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Observer Design for Ecological Monitoring

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***Abstract*—**Monitoring of ecological systems is one of the major issues in ecosystem research. The concepts and methodology of mathematical systems theory provide useful tools to face this problem. In many cases, state monitoring of a complex ecological system consists in observation (measurement) of certain state variables, and the whole state process has to be determined from the observed data. The solution proposed in the paper is the design of an observer system, which makes it possible to approximately recover the state process from its partial observation. The method is illustrated with a trophic chain of *resource – producer – primary consumer* type and a numerical example is also presented.

***Keywords*—**Monitoring, observer system, trophic chain.

I. Introduction

THE problem of sustainability of economic and social development in a broader sense also involves conservation aspects of ecology. The problem of state monitoring of population systems, even under natural conditions, is an important issue in conservation ecology. Nearly natural populations are often exposed to a strong human intervention, e.g. by wildlife management, fisheries or environmental pollution. This means that human activity may improve or break the equilibrium of the population system in question, it may also increase or decrease the genetic variability of the given populations. One of the main tasks of conservation biology is to preserve the diversity of population systems and genetic variability of certain populations. These problems make it necessary to extend the traditional approach of theoretical biology focusing only on a biological object, to the study of the system “biological object – man”. This, in dynamic situation, i.e. in case of a long-term human intervention, typically requires the approach of mathematical systems theory (in frequently used terms, state-space modelling). On the state-space approach to modelling in population biology, [1] is an early reference, see also [2].

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Mathematical systems theory offers a methodology to answer this question. This discipline had been developed by the 1960s to solve variety of problems faced in engineering and industry. A basic reference is [3], see also [4]. A recent reference is [5]. While by now, mathematical systems theory became quite familiar to system engineers, observability and controllability analysis of dynamic models in population biology is relatively new. In many cases, state monitoring of a complex ecological system consists in observation (measurement) of certain state variables, and the whole state process has to be determined from the observed data. In a more general setting, the state process is a system of differential equations, and instead of its concrete solution only a transform (in particular a subset of the components) of it is known (measured). The considered system is called (locally) observable, if from the observation, the underlying state process can be uniquely recovered (near an equilibrium state). Based on the a sufficient condition for nonlinear observation systems published in [6], for different coexisting Lotka-Volterra type population systems, local observability results have been obtained in part by some of the coauthors of the of the present paper in [7], [8] and [9]. Later on, in addition to these theoretical results, for *Lotka-Volterra systems* even a so-called observer systems has been constructed that made it possible to numerically recover the state process from the observation data, see [8], [9], [10], [11] and [12].

In the present paper ecological systems of *non Lotka-Volterra* type will be considered. Until now in [13], only observability results have been obtained for systems of type *resource – producer – primary consumer*. In Section II, from [13], the model setup and basic conditions for the existence of an equilibrium of the system are shortly recalled. Section III is the main body of the paper. First the theoretical background of the observer design is set up. Then the construction of the observer and the asymptotic recovery of the state process is illustrated with a numerical example. Section 4 is devoted to the discussion of the results.

II. Description of the Dynamic Model

In order to illustrate the application of the methodology of mathematical systems theory, a relatively simple food web, a trophic chain has been chosen, that in addition to populations also involves a resource (energy or nutrient). In the following, the model setup is shortly recalled from [13], see also [14] and [15]. For further details on trophic chains (and genaral food webs) see e.g. [16] and [17].

The considered model describes how a resource moves through a trophic chain. A typical terrestrial trophic chains consists in the following components:

resource, the *0th trophic level* (solar energy or inorganic nutrient),

which is incorporated by

a plant population, the *1st trophic level* (*producer*),

which transfers it to

a herbivorous animal population, the *2nd trophic level* (*primary consumer*).

Let it be noted that, in a longer trophic chain, the herbivores can be consumed by a predator population the *3rd trophic level* (*secondary consumer*), which can be followed by top predator population (*tertiary consumers*). In the present paper, for technical simplicity only trophic chains of the type *resource – producer – primary consumer* will be studied. According to the possible types of 0th level (energy or nutrient), two types of trophic chains will be considered: *open chains* (without recycling) and *closed chains* (with recycling). At the *0th trophic level, resource* is the common term for energy and nutrient.

Let x_0 denote of the time-varying quantity of resource present in the system, x_1 and x_2, in function of time, the biomass (or density) of the producer (species 1) and the primary consumer (species 2), respectively. Let Q be the resource supply considered constant in the model. Let $\alpha_0 x_0$ be the velocity at which a unit of biomass of species 1 consumes the resource, and it is assumed that this consumption increases the biomass of this species at rate k_1. A unit of biomass of species 2 consumes the biomass of species 1 at velocity $\alpha_1 x_1$, converting it into biomass at rate k_2. Both the plant and the animal populations are supposed to decrease exponentially in the absence of the resource and the other species, with respective rates of decrease (Malthus parameters) m_1 and m_2.

Finally, in a *closed system* the dead individuals of species 1 and 2 are recycled into nutrient at respective rates $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$, while for an *open system* (where there is no natural recycling) $\beta_1 = 0,\ \beta_2 = 0$ holds. Then with model parameters

$$Q, \alpha_0, \alpha_1, m_1, m_2 > 0;\ k_1, k_2 \in]0,1[;\ \beta_1, \beta_2 \in [0,1[,$$

for the trophic chain the following dynamic model can be set up:

$$\dot{x}_0 = Q - \alpha_0 x_0 x_1 + \beta_1 m_1 x_1 + \beta_2 m_2 x_2 \quad (2.0)$$

$$\dot{x}_1 = x_1(-m_1 + k_1\alpha_0 x_0 - \alpha_1 x_2) \quad (2.1)$$

$$\dot{x}_2 = x_2(-m_2 + k_2\alpha_1 x_1)\,. \quad (2.2)$$

Let function f be defined in terms of the right-hand side of this system:

$$f : \mathbf{R}^3 \to \mathbf{R}^3,$$

$$f(x) = f(x_0, x_1, x_2) := \begin{bmatrix} Q - \alpha_0 x_0 x_1 + \beta_1 m_1 x_1 + \beta_2 m_2 x_2 \\ x_1(-m_1 + k_1\alpha_0 x_0 - \alpha_1 x_2) \\ x_2(-m_2 + k_2\alpha_1 x_1) \end{bmatrix}$$

.

In [13], a necessary and sufficient condition were found for the existence of a non-trivial ecological equilibrium x^* of dynamic system (2.0)-(2.2), where all components are present: system (2.0)-(2.2) has a unique equilibrium $x^* = (x_0^*, x_1^*, x_2^*) > 0$ if and only if the resource supply is high enough, i.e.

$$Q > Q_2 := \frac{m_1 m_2}{\alpha_1 k_1 k_2} - \frac{\beta_1 m_1 m_2}{\alpha_1 k_2}\,. \quad (2.3)$$

Throughout the paper condition (2.3) will be supposed.

III. Concept of Observability. Construction of an Observer System for a Trophic Chain

Given positive integers *m, n,* and continuously differentiable functions

$$f : \mathbf{R}^n \to \mathbf{R}^n, \quad h : \mathbf{R}^n \to \mathbf{R}^m,$$

the following observation system is considered

$$\dot{x} = f(x) \quad (3.1)$$

$$y = h(x)\ , \quad (3.2)$$

where y is called the *observed function*. It is supposed that $x^* \in \mathbf{R}^n$ is an equilibrium with zero observation: $f(x^*) = 0$ and $h(x^*) = 0$.

Definition 3.1. Observation system (3.1)-(3.2) is called *locally observable* near equilibrium x^*, over a given time interval $[0,T]$, if there exists $\varepsilon > 0$, such that for any two different solutions x and $\bar{x}$ of system (1) with $|x(t) - x^*| < \varepsilon$ and $|\bar{x}(t) - x^*| < \varepsilon \ (t \in [0,T])$, the observed functions $h \circ x$ and $h \circ \bar{x}$ are different. ($\circ$ denotes the composition of functions. For brevity, the reference to $[0,T]$ will be suppressed).

For the formulation of a sufficient condition for local observability of the observation system (3.1)-(3.2), the latter is linearized in terms of the corresponding Jacobians
$A := f'(x^*)$ and $C := h'(x^*)$.

Theorem 3.2. (Lee and Markus, 1971) Suppose that rank condition

$$rank[C \,|\, CA \,|\, CA^2 \,|\, ... \,|\, CA^{n-1}]^T = n \qquad (3.3)$$

holds. Then the observation system (3.1)-(3.2) is locally observable near equilibrium x^*.

Now, the construction of an observer system for system (3.1)-(3.2) will be based on [18].

Definition 3.3. Given a continuously differentiable function $G : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, system

$$\dot{z} = G(z, y) \qquad (3.4)$$

is called a *local asymptotic (respectively, exponential) observer for observation system* (3.1)-(3.2) if the composite system (3.1)-(3.2), (3.4) satisfies the following two requirements.

i) If $x(0) = z(0)$, then $x(t) = z(t)$, for all $t \geq 0$.

ii) There exists a neighbourhood V of the equilibrium x^* of $\mathbf{R}^n$ such that for all $x(0), z(0) \in V$, the estimation error $z(t) - x(t)$ decays asymptotically (respectively, exponentially) to zero.

Theorem 3.4. (Sundarapandian, 2002) Suppose that x^* is a Lyapunov stable equilibrium of system (3.1), and that there exists a matrix H such that matrix *A-HC* is Hurwitz (i.e. all its eigenvalues have negative real parts). Then dynamic system defined by

$$\dot{z} = f(z) + K[y - h(z)] \qquad (3.5)$$

is a local exponential observer for observation system (3.1)-(3.2).

For the case of observation of a single state variable, *local observability* of system (2.0)-(2.2) was proved in [13]. However, this only means that from the observation of one component, *in principle*, the whole state process can be uniquely recovered near the equilibrium. Below, by the construction of the observer (or state estimator) system, the original trajectory will be also *numerically estimated*. For the application of Theorem 3.4, first, under condition (2.3), the system matrix of the linearization of dynamic system (2.0)-(2.2) around the unique positive equilibrium x^* is calculated:

$$A := f'(x^*) = \begin{bmatrix} -\alpha_0 x_1^* & -\alpha_0 x_0^* + \beta_1 m_1 & \beta_2 m_2 \\ k_1 \alpha_0 x_1^* & 0 & -\alpha_1 x_1^* \\ 0 & k_2 \alpha_1 x_2^* & 0 \end{bmatrix}.$$

It is supposed that from the state vector x, only the time-varying quantity x_0 of the resource of system (2.0)-(2.2) is observed, i.e. the observation function is

$$h(x) := x_0 - x_0^*. \qquad (3.6)$$

(For technical reason, instead of x_0, its deviation from its equilibrium value x_0^* is supposed to be observed.) Hence the linearization of the observation function is

$$C := h'(x^*) = (1, 0, 0).$$

For the construction the local observer for the considered observation system, a matrix $H = col(h_1, h_2, h_3)$ is needed such that matrix *A-HC* is Hurwitz, i.e., all its eigenvalues have negative real parts. According to the Hurwitz criterion (see e.g. Chen et al. (2004)), in terms of the normed characteristic polynomial of *A-HC*, the following necessary and sufficient condition holds:

$$p(\lambda) = \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 \ \textit{is Hurwitz} \Leftrightarrow$$
$$a_0, a_1, a_2 > 0 \quad \textit{and} \quad a_2 \cdot a_1 > a_0. \qquad (3.7)$$

For the case when the resource supply is high enough, this matrix H can be determined from the following theorem:

Theorem 3.5. Suppose that inequality $Q > \dfrac{m_1 m_2}{\alpha_1 k_1 k_2}$ holds and matrix

$$H = \begin{pmatrix} h_1 \\ 0 \\ 1 \end{pmatrix},$$

is chosen such that $h_1 > \max\{ \dfrac{m_1 x_1^*}{m_2 x_2^*}, \dfrac{\alpha_0 x_0^*}{\beta_2 k_2} \}$. Then dynamic system defined by

$$\dot{z} = f(z) + H[y - h(z)]$$

is a local exponential observer for system (2.0)-(2.2) with the observation h defined in (3.6).

Proof. It is sufficient to show that under the conditions of the theorem, x^* is a Lyapunov stable equilibrium of system (3.1), and matrix *A-HC* is Hurwitz. Then the proof can be concluded applying Theorem 3.4.

First, from $Q > \frac{m_1 m_2}{\alpha_1 k_1 k_2}$ inequality $Q > Q_2$ also follows, which on the one hand, as quoted at the end of Section 2, implies the existence of a unique positive equilibrium. On the other hand, in [13], [14] it was proved, that both in open systems (with $\beta_1 = 0,\ \beta_2 = 0$) and in partially or totally closed systems (at least one of inequalities $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$ holds) condition $Q > Q_2$ also implies (asymptotic) stability of the equilibrium.

From (2.0)-(2.2) the coordinates of the positive equilibrium x^* are

$$x_0^* = \frac{-\alpha_1 Q - \frac{\beta_1 m_1 m_2}{k_2} + \beta_2 m_1 m_2}{-\frac{\alpha_0 m_2}{k_2} + \alpha_0 \beta_2 k_1 m_2},$$

$$x_1^* = \frac{m_2}{k_2 \alpha_1},$$

$$x_2^* = \frac{-\alpha_0 k_1 Q + -\frac{\alpha_0 m_1 m_2}{k_2 \alpha_1} - \frac{\beta_1 k_1 \alpha_0 m_1 m_2}{k_2 \alpha_1}}{-\frac{\alpha_0 m_2}{k_2} + \alpha_0 \beta_2 k_1 m_2}.$$

Now it will be proved that for the coefficients of the normed characteristic polynomial of *A-HC* conditions (3.7) hold. To cut short the rather tedious calculations, the following statements can be checked: Hypothesis $Q > \frac{m_1 m_2}{\alpha_1 k_1 k_2}$ and $k_1, k_2 \in]0,1[;\ \beta_1, \beta_2 \in [0,1[$ implies $Q > \frac{m_1 m_2}{\alpha_1}$ and also $\alpha_0 x_0^* - \beta_1 m_1 > 0$, and the latter is sufficient for $a_1 > 0$ and also used in the proof of $a_2 \cdot a_1 - a_0 > 0$. On the other hand,

$$h_1 > \frac{m_1 x_1^*}{m_2 x_2^*} \Rightarrow \alpha_1 h_1 k_2 x_2^* - \beta_1 m_1 > 0,$$

to be used in the proof of $a_0 > 0$;

$$h_1 > \frac{\alpha_0 x_0^*}{\beta_2 k_2} \Rightarrow \beta_2 m_2 h_1 - \alpha_0 \alpha_1 x_0^* x_1^* > 0 \Rightarrow$$

$$a_2 \cdot a_1 - a_0 > 0.$$

From $x_1^* = \frac{m_2}{k_2 \alpha_1}$ and $k_1, k_2 \in]0,1[;\ \beta_2 \in [0,1[$ in inequality $\alpha_1 k_2 x_1^* x_2^* - \beta_2 k_1 k_2 m_2 x_2^* > 0$ can be derived, which implies $a_0 > 0$. Finally, inequalities $h_1, a_0, x_1^* > 0$ directly imply $a_2 > 0$. Summing up, all inequalities conditions (3.7) hold for $p(\lambda)$. Therefore matrix *A-HC* is Hurwitz, and the application of Theorem 3.4 concludes the proof.

Example 3.6. For a numerical example, the following set of parameters is considered:

$$Q := 10;\ \alpha_0 := 0.3;\ \alpha_1 := 0.1;\ \beta_1 := 0.2;$$
$$\beta_2 := 0.3;\ m_1 := 0.1;\ m_2 := 0.4;\ k_1 := 0.5;\ k_2 := 0.5.$$

In this case the system (2.0)-(2.2) has a positive equilibrium $x^* = (4.52, 8, 5.78)$, and with matrix

$$H = \begin{pmatrix} 2.8 \\ 0 \\ 1 \end{pmatrix},$$

conditions of Theorem 3.5 are satisfied. Therefore, the following must be an observer system:

$$\begin{aligned} \dot{z}_0 &= 10 - 0.3 z_0 z_1 + 0.2 \cdot 0.1 z_1 + 0.3 \cdot 0.4 z_2 + 2.8[y - (z_0 - x_0^*)] \\ \dot{z}_1 &= z_1(-0.1 + 0.5 \cdot 0.3 z_0 - 0.1 z_2) \\ \dot{z}_2 &= z_2(-0.4 + 0.5 \cdot 0.1 z_1) + 1[y - (z_0 - x_0^*)]. \end{aligned} \tag{3.8}$$

Indeed, let $x(0) := (3, 7, 2)$ be the initial value for system (2.0)-(2.2), near the equilibrium, and similarly, $z(0) := (1.3, 7.9, 1)$ another nearby initial value for the observer system (3.8). Figure 1 shows that the corresponding solution z tends to the solution x of the original system, providing a rather good asymptotic estimate of the state process.

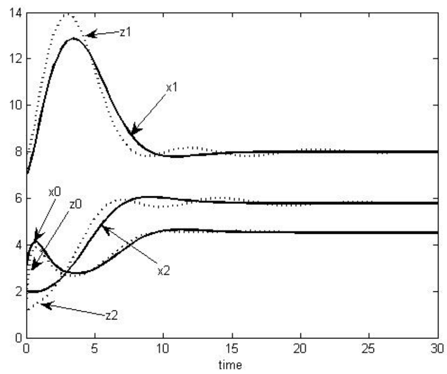

Fig. 1 Solutions of systems (2.0)-(2.2) and (3.8)

IV. DISCUSSION

In the paper the construction of an observer system was applied for the state monitoring of a simple trophic chain of the type *resource – producer – primary consumer*, recovering the whole state process from the only observation of the resource. The applied methodology can also be extended to more complex models of food webs, involving the observation of certain abiotic environmental components and/or certain indicator species. A similar approach may be also useful for the monitoring of population systems in changing environment. (In [7] only observability of such ecological systems was discussed.)

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