\begin{aligned}
 \mathbf{P} &= \{(x, y, 1) : x, y \in \mathbf{R}\} \\
 &\cup \{(1, y, z) : y \in \mathbf{R}, z \in \mathbf{I}\} \\
 &\cup \{(w, 1, z) : w, z \in \mathbf{I}\}, \\
 \mathbf{L} &= \{[m, 1, p] : m, p \in \mathbf{R}\} \\
 &\cup \{[1, n, p] : p \in \mathbf{R}, n \in \mathbf{I}\} \\
 &\cup \{[q, n, 1] : q, n \in \mathbf{I}\} \\
 [m, 1, p] &= \{(x, xm + p, 1) : x \in \mathbf{R}\} \\
 &\cup \{(1, zp + m, z) : z \in \mathbf{I}\} \\
 [1, n, p] &= \{(yn + p, y, 1) : y \in \mathbf{R}\} \\
 &\cup \{(zp + n, 1, z) : z \in \mathbf{I}\} \\
 [q, n, 1] &= \{(1, y, yn + q) : y \in \mathbf{R}\} \\
 &\cup \{(w, 1, wq + n) : w \in \mathbf{I}\}
 \end{aligned}$$

and

$$\begin{aligned}
 P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q &\Leftrightarrow \\
 x_i - y_i \in \mathbf{I} \ (i = 1, 2, 3), \forall P, Q \in \mathbf{P} \\
 g = [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h &\Leftrightarrow \\
 x_i - y_i \in \mathbf{I} \ (i = 1, 2, 3), \forall g, h \in \mathbf{L}.
 \end{aligned}$$

Now it is time to give the following theorem from [3].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let \mathbf{A} be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where $a_i, b_i \in \mathbf{A}$ for $i = 1, 2$. Then \mathcal{A} is a local alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of \mathcal{A} is denoted as \mathbf{I}^{-1} . Calculations with the elements of \mathbf{I}^{-1} are defined as follows [7]:

$$\begin{aligned}
 (a\varepsilon)^{-1} + t &:= (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1} \\
 q(a\varepsilon)^{-1} &:= (aq^{-1}\varepsilon)^{-1} \\
 (a\varepsilon)^{-1}q &:= (q^{-1}a\varepsilon)^{-1} \\
 ((a\varepsilon)^{-1})^{-1} &:= a\varepsilon,
 \end{aligned}$$

where $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}$, $t \in \mathcal{A}$, $q \in \mathcal{A} \setminus \mathbf{I}$. (Other terms are not defined.). For more information about \mathcal{A} and its relation to MK-planes, the reader is referred to the papers of Blunck [7], [8]. In [8], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of \mathcal{A} which is commuting and associating with all elements of \mathcal{A} . It is $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$, where $\mathbf{Z} = \{z \in \mathbf{A} : za = az, \forall a \in \mathbf{A}\}$ is the centre of \mathbf{A} . If \mathbf{A} is not associative, then \mathbf{A} is a Cayley division algebra over its centre \mathbf{Z} .

Throughout this paper we assume $\text{char} \mathbf{A} \neq 2$ and we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$

Blunck [8] gives the following algebraic definition of the cross-ratio for the points on the line $g := [1, 0, 0]$ in $\mathbf{M}(\mathcal{A})$.

$$\begin{aligned}
 (A, B; C, D) &:= (a, b; c, d) \\
 &= \langle \left((a - d)^{-1} (b - d) \right) \left((b - c)^{-1} (a - c) \right) \rangle > \\
 (Z, B; C, D) &:= (z^{-1}, b; c, d) \\
 &= \langle \left((1 - dz)^{-1} (b - d) \right) \left((b - c)^{-1} (1 - cz) \right) \rangle > \\
 (A, Z; C, D) &:= (a, z^{-1}; c, d) \\
 &= \langle \left((a - d)^{-1} (1 - dz) \right) \left((1 - cz)^{-1} (a - c) \right) \rangle > \\
 (A, B; Z, D) &:= (a, b; z^{-1}, d) \\
 &= \langle \left((a - d)^{-1} (b - d) \right) \left((1 - zb)^{-1} (1 - za) \right) \rangle > \\
 (A, B; C, Z) &:= (a, b; c, z^{-1}) \\
 &= \langle \left((1 - za)^{-1} (1 - zb) \right) \left((b - c)^{-1} (a - c) \right) \rangle >,
 \end{aligned}$$

where $A = (0, a, 1), B = (0, b, 1), C = (0, c, 1), D = (0, d, 1), Z = (0, 1, z)$ are pairwise non-neighbour points of g and $\langle x \rangle = \{y^{-1}xy : y \in \mathcal{A}\}$.

In [7, Theorem 2], it is shown that the transformations

$$\begin{aligned}
 t_u(x) &= x + u; u \in \mathcal{A} \\
 r_u(x) &= xu; u \in \mathcal{A} \setminus \mathbf{I} \\
 i(x) &= x^{-1} \\
 l_u(x) &= ux = (ir_u^{-1}i)(x); u \in \mathcal{A} \setminus \mathbf{I}
 \end{aligned}$$

which are defined on the line g preserve cross-ratios. In [6, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by Λ , equals to the group of projectivities of a line in $\mathbf{M}(\mathcal{A})$. The elements preserving cross-ratio of the group Λ defined on g will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in $\mathbf{M}(\mathcal{A})$.

Theorem 2.2: Let $\{O, U, V, E\}$ be the basis of $\mathbf{M}(\mathcal{A})$ where $O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)$ (see [3, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line l can be calculated as follows:

If A, B, C, D, Z are the pairwise non-neighbour points

- of the line $l = [m, 1, k]$, where $A = (a, am + k, 1), B = (b, bm + k, 1), C = (c, cm + k, 1), D = (d, dm + k, 1)$ are not near to the line $UV = [0, 0, 1]$ and $Z = (1, m + zp, z)$ is near to UV ;
- of the line $l = [1, n, p]$, where $A = (an + p, a, 1), B = (bn + p, b, 1), C = (cn + p, c, 1), D = (dn + p, d, 1)$ are not neighbour to V and $Z = (n + zp, 1, z) \sim V$;
- of the line $l = [q, n, 1]$, where $A = (1, a, q + an), B = (1, b, q + bn), C = (1, c, q + cn), D = (1, d, q + dn)$ are not neighbour to V and $Z = (z, 1, zq + n) \sim V$;

then

$$\begin{aligned}(A, B; C, D) &= (a, b; c, d) \\ (Z, B; C, D) &= (z^{-1}, b; c, d) \\ (A, Z; C, D) &= (a, z^{-1}; c, d) \\ (A, B; Z, D) &= (a, b; z^{-1}, d) \\ (A, B; C, Z) &= (a, b; c, z^{-1}).\end{aligned}$$

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

Theorem 2.3: In $M(\mathcal{A})$, perspectivities preserve cross-ratios.

Now we give a definition in $M(\mathcal{A})$, well known from the case of Moufang planes [10]. In $M(\mathcal{A})$, any pairwise non-neighbour four points $A, B, C, D \in l$ are called as *harmonic* if $(A, B; C, D) = -1$ and we let $h(A, B, C, D)$ represent the statement: A, B, C, D are harmonic.

III. ON CROSS-RATIO IN $M(\mathcal{A})$.

In this section we will give a collineation of $M(\mathcal{A})$, from [2]. Next, we show that the collineation preserve cross-ratios. Now we start with giving the collineation of $M(\mathcal{A})$, where $w, z, q, n \in \mathbf{A}$: For any $s \notin \mathbf{I}$, the map J_s transforms points and lines as follows:

$$\begin{aligned}(x, y, 1) &\rightarrow (ys^{-1}, xs, 1) \\ (1, y, z\varepsilon) &\rightarrow (1, sy^{-1}s, s(y^{-1}z)) \text{ if } y \notin \mathbf{I} \\ (1, y, z\varepsilon) &\rightarrow (s^{-1}ys^{-1}, 1, s^{-1}z) \text{ if } y \in \mathbf{I} \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow (1, sws, sz)\end{aligned}$$

and

$$\begin{aligned}[m, 1, k] &\rightarrow [sm^{-1}s, 1, -(km^{-1})s] \text{ if } m \notin \mathbf{I} \\ [m, 1, k] &\rightarrow [1, s^{-1}ms^{-1}, ks^{-1}] \text{ if } m \in \mathbf{I} \\ [1, n\varepsilon, p] &\rightarrow [sns, 1, ps] \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [sn, s^{-1}q, 1].\end{aligned}$$

Now we are ready to give the following

Theorem 3.1: The collineation J_s preserve cross-ratio.

Proof: Let A, B, C, D and Z be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

$$\begin{aligned}(A, B; C, D) &= (a, b; c, d) \\ (Z, B; C, D) &= (z^{-1}, b; c, d) \\ (A, Z; C, D) &= (a, z^{-1}; c, d) \\ (A, B; Z, D) &= (a, b; z^{-1}, d) \\ (A, B; C, Z) &= (a, b; c, z^{-1}),\end{aligned}\quad (1)$$

where $z \in \mathbf{I}$. In this case we must find the effect of φ to the points of any line where φ is the collineations J_s .

Let $\varphi = J_s$. If $l = [m, 1, k]$, then

$$\begin{aligned}\varphi(X) &= \varphi(x, xm + k, 1) \\ &= ((xm + k)s^{-1}, xs, 1) \\ \varphi(Z) &= \varphi(1, m + zk, z) \\ &= (1, s(m + zk)^{-1}s, s((m + zk)^{-1}z)) \\ &\quad \text{for } m + zk \notin \mathbf{I} \\ \varphi(Z) &= \varphi(1, m + zk, z) \\ &= (s^{-1}(m + zk)s^{-1}, 1, s^{-1}z), \\ &\quad \text{for } m + zk \in \mathbf{I} \\ \varphi(l) &= [sm^{-1}s, 1, -(km^{-1})s] \text{ for } m \notin \mathbf{I} \\ \varphi(l) &= [1, s^{-1}ms^{-1}, ks^{-1}] \text{ for } m \in \mathbf{I}.\end{aligned}$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[sm^{-1}s, 1, -(km^{-1})s]$ is as follows:

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= ((am + k)s^{-1}, (bm + k)s^{-1}; \\ &\quad (cm + k)s^{-1}, (dm + k)s^{-1}) \\ &= {}^\sigma(a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= ((s((m + zk)^{-1}z))^{-1}, (bm + k)s^{-1}; \\ &\quad (cm + k)s^{-1}, (dm + k)s^{-1}) \\ &= {}^\sigma(z^{-1}, b; c, d),\end{aligned}$$

where $\sigma = r_{m^{-1}} \circ t_{-k} \circ r_s \in \Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $[1, s^{-1}ms^{-1}, ks^{-1}]$ is as follows:

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (as, bs; cs, ds) = {}^\sigma(a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}s, bs; cs, ds) = {}^\sigma(z^{-1}, b; c, d),\end{aligned}$$

where $\sigma = r_{s^{-1}} \in \Lambda$.

If $l = [1, n, p]$, then

$$\begin{aligned}\varphi(X) &= \varphi(xn + p, x, 1) = (xs^{-1}, (xn + p)s, 1) \\ \varphi(Z) &= \varphi(n + zp, 1, z) = (1, s(n + zp)s, sz)\end{aligned}$$

and

$$\varphi(l) = [sns, 1, ps].$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[sns, 1, ps]$ is as follows:

$$\begin{aligned}(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (as^{-1}, bs^{-1}; cs^{-1}, ds^{-1}) = {}^\sigma(a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}s^{-1}, bs^{-1}; cs^{-1}, ds^{-1}) = {}^\sigma(z^{-1}, b; c, d),\end{aligned}$$

where $\sigma = r_s \in \Lambda$. If $l = [q, n, 1]$, then

$$\begin{aligned}\varphi(X) &= \varphi(1, x, q + xn) = (1, sx^{-1}s, s(x^{-1}(q + xn))) \\ &\quad \text{for } x \notin \mathbf{I} \\ \varphi(X) &= \varphi(1, x, q + xn) = (s^{-1}xs^{-1}, 1, s^{-1}(q + xn)) \\ &\quad \text{for } x \in \mathbf{I} \\ \varphi(Z) &= \varphi(z, 1, zq + n) = (1, szs, s(zq + n))\end{aligned}$$

and

$$\varphi(l) = [sn, s^{-1}q, 1].$$

In this case, from (c) of Theorem 2.2, the cross-ratio of the points of $[sn, s^{-1}q, 1]$ is as follows:

$$\begin{aligned} & (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \\ &= (sa^{-1}s, sb^{-1}s; sc^{-1}s, sd^{-1}s) \\ &= {}^{\sigma}(a, b; c, d), \\ & (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \\ &= (szs, sb^{-1}s; sc^{-1}s, sd^{-1}s) \\ &= {}^{\sigma}(z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = i \circ l_{s^{-1}} \circ r_{s^{-1}} \in \Lambda$. Consequently, by considering other all cases we get

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}, b; c, d) \\ (\varphi(A), \varphi(Z); \varphi(C), \varphi(D)) &= (a, z^{-1}; c, d) \\ (\varphi(A), \varphi(B); \varphi(Z), \varphi(D)) &= (a, b; z^{-1}, d) \\ (\varphi(A), \varphi(B); \varphi(C), \varphi(Z)) &= (a, b; c, z^{-1}) \end{aligned}$$

for collineation φ . Combining the last result and the result of (1), the proof is completed. ■

Now we are ready to give the other results of the paper. On \mathcal{A} we give the following theorem, an alternate definition of harmonicity and given for an alternative ring \mathbf{A} with $\text{char } \mathbf{A} \neq 2$.

Theorem 3.2: Let $a, b, c, d \in \mathcal{A}$. Then $h(a, b, c, d)$ if and only if

- 1) if $a, b, c, d \in \mathcal{A}$, $2(a-b)^{-1} = (a-c)^{-1} + (a-d)^{-1}$.
- 2) if $a = z^{-1}$, $2(d-c)^{-1} + (c-b)^{-1} = z \in \mathbf{I}$.
- 3) if $b = z^{-1}$, $2(c-d)^{-1} + (d-a)^{-1} = z \in \mathbf{I}$.
- 4) if $c = z^{-1}$, $2(b-a)^{-1} + (d-b)^{-1} = z \in \mathbf{I}$.
- 5) if $d = z^{-1}$, $2(a-b)^{-1} + (c-a)^{-1} = z \in \mathbf{I}$.

Proof: 1. From the definition of cross-ratio,

$$h(a, b, c, d) = \left((a-d)^{-1}(b-d) \right) \left((b-c)^{-1}(a-c) \right) = -1.$$

By direct computation (with Lemma 2.1),

$$\begin{aligned} (a-d)^{-1}(b-d) &= -(a-c)^{-1}(b-c) \\ (a-d)^{-1}(b-a+a-d) &= -(a-c)^{-1}(b-a+a-c) \\ (a-d)^{-1}(b-a)+1 &= -(a-c)^{-1}(b-a)-1 \\ 2 &= -(a-c)^{-1}(b-a) - (a-d)^{-1}(b-a) \\ 2(a-b)^{-1} &= (a-c)^{-1} + (a-d)^{-1}. \end{aligned}$$

2. From the definition of cross-ratio,

$$\begin{aligned} h(z^{-1}, b, c, d) \\ = \left((1-dz)^{-1}(b-d) \right) \left((b-c)^{-1}(1-cz) \right) = -1. \end{aligned}$$

By direct computation (Lemma 2.1),

$$\begin{aligned} (b-c)^{-1}(1-cz) &= -(b-d)^{-1}(1-dz) \\ (b-c)^{-1}(1-cz) &= -(b-d)^{-1}(1-cz+cz-dz) \\ (b-c)^{-1}(1-cz) &= -(b-d)^{-1}(1-cz) \\ &\quad - (b-d)^{-1}((c-d)z) \\ \left((b-c)^{-1} + (b-d)^{-1} \right) (1-cz) &= -(b-d)^{-1}((c-d)z) \\ (b-c)^{-1} + (b-d)^{-1} &= - \left((b-d)^{-1}((c-d)z) \right) (1+cz) \\ (b-c)^{-1} + (b-d)^{-1} &= -(b-d)^{-1}((c-d)z) \\ (b-d)(b-c)^{-1} + 1 &= -(c-d)z \\ (b-c+c-d)(b-c)^{-1} + 1 &= -(c-d)z \\ 2 + (c-d)(b-c)^{-1} &= -(c-d)z \\ 2(c-d)^{-1} + (b-c)^{-1} &= -z \\ 2(d-c)^{-1} + (c-b)^{-1} &= z \in \mathbf{I}, \end{aligned}$$

where $zz = 0$ since $z \in \mathbf{I}$.

3. The proof is same the proof of 2.

4. From the definition of cross-ratio,

$$\begin{aligned} h(a, b, z^{-1}, d) \\ = \left((a-d)^{-1}(b-d) \right) \left((1-zb)^{-1}(1-za) \right) = -1. \end{aligned}$$

By direct computation (Lemma 2.1),

$$\begin{aligned} (1-zb)^{-1}(1-za) &= -(b-d)^{-1}(a-d) \\ (1+zb)(1-za) &= -(b-d)^{-1}(a-b+b-d) \\ 1+zb-za &= -(b-d)^{-1}(a-b)-1 \\ 2+zb-a &= -(b-d)^{-1}(a-b) \\ 2(b-a)^{-1} + z &= (b-d)^{-1} \\ 2(b-a)^{-1} + (d-b)^{-1} &= z \in \mathbf{I}, \end{aligned}$$

where $(1-zb)^{-1} = 1+zb$ and $zz = 0$.

5. The proof is same the proof of 4. ■

Now, we give the following theorem, given as without proof in [10] for \mathbf{A} .

Theorem 3.3: On \mathcal{A} , the followings is valid:

- 1) $h\left(0, a, 0^{-1}, \frac{a}{2}\right)$
- 2) $h\left(a, b, 0^{-1}, \frac{a+b}{2}\right)$
- 3) $h\left(a, -a, 0^{-1}, 0\right)$
- 4) $h\left(1, -1, a, a^{-1}\right)$
- 5) $h\left(a^2, 1, a, -a\right)$

Proof: 1. By the definition of cross-ratio, since

$$\left(0, a, 0^{-1}, \frac{a}{2}\right) = \left(0 - \frac{a}{2}\right)^{-1} \left(a - \frac{a}{2}\right) = \frac{-2}{a} \frac{a}{2} = -1,$$

then $h\left(0, a, 0^{-1}, \frac{a}{2}\right)$.

2. By the definition of cross-ratio, since

$$\begin{aligned} \left(a, b, 0^{-1}, \frac{a+b}{2}\right) &= \left(a - \frac{a+b}{2}\right)^{-1} \left(b - \frac{a+b}{2}\right) \\ &= \left(\frac{a-b}{2}\right)^{-1} \left(\frac{b-a}{2}\right) = -1, \end{aligned}$$

then $h(a, b, 0^{-1}, \frac{a+b}{2})$.

3. By the definition of cross-ratio, since

$$(a, -a, 0^{-1}, 0) = (a - 0)^{-1} (-a - 0) = -1,$$

then $h(a, -a, 0^{-1}, 0)$.

4. By the definition of cross-ratio, since

$$\begin{aligned} (1, -1, a, a^{-1}) &= \left((1 - a^{-1})^{-1} (-1 - a^{-1}) \right) \\ &\quad \left((-1 - a)^{-1} (1 - a) \right) \\ &= \left((a^{-1} - 1)^{-1} - (1 - a^{-1})^{-1} a^{-1} \right) \\ &\quad \left((-1 - a)^{-1} + (1 + a)^{-1} a \right) \\ &= \left((a^{-1} - 1)^{-1} - (a(1 - a^{-1}))^{-1} \right) \\ &\quad \left((-1 - a)^{-1} + (a^{-1}(1 + a))^{-1} \right) \\ &= \left((a^{-1} - 1)^{-1} - (a - 1)^{-1} \right) \\ &\quad \left(-(1 + a)^{-1} + (a^{-1} + 1)^{-1} \right) \\ &= (a^{-1} - 1)^{-1} (a^{-1} + 1)^{-1} - (1 + a)^{-1} \\ &\quad - (a - 1)^{-1} \left((a^{-1} + 1)^{-1} - (1 + a)^{-1} \right) \\ &= (a^{-1} - 1)^{-1} (a^{-1} + 1)^{-1} - (a^{-1} - 1)^{-1} \\ &\quad (1 + a)^{-1} - (a - 1)^{-1} (a^{-1} + 1)^{-1} \\ &\quad + (a - 1)^{-1} (1 + a)^{-1} \\ &= ((a^{-1} + 1)(a^{-1} - 1))^{-1} \\ &\quad - ((1 + a)(a^{-1} - 1))^{-1} \\ &\quad - ((a^{-1} + 1)(a - 1))^{-1} \\ &\quad + ((1 + a)(a - 1))^{-1} \\ &= (a^{-1}a^{-1} - a^{-1} + a^{-1} - 1)^{-1} \\ &\quad - (a^{-1} - 1 + 1 - a)^{-1} \\ &\quad - (1 - a^{-1} + a - 1)^{-1} + (a - 1 + aa - a)^{-1} \\ &= (a^{-1}a^{-1} - 1)^{-1} - (a^{-1} - a)^{-1} \\ &\quad - (-a^{-1} + a)^{-1} + (-1 + aa)^{-1} \\ &= (a^{-1}(a^{-1} - a))^{-1} - (a^{-1} - a)^{-1} \\ &\quad + (a^{-1} - a)^{-1} - (a(a^{-1} - a))^{-1} \\ &= (a^{-1} - a)^{-1} a - (a^{-1} - a)^{-1} a^{-1} \\ &= (a^{-1} - a)^{-1} (a - a^{-1}) \\ &= -1, \end{aligned}$$

then $h(1, -1, a, a^{-1})$.

5. By the definition of cross-ratio, since

$$\begin{aligned} (a^2, 1, a, -a) &= \left((a^2 + a)^{-1} (1 + a) \right) \left((1 - a)^{-1} (a^2 - a) \right) \\ &= \left(((a + 1)a)^{-1} (1 + a) \right) \left((1 - a)^{-1} ((a - 1)a) \right) \\ &= \left(a^{-1}(a + 1)^{-1} (1 + a) \right) \left((1 - a)^{-1} (a - 1)a \right) \\ &= a^{-1}(-a) \\ &= -1, \end{aligned}$$

then $h(a^2, 1, a, -a)$. ■

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