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Numerical Treatment of Block Method for the Solution of Ordinary Differential Equations

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Abstract**—**Discrete linear multistep block method of uniform order for the solution of first order initial value problems (IVPs) in ordinary differential equations (ODEs) is presented in this paper. The approach of interpolation and collocation approximation are adopted in the derivation of the method which is then applied to first order ordinary differential equations with associated initial conditions. The continuous hybrid formulations enable us to differentiate and evaluate at some grids and off – grid points to obtain four discrete schemes, which were used in block form for parallel or sequential solutions of the problems. Furthermore, a stability analysis and efficiency of the block method are tested on ordinary differential equations, and the results obtained compared favorably with the exact solution.

Keywords**—**Block Method, First Order Ordinary Differential Equations, Hybrid, Self starting.

I. Introduction

LET us consider the numerical solution of the first order ordinary differential equation of the form

$$y' = f(x,y),\ \ a \le x \le b \tag{1}$$

with associated initial or boundary conditions. The mathematical models of most physical phenomena especially in mechanical systems without dissipation leads to initial value problem of type (1). Solutions to initial value problem of type (1) according to Fatunla [1], [2] are often highly oscillatory in nature and thus, severely restrict the mesh size of the conventional linear multistep method. Such system often occurs in mechanical systems without dissipation, satellite tracking, and celestial mechanics.

Phenomena in many disciplines are modeled by first-order differential equations such as in mechanical system, electrical circuits, population models, Newton's law of cooling, compartmental analysis, Garity [3]

Lambert [4] and several authors such as Onumanyi et al. [5], Agbeboh et al. [6], and Sunday & Odekunle [7], have written on conventional linear multistep method:

$$\sum_{j=0}^{k} \propto_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j}\,, k \ge 2 \tag{2}$$

or compactly in the form

$$\rho(E)y_n = h\delta(E)f_n \tag{3}$$

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where E is the shift operator specified by $E^j y_n = y_{n+j}$ while ρ and δ are characteristics polynomials and are given as

$$\rho(\xi) = \sum_{j=0}^{k} \propto_j \xi^j,\ \ \delta(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \tag{4}$$

y_n is the numerical approximation to the theoretical solution y(x) and $f_n = f(x_n, y_n)$.

In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, i.e. obtaining the most accuracy per unit of computational effort, that can be secured with the group of methods proposed in this paper over Agbeboh et al. [6], and Sunday & Odekunle [7].

A. Definition: Consistent, Lambert [4]

The linear multistep method (2) is said to be consistent if it has order $p \ge 1$, that is if

$$\sum_{j=0}^{k} \propto_j = 0 \text{ and } \sum_{j=0}^{k} j \propto_j - \sum_{j=0}^{k} \beta_j = 0 \tag{5}$$

Introducing the first and second characteristics polynomials (4), we have from (5) LMM type (2) is consistent if

$$\rho(1) = 0,\ \rho^1(1) = \delta(1)$$

B. Definition: Zero Stability, Lambert [4]

A linear multistep method type (2) is zero stable provided the roots $\xi_j, j = 0(1)k$ of first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = \det\left|\sum_{j=0}^{k} A(i)\xi^{(k-i)}\right| = 0$ satisfies $|\xi_j| \le 1$ and for those roots with $|\xi_j| = 1$ the multiciplicity must not exceed two. The principal root of $\rho(\xi)$ is denoted by $\xi_1 = \xi_2 = 1$.

C. Definition: Convergence, Lambert [4]

The necessary and sufficient conditions for the linear multistep method type (2) is said to be convergent if it is consistent and zero stable.

D. Definition: Order and Error Constant, Lambert [4]

The linear multistep method type (2) is said to be of order p if $c_0 = c_1 = c_2 \dots c_P = 0$ but $c_{p+1} \ne 0$ and c_{p+1} is called the error constant, where

$$c_0 = \sum_{j=0}^{k} \propto_j = \propto_0 + \propto_1 + \propto_2 + \ \dots\ + \propto_k$$
$$c_1 = \sum_{j=0}^{k} j \propto_j = (\propto_1 + 2\propto_2 + 3\propto_3 + \ \dots\ + k\propto_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$
$$c_2 = \sum_{j=0}^{k} \frac{1}{2!} j^2 \propto_j - \sum_{j=0}^{k} \beta_j$$

$$= \begin{Bmatrix} \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^3\alpha_3 + \dots + k^2\alpha_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \dots + k\beta_k) \end{Bmatrix}$$

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$$c_q = \sum_{j=1}^{k}\{\frac{1}{q!}j^q\alpha_j - \frac{1}{(q-2)!}j^{q-2}\beta_j\}$$

$$= \begin{Bmatrix} \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \dots + k^q\alpha_k) \\ -\frac{1}{(q-1)!}(\beta_1 + 2^{(q-1)}\beta_2 + 3^{(q-1)}\beta_3 + \dots + k^{(q-1)}\beta_k) \end{Bmatrix} \quad (6)$$

E. Theorem: Lambert, [4]

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $\{(x, y) : a \le x \le b, -\infty < y < \infty\}$ where *a and b* finite, and let there exist a constant L such that for every x, y, y^* such that (x, y) *and* (x, y^*) are both in D:

$$|f(x, y) - f(x, y^*)| \le L|y - y^*| \quad (7)$$

Then if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1), where $y(x)$ is continuous and differentiable for all (x, y) *in D*. The inequality (7) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

II. Derivation of the Proposed Method

We proposed an approximate solution to (1) in the form

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^i = y_{n+j}, \quad (8)$$
$$i = 0(1)m + t - 1$$

$$y'(x) = \sum_{j=0}^{t+m-1} i(i-1)a_j x^{i-2} = f_{n+j}, \quad (9)$$
$$i = 2(3)m + t - 1$$

with m = 1, t = 4 and p = m+t-1 where the $a_j, j = 0,1,(m + t - 1)$ are the parameters to be determined, t and m are points of interpolation and collocation respectively where P, is the degree of the polynomial interpolant of our choice.

Specifically, we collocate (9) at $x = x_{n+j}, j = 0(1)k$ and interpolate (8) at $x = x_{n+j}$,

$j = 0(1)k - 2$ using the method described above.

The matrix D of the proposed method is expressed as:

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_n + h & (x_n + h)^2 & (x_n + h)^3 & (x_n + h)^4 \\ 1 & x_n + 2h & (x_n + 2h)^2 & (x_n + 2h)^3 & (x_n + 2h)^4 \\ 1 & x_n + \frac{1}{2}h & (x_n + \frac{1}{2}h)^2 & (x_n + \frac{1}{2}h)^3 & (x_n + \frac{1}{2}h)^4 \\ 0 & 1 & 2x_n + 6h & 3(x_n + 3h)^2 & 4(x_n + 3h)^3 \end{bmatrix} \quad (10)$$

Matrix D in (10), which when solved either by matrix inversion techniques, or Gaussian elimination method to obtain the values of parameters α_j's, j = 0, 1, … which is substituted in (8) yields, after some algebraic manipulation, the new continuous form of Block Hybrid Backward Differentiation Formulae for the solution

$\bar{y}(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j}$ is obtain as:

$$\bar{y}(x) = \Big\{\Big(\frac{1}{134}\Big(\frac{134h^4+507x_nh^3+602x_n^2h^2+267x_n^3h+38x_n^4}{h^4}\Big) - \frac{1}{134}\Big(\frac{507h^3+1204x_nh^2+801x_n^2h+152x_n^3}{h^4}\Big)x + \frac{1}{134}\Big(\frac{602h^2+801x_nh+228x_n^2}{h^4}\Big)x^2 - \frac{1}{134}\Big(\frac{267h+152x_n}{h^4}\Big)x^3 + \frac{19}{67h^4}x^4\Big)y_n + \Big(\frac{1}{67}\Big(\frac{x_n(186h^3+517x_nh^2+316x_n^2h+52x^3)}{h^4}\Big) - \frac{2}{67}\frac{93h^3+517x_nh^2+474x_n^2h+104x_n^3}{h^4}x + \frac{1}{67}\frac{517h^2+948x_nh+312x_n^2}{h^4}x^2 - \frac{4}{67}\frac{79h+52x_n}{h^4}x^3 + \frac{52}{67h^4}x^4\Big)y_{n+1} + \Big(-\frac{1}{402}\frac{x_n(460x_nh^2+393x_n^2h+74x_n^3+141h^3)}{h^4} + \frac{1}{402}\frac{920x_nh^2+1179x_n^2h+296x_n^3+141h^3}{h^4}x - \frac{1}{402}\frac{406h^2+1179x_nh+444x_n^2}{h^4}x^2 + \frac{1}{402}\frac{393h+296x_n}{h^4}x^3 - \frac{37}{201}x^4\Big)y_{n+2} + \Big(-\frac{16}{201}\frac{x_n(11x_n^3+72x_n^2h+139x_nh^2+78h^3)}{h^4} + \frac{32}{201}\frac{22x_n^3+108x_n^2h+139x_nh^2+39h^3}{h^4}x - \frac{16}{201}\frac{66x_n^2+216x_nh+139h^2}{h^4}x^2 + \frac{11x_n+18h}{h^4}x^3 - \frac{176}{201h^4}x^4\Big)y_{n+\frac{1}{2}} + \Big(\frac{1}{67}\frac{x_n(2x_n^3+7x_n^2h+7x_nh^2+2h^3}{h^3} - \frac{1}{67}\frac{8x_n^3+21x_n^2h+14x_nh^2+2h^3}{h^3}x + \frac{1}{67}\frac{12x_n^2+21x_nh+7h^2}{h^3}x^2 - \frac{1}{67}\frac{8x_n+7h}{h^3}x^3 + \frac{2}{67h^3}x^4\Big)f_{n+3}\Big\} \quad (11)$$

Evaluating (11) at $x = x_{n+3}$ and its first derivative at $x = x_{n+1}$, $x = x_{n+2}$, $x = x_{n+\frac{1}{2}}$ yield the following four discrete hybrid schemes which are used as a block integrator;

$$\text{(a)}\ \frac{225}{67}y_{n+1} - \frac{150}{67}y_{n+2} + y_{n+3} - \frac{192}{67}y_{n+\frac{1}{2}} + \frac{50}{67}y_n = \frac{30}{67}hf_{n+3}$$

$$\text{(b)}\ y_{n+1} + \frac{52}{324}y_{n+2} - \frac{448}{324}y_{n+\frac{1}{2}} + \frac{72}{324}y_n = \frac{1}{324}h\{201f_{n+1} + 3f_{n+3}\}$$

$$\text{(c)}\ \frac{1476}{649}y_{n+1} - y_{n+2} - \frac{1088}{649}y_{n+\frac{1}{2}} + \frac{261}{649}y_n = \frac{1}{649}h\{-402f_{n+2} + 36f_{n+3}\}$$

$$\text{(d)}\ -\frac{2880}{1600}y_{n+1} + \frac{245}{1600}y_{n+2} + y_{n+\frac{1}{2}} + \frac{1035}{1600}y_n = \frac{1}{1600}h\Big\{-1608f_{n+\frac{1}{2}} + 18f_{n+3}\Big\} \quad (12)$$

Equations (12) constitute the member of a zero stable block integrators of order $(4,4,4,4)^T$ with $c_5 = \Big(-\frac{15}{268}, \frac{41}{4020}, -\frac{97}{2680}, -\frac{79}{8576}\Big)$. The application of the block integrators with n = 0 gives the accurate values as shown in Tables I and II of forth section of this paper. To start the IVP integration on the sub interval $[X_0, X_3]$, we use (12) when n = 0 to produce simultaneously values for y_1, y_2, y_3 and $y_{\frac{1}{2}}$ without recourse to any predictor – corrector method to provide y_1 and y_2 in the main method. Hence, this is an improvement over other cited works.

III. Stability Analysis

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus, several definitions, which call for the method to posses some "adequate" region of absolute stability, can be found in several literatures. See Lambert [4], Fatunla [1], [2] etc.

The four discrete schemes proposed in this report in (12) are put using Fatula's approach [ ]

i.e. $A^{(0)}Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h\sum_{i=0}^{k} B^{(i)} F_{m-i}$ (13)

where h is a fixed mesh size within a block, $A^i, B^i, i = 0(1)k$ are r by r matrix coefficients, $A^{(0)}$ is r by r identity matrix, Y_m, Y_{m-i} and F_{m-i} are vectors of numerical estimates described by

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+r} \end{bmatrix}, \quad Y_{m-i} = \begin{bmatrix} y_{n-r} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+1} \\ y_n \end{bmatrix}, \quad F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ \cdot \\ \cdot \\ \cdot \\ f_{n+r} \end{bmatrix}, \quad F_{m-i} = \begin{bmatrix} f_{n-r} \\ \cdot \\ \cdot \\ \cdot \\ f_{n+1} \\ f_n \end{bmatrix}$$

For n = mr and for some integer $m \geq 0$
This give rise to:

$$\begin{bmatrix} -\frac{192}{67} & \frac{225}{67} & -\frac{150}{67} & 1 \\ \frac{448}{324} & -1 & -\frac{52}{324} & 0 \\ -\frac{1088}{649} & \frac{1476}{649} & -1 & 0 \\ 1 & -\frac{2880}{1600} & \frac{245}{1600} & 0 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{50}{67} \\ 0 & 0 & 0 & \frac{72}{324} \\ 0 & 0 & 0 & -\frac{261}{649} \\ 0 & 0 & 0 & -\frac{1035}{1600} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \left\{ \begin{bmatrix} 0 & 0 & 0 & \frac{30}{67} \\ 0 & \frac{201}{324} & 0 & -\frac{3}{324} \\ 0 & 0 & -\frac{402}{649} & \frac{36}{649} \\ -\frac{1608}{1600} & 0 & 0 & \frac{18}{1600} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} \right\} \quad (14)$$

with $y_0 = \begin{pmatrix} y_0 \\ hz_0 \end{pmatrix}$ usually giving along the initial value problem.

Equation (14) is the 1- block 4 point method. The first characteristics polynomial of the proposed 1- block 4 – point method is given by

$$\rho(\lambda) = \det[\lambda I - A_1^{(1)}] \quad (15)$$

$$\rho(\lambda) = \det \begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 1 & \lambda - 1 \end{bmatrix} \quad (16)$$

Solving the determinant of (16), yields $\rho(\lambda) = \lambda^3(\lambda - 1)$, which implies, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ or $\lambda_4 = 1$

By definition of zero stable and (16), the hybrid method is zero stable and is also consistent as its order $(4,4,4,4)^T > 1$, thus, it is convergent following Henrici [8] and Fatunla [2].

IV. Implementation of the Method

This section deals with numerical experiments by considering the derived discrete schemes in block form for solution of differential equations of first order initial value problems. The idea is to enable us see how the proposed methods performs when compared with exact solutions. The results are summarized in Tables I & II.

A. Numerical Experiment

From Agbeboh et al. [6]; consider the IVP $y' = -y$, $y(0) = 1, x \in [0,1]$, $h = 0.1$, whose exact solution is $y = e^{-x}$

TABLE I
Results for the Proposed Method

x	Exact Solution	Approximate Solution	Error of Proposed Method	Error of Agbeboh et al. [6] Method
0.1	0.9048374190	0.9048374190	0.0000000000E+00	0.2369581120E-07
0.2	0.8187307548	0.8187307547	1.0000000827E-10	0.4288171396E-07
0.3	0.7408182230	0.7408182186	4.4000000310E-09	0.5820146953E-07
0.4	0.6703200488	0.6703200387	1.0099999947E-08	0.7021715764E-07
0.5	0.6065306629	0.6065306501	1.2800000060E-08	0.7941889080E-07
0.6	0.5488116395	0.5488116201	1.9400000051E-08	0.8623342206E-07
0.7	0.4965853074	0.4965852654	4.2000000033E-08	0.9103176596E-07
0.8	0.4493289679	0.4493288879	8.0000000013E-08	0.9413594187E-07
0.9	0.4065696636	0.4065695729	9.0700000011E-08	0.9582493909E-07
1.0	0.3678794450	0.3678793518	9.3199999995E-08	0.9633999071E-07

B. Numerical Experiment

From Sunday and Odekunle [7];

Consider the differential equation of growth model of the form of $y' = \alpha y$, $y(0) = 1000$, $t \in [0,1]$, $h = 0.1$ (*)

Equation (*) represents the rate of growth of bacteria in a colony. We shall assume that the model grows continuously and without restriction. One may ask how many bacterial are in the colony after some hours if an individual produces an average of 0.2 offspring every hour?

We assume that y(t) is the population size at time t. This therefore implies that equation (*) may be written as: $y' = 0.2y$, $y(0) = 1000$, $t \in [0,1]$, $h = 0.1$

TABLE II
RESULTS FOR THE PROPOSED METHOD

x	Exact Solution	Approximate Soln.	Error of Proposed Method	Error of Sunday and Odekunle [7] Method
0.0	1000.000000	1000.000000	0.00000000E+00	0.00000000E+00
0.1	1020.201340	1020.201340	0.00000000E+00	0.00000000E+00
0.2	1040.810774	1040.810774	0.00000000E+00	0.00000000E+00
0.3	1061.836547	1061.836547	0.00000000E+00	0.00000000E+00
0.4	1083.287068	1083.287068	0.00000000E+00	0.00000000E+00
0.5	1105.170918	1105.170918	0.00000000E+00	0.00000000E+00
0.6	1127.496852	1127.496852	0.00000000E+00	0.00012207E-04
0.7	1150.273799	1150.273798	1.000000000E-06	0.00012207E-04
0.8	1173.510871	1173.510823	4.800000000E-05	0.00012207E-04
0.9	1197.217363	1197.217312	5.100000000E-05	0.00024414E-04
1.0	1221.402758	1221.402692	6.600000000E-05	0.00024414E-04

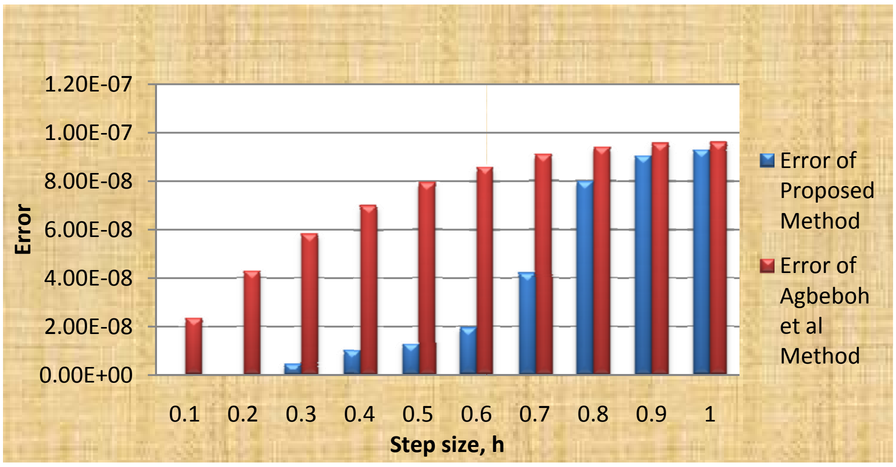

Fig. 1 Comparative Error Analysis of Table I

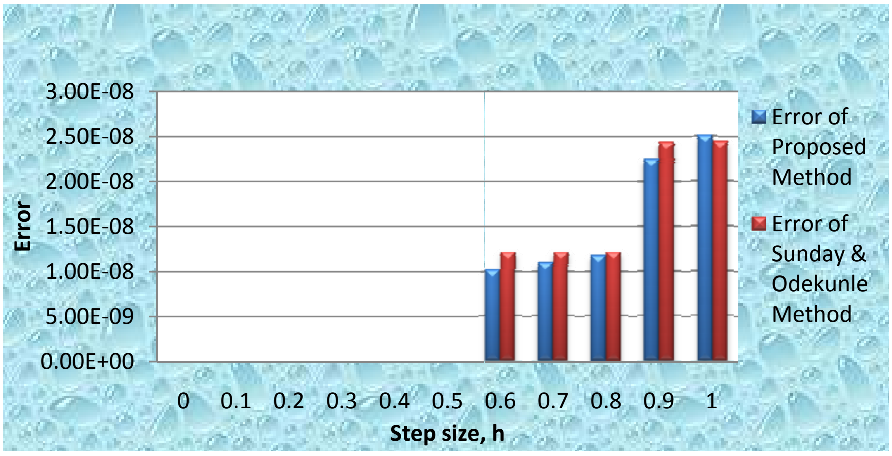

Fig. 2 Comparative Error Analysis of Table II

V. CONCLUSION

In this paper, a new block method with uniform order was developed. The resultant numerical integrator posses the following desirable properties:

- Being self – starting as such it eliminate the use of predictor – corrector method
- Convergent schemes
- Facility to generate solutions at 4 points simultaneously
- Produce solution over sub intervals that do not overlaps.
- Zero stability

In addition, the new schemes compares favorably with the theoretical solution and the results are more accurate than the result of Agbeboh et al. [6], and Sunday & Odekunle [7], see

Tables I and II respectively. Hence, this work is an improvement over other cited works.

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