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# On a Way for Constructing Numerical Methods on the Joint of Multistep and Hybrid Methods

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***Abstract*****—**Taking into account that many problems of natural sciences and engineering are reduced to solving initial-value problem for ordinary differential equations, beginning from Newton, the scientists investigate approximate solution of ordinary differential equations. There are papers of different authors devoted to the solution of initial value problem for ODE. The Euler's known method that was developed under the guidance of the famous scientists Adams, Runge and Kutta is the most popular one among these methods.

Recently the scientists began to construct the methods preserving some properties of Adams and Runge-Kutta methods and called them hybrid methods. The constructions of such methods are investigated from the middle of the XX century. Here we investigate one generalization of multistep and hybrid methods and on their base we construct specific methods of accuracy order $p = 5$ and $p = 6$ for $k = 1$ ( $k$ is the order of the difference method).

***Keywords*****—**Multistep and hybrid methods, initial value problem, degree and stability of hybrid methods

## I. Introduction

ACTUALITY, urgency of investigation of numerical solution of the initial value problem for ordinary differential equations does not give rise to doubts. Therefore consider the following problem

$$y' = f(x, y), \ y(x_0) = y_0, \ x \in [x_0, X] \qquad (1)$$

Suppose that problem (1) has a unique continuous solution determined on the segment $[x_0, X]$ . For finding the numerical solution of problem (1) we divide the segment $[x_0, X]$ into $N$ equal parts by means of constant step $h > 0$ and define the partition points in the form $x_i = x_0 + ih \quad (i = 0,1,2,..., N)$.

As it was noted in [1] the first numerical method for solving problem (1) was constructed by Clairaunt and was applied to the investigation of the orbit of Halley's comet. Later on this method was developed by D'Alambert and other scientists. However, these methods were oblique – numerical (see [1]). Recently such methods are called analytic-numerical. The main deficiency of these methods was determined by Euler and as a result he suggested constructing direct methods. The first direct method was constructed by Euler and is used up to day. Some the accuracy of the Euler method is low, the scientists have developed Euler's idea and constructed new classes of numerical methods. The Adams and Runge-Kutta methods are most popular among them. The known methods Adams and Runge-Kutta hadn't high accuracy and while applying them to solving some applied problems they couldn't give satisfactory results.

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Therefore at the middle of the XX century the scientists have constructed a new class of methods preserving some properties of Adams and Runge-Kutta methods, and having high accuracy. They called it a hybrid method. One of the first hybrid methods constructed on the base of multistep methods is the special case of the following method (see [2], [3]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h\gamma f_{n+k-\theta}, \ (|\theta| < 1) \qquad (2)$$

In references this method is called Gear's method. Then the Gear method was modified in the following form:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=1}^{k-1} \beta_i f_{n+i} + h\gamma f_{n+\theta} + h\gamma_1 f_{n+k-\theta}, \qquad (3)$$
$$(|\theta| < 1)$$

Such schemes were called symmetric. The scientists from diferend countries constructed hybrid type one and multistep methods (see [4], [6]), whose generalizations were formulated in the form (see [7]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \gamma_i f_{n+i+\nu_i}, \qquad (4)$$
$$(|\nu_i| < 1, \ i = 0,1,..., k).$$

For finding a $\alpha_i, \gamma_i \ (i = 0,1,..., k)$ coefficients of method (4) the some place the system of nonlinear algebraic equations was suggested and the method with accuracy order four were constructed.

Note that while constructing the method $p = 6$ turned out so that $\nu_1 \neq 0$. However, if in method (4) we assume $\nu_i = 0 \ (i = 0,1,2,..., k)$ then the $k$-step method with constant coefficients and that was well investigated by many authors follows from (4).

*Here we suggest investigating the following method:*

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \gamma_i f_{n+i+\nu_i}, \qquad (5)$$

that combines multistep and hybrid methods. Indeed if in (5) we assume $\gamma_i = 0 \ (i = 0,1,2,..., k)$ then we get a multistep method, if we assume $\beta_i = 0 \ (i = 0,1,2,..., k)$ method (4) follows from (5).

Here we give a way for defining the coefficients of method (5) and construct specific methods with degree $p = 5,6$ for $k = 1$.

The degree of method (4) is determined by the scheme suggested in [8], i.e. it is said that if for rather smooth function $y(x)$ it holds

$$\sum_{i=0}^{k} (\alpha_i y(x + ih) - h\gamma_i y'(x + (i + \nu_i)h) = \qquad (6)$$
$$= O(h^{p+1}), \qquad h \to 0,$$

the degree of method (4) equals p, here $x$ denote the fixed point $x = x_0 + nh$

## II. One way for constructing hybrid methods

The properties of numerical methods depend on the value of their coefficients and different methods are used for their definition. Dependently on the way of determination of coefficients the methods are called multistep, Adams type or difference methods. For determining the coefficients of methods (5) we use the method with undetermined coefficients and to this end consider the following expansion:

$$y(x+ih) = y(x) + ihy'(x) + \\ + \frac{(ih)^2}{2!} y''(x) + ... + \frac{(ih)^p}{p!} y^{(p)}(x) + O(h^{p+1}), \quad (7)$$

$$y'(x+l_i h) = y'(x) + l_i hy''(x) + \\ + \frac{(l_i h)^2}{2!} y'''(x) + ... + \frac{(l_i h)^{p-1}}{(p-1)!} y^{(p)}(x) + O(h^p), \quad (8)$$

Suppose that method (5) has the degree $p$. Then asymptotic equality (6) holds. In the left hand side of asymptotic equality (6) we take into account expansion (7) and (8). Then we have

$$\left(\sum_{i=0}^{k} \alpha_i\right) y(x) + h \sum_{i=0}^{k} (i\alpha_i - \beta_i - \gamma_i) y'(x) + \\ + h \sum_{i=0}^{k} \left(\frac{i^2}{2}\alpha_i - i\beta_i - l_i\gamma_i\right) y'(x) + ... + \\ + h^p \sum_{i=0}^{k} \left(\frac{i^p}{p!}\alpha_i - \frac{i^{p-1}}{(p-1)!}\beta_i - \frac{l_i^{p-1}}{(p-1)!}\gamma_i\right) \times \\ \times y^{(p)}(x) = O(h^{p+1}), \qquad h \to 0, \quad (9)$$

where $l_i = i + \nu_i \quad (i = 0,1,2,...,k)$.

If we take into account that method (5) has the degree $p$, then from (9) we have:

$$\sum_{i=0}^{k} \alpha_i = 0, \ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k} (\beta_i + \gamma_i), \\ \sum_{i=0}^{k} \frac{i^m}{m!}\alpha_i = \sum_{i=0}^{k} \left(\frac{i^{m-1}}{(m-1)!}\beta_i + \frac{l_i^{m-1}}{(m-1)!}\gamma_i\right), \quad (10) \\ (m = 2,3,..., p).$$

It is easily determined that under the values $\nu_i = 0 \ (i = 0,1,...,k)$ system (10) in linear and coincides with the known systems used for determined the coefficients of the multistep method with constant coefficients. While fulfilling the conditions $|\nu_0| + |\nu_1| + ... + |\nu_k| \neq 0$ system (10) is nonlinear. By solving system (10) we determine the coefficients of the method of type (5). In this system the amount of the unknowns equals $4k+4$, the amount of the equations equals $p+1$. Since system (10) is homogeneous naturally, it will always have a zero solution. For system (10) have a non-zero solution, the condition $4k+4 > p+1$ should be fulfilled between the quantities $p$ and $k$. Hence we get $p \le 4k+2$.

Note that if we take $\beta_i = 0 \ (i = 0,1,2,...,k)$, the relation between the degree and order of method (5) or (4) will be of the form: $p \le 3k+1$.

It is known that if we consider the case $\gamma_i = 0 \ (i = 0,1,2,...,k)$, the degree of method (3) satisfied the condition $p \le 2k$.

The relation between the degree and order of hybrid methods of type (5) are more accurate than the known multistep methods. Consequently the methods of type (5) show, that the hybrid methods are both of theoretical and practical interest. Therefore we consider the investigation of method (5) and impose some restrictions of the coefficients of method (5) that are determined in the same way as in the paper [8].

A: The coefficients $\alpha_i, \beta_i, \gamma_i, \nu_i \ (i = 0,1,2,...,k)$ are some real numbers, moreover, $\alpha_k \neq 0$.

B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i,$$

$$\sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i;$$

$$\gamma(\lambda) \equiv \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i}.$$

have no common multipliers different from the constant.

C: $\sigma(1) + \gamma(1) \neq 0$ and $p \ge 1$.

The condition A is obvious.

Condition B. Assume the contrary and denote by $\varphi(\lambda)$ the common multiplier of these polynomials. Then by means of the shift operator

$$E^l y(x) = y(x+lh).$$

we rewrite method (5) in the form:

$$\rho(E)y_n - h\nu(E)y'_n - h\gamma(E)y'_n = 0. \quad (11)$$

It is known from the theory of finite-difference equations that for the existence of the function $y(x)$ should be known initial values $y_0, y_1, ..., y_{k-1}$ of the function $y(x)$. Equation (11) can be rewrite in the following form:

$$\varphi(E)(\rho_1(E)y_n - h\nu_1(E)y'_n - h\gamma_1(E)y'_n) = 0, \quad (12)$$

where

$$\rho_1(\lambda) = \rho(\lambda)/\varphi(\lambda), \\ \nu_1(\lambda) = \nu(\lambda)/\varphi(\lambda), \\ \gamma_1(\lambda) = \gamma(\lambda)/\varphi(\lambda).$$

Hence it follows that

$$\rho_1(E)y_n + h\nu_1(E)y'_n - h\gamma_1(E)y'_n = 0, \quad (13)$$

since $\varphi(\lambda) \neq const$.

Obviously the order of difference equation (13) is at most $k-1$. Therefore, taking into account the equivalent of equation (12), (13) we get that equation (11) has a unique solution if the $k-1$ initial conditions that contradict general theory of difference equations are known. Consequently the condition B holds.

Now proof that if method (5) converges, the condition C holds. Using the shift operator and the necessary condition $\rho(1)=0$ of convergence of the multistep method we can write relation (5) in the form

$$\begin{gathered}\rho_1(E)(y_i-y_{i-1})\\ \rho(E)y_i-h\nu(E)y_i'-h\gamma(E)y_i'=0\\ (\rho_1(\lambda)=\rho(\lambda)/(\lambda-1))\end{gathered} \tag{14}$$

Here we change the parameter I from 1 to n, and summing the obtained equalities. Then we get

$$\rho_1(E)(y_n-y_0)-\nu(E)h\sum_{j=1}^{n}y_j'-\gamma(E)h\sum_{j=1}^{n}y_j'=0. \tag{15}$$

Passing to limit as $h\to 0$, we have:

$$\rho_1(1)(y(x)-y_0)=(\nu(1)+\gamma(1))\int_{x_0}^{x}f(s,y(s))ds,$$

Here $x=x_0+nh$ is a fixed point.

From here we get

$$\rho_1(1)=\nu(1)+\gamma(1). \tag{16}$$

Considering

$$\rho_1(\lambda)=\rho(\lambda)/(\lambda-1) \text{ и } \lim_{\lambda\to 1}\rho_1(\lambda)=\rho'(1).$$

We rewrite equality (16) in the form

$$\rho_1'(1)=\nu(1)+\gamma(1).$$

Thus we obtain that $\rho(1)=0$ and $\rho'(1)=\nu(1)+\gamma(1)$, that coincide with the first two equations of system (10) where from it follows that $p\geq 1$. Note that if $\nu(1)+\gamma(1)=0$, we get, $\rho'(1)=0$. Hence it follows that method (5) is not stable. But it is known that the stability is a necessary and sufficient condition for the convergence the method's (5). Consequently the condition C holds. Method (5) is called stable if the roots of the polynomial $\rho(\lambda)$ lie interior to a unit circle whose boundary the polynomial $\rho(\lambda)$ has no multiple roots (see [8]).

Thus we proved that the restrictions A,B,C are natural and suppose that they holds every where.

As was noted above, if method (5) is stable and has the degree p, then $p\leq 4k+2$. The method with the degree $p=6$, obtained for $k=1$ is stable. For $k=2$ one can construct stable methods with the degree $p=8$ and $p=9$.

Consider the investigation of method (5) for $k=1$. In this case, under the assumption, a $\alpha_1=-\alpha_0=1$, system (10) has the following form:

$$\begin{aligned}&\beta_0+\beta_1+\gamma_0+\gamma_1=1,\\ &\beta_1+l_0\gamma_0+l_1\gamma_1=1/2,\\ &\beta_1+l_0^2\gamma_0+l_1^2\gamma_1=1/3,\\ &\beta_1+l_0^3\gamma_0+l_1^3\gamma_1=1/4,\\ &\beta_1+l_0^4\gamma_0+l_1^4\gamma_1=1/5,\\ &\beta_1+l_0^5\gamma_0+l_1^5\gamma_1=1/6.\end{aligned} \tag{17}$$

By solving the system of nonlinear algebraic equations we get the following solution:

$$\beta_0=\beta_1=1/12,\ \gamma_0=\gamma_1=5/12,$$
$$l_0=1/2-\sqrt{5}/10, l_1=1/2+\sqrt{5}/10.$$

The method with the degree $p=6$ is on the form:

$$\begin{aligned}y_{n+1}&=y_n+h(f_{n+1}+f_n)/12+\\ &+5h(f_{n+1/2-\sqrt{5/10}}+f_{n+1/2+\sqrt{5/10}})/12\end{aligned}. \tag{18}$$

For applying hybrid method (18) to the solution of some problems we shoued know the values of the quantities $y_{n+1/2-\sqrt{5/10}}$ and $y_{n+1/2+\sqrt{5/10}}$, and the accuracy of these values should be at least $O(h^6)$.

Note that hybrid method (18) is implicit. In order to use such methods, the predictor-corrector scheme that contains even of one explicit method that in one variant has the form

$$\begin{aligned}y_{n+1}&=y_n+hf_n/9+h((16+\sqrt{6})f_{n+(6-\sqrt{6})/10}+\\ &+(16-\sqrt{6})f_{n+(6+\sqrt{6})/10})/36.\end{aligned} \tag{19}$$

This method is explicit and has the degree $p=5$.

For using method (19) the following algorithm is used.

Suppose that the approximate values $y_n$ and $y_{n+1/2}$ were found with accuracy $O(h^4)$ and in order to use method (19) we use the following block method:

Step I. $\hat{y}_{n+1}=y_n+hy'_{n+1/2}$,

Step II $y_{n+1}=y_n+h(\hat{y}'_{n+1}+4y'_{n+1/2}+y'_n)/6$,

Step III $y_{n+3/2}=y_{n+1/2}+h(7y'_{n+1}-2y'_{n+1/2}+y'_n)/6$,

Step IV

$$\begin{aligned}y_{n+2}=y_n+\alpha hy'_n+\alpha^2h((\alpha^2-12\alpha+6)y'_{n+3/2}-\\ -(3\alpha^2-48\alpha+27)y'_{n+1}+\\ +(3\alpha^2-60\alpha+54)y'_{n+1/2}-\\ -(\alpha^2-24\alpha+33y'_n)/18\end{aligned}$$

for $\alpha=\dfrac{6-\sqrt{6}}{10}$ и $\alpha=\dfrac{6+\sqrt{6}}{10}$.

Step V

$$\begin{aligned}y_{n+1}&=y_n+hf_n/9+h((16+\sqrt{6})f_{n+(6-\sqrt{6})/10}+\\ &+(16-\sqrt{6})f_{n+(6+\sqrt{6})/10})/36.\end{aligned}$$

For calculating the initial values of $y_0$ and $y_{1/2}$ we use the initial condition of the problem (1) and the following scheme:

$$\bar{y}_{1/2}=y_0+\frac{h}{2}f_0,\quad 0(h^2),$$
$$\hat{y}_{1/2}=y_0+h(f_0+f(h/2,\bar{y}_{1/2}))/4,\quad 0(h^3),$$
$$\bar{y}_{n+1/6}=y_n+hf_n/6,$$
$$\hat{y}_{n+1/6}=y_n+h(f_n+f_{n+1/6})/12,$$
$$\hat{y}_{n+1/2}=y_n+h(3\hat{f}_{n+1/6}+\hat{f}'_{n+1/2})/4.$$

## III. Conclusion

We partly investigated method (5) showed that it is more precise that the known methods. However we didn't consider definition of the stability domain and the stability connected with its application to specific problems and etc.

The construction of the block method for using method (19) shows that while increasing the accuracy of the method there arise some difficulties for their use. Note that for using method (19) we can use iterative processes, however in this case there arise more complicated questions that are difficult to solve. Therefore, here are chose the block methods being one of the variants of the predictor-corrector scheme.

Here the algorithm chose for using method (15) is the first step in this field and therefore needs some corrections.

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