Numerical study of some coupled PDEs by using differential transformation method

Reza Abazari, Rasool Abazari

Abstract—In this paper, the two-dimension differential transformation method (DTM) is employed to obtain the closed form solutions of the three famous coupled partial differential equation with physical interest namely, the coupled Korteweg-de Vries(KdV) equations, the coupled Burgers equations and coupled nonlinear Schrödinger equation. We begin by showing that how the differential transformation method applies to a linear and non-linear part of any PDEs and apply on these coupled PDEs to illustrate the sufficiency of the method for this kind of nonlinear differential equations. The results obtained are in good agreement with the exact solution. These results show that the technique introduced here is accurate and easy to apply.

Keywords—Coupled Korteweg-de Vries(KdV) equation; Coupled Burgers equation; Coupled Schrödinger equation; Differential transformation method.

I. INTRODUCTION

NONLINEAR coupled partial differential equations (cPDEs) such as the nonlinear coupled Korteweg-de Vries (KdV) equation, coupled Burger's equation and coupled nonlinear Schrödinger equation arise in a large number of mathematical and engineering problems. These include solid state physics, fluid mechanics, chemical physics, plasma physics, optic, etc.(see [1], [2], [3] and the references therein). The coupled KdV equations, introduced by HirotaSatsuma [4] is an important class of nonlinear equations with many applications in physical sciences. Coupled KdV equations describe an interaction of the two long waves with different dispersion relation, while the Burger's equations describe phenomena such as a mathematical model of turbulence [5]. The coupled nonlinear Schrödinger equation [3] represents propagation of pulses with equal mean frequencies in birefringent nonlinear fiber.

Recently many authors have studied the numerical and approximate solution of the nonlinear coupled PDEs by using various techniques. Some of them are: the MQ quasi-interpolation method [6], the local discontinuous Galerkin method [7], the adomian decomposition method [8], the Hes variational iteration method [9], the homogeneous balance method [10], the trigonometric function transform method [11], the F-expansion transform method [12], the Chebyshev spectral collocation(ChSc) method [13] and the homotophy perturbation method [14].

On the other hand, in recent years, the differential transform method(DTM) is a semi–numerical-analytic-technique that formalizes the Taylor series in a totally different manner.

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The DTM was first introduced by J.K. Zhou in a study about electrical circuits [15]. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylors series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. With this technique, the given partial differential equation and related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear ordinary and partial differential equations. There is no need for linearization or perturbations, large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. It is possible to solve system of differential equations [16], differentialalgebraic equations [17], difference equations [18], differential difference equations [19], partial differential equations [2], [3], [5], [20], fractional differential equations [21], pantograph equations [22], onedimensional Volterra integral and integro-differential equations [23] and matrix differential equations [24] by using this method.

The purpose of this paper is to employ the differential transformation method(DTM)to solve the following classes of PDEs:

Class A: Coupled KdV equations:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \alpha \frac{\partial^3 u}{\partial x^3} + 6\alpha u \frac{\partial u}{\partial x} - 2\gamma\, v\, \frac{\partial v}{\partial x} = 0, \\
&\frac{\partial v}{\partial t} + \beta \frac{\partial^3 v}{\partial x^3} + 3\beta\, u\, \frac{\partial v}{\partial x} = 0,
\end{aligned} \tag{1}$$

where α, β, and γ are real parameters.

Class B: Coupled Burgers equations:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} + \alpha \frac{\partial(uv)}{\partial x} = 0, \\
&\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + \beta \frac{\partial(uv)}{\partial x} = 0,
\end{aligned} \tag{2}$$

where α, β, are real parameters.

Class C: Coupled Schrödinger equation:

$$\begin{aligned}
&i\left(\frac{\partial \Phi}{\partial t} + \eta \frac{\partial \Phi}{\partial x}\right) + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2} + \left(|\Phi|^2 + e|\Psi|^2\right)\Phi = 0, \\
&i\left(\frac{\partial \Psi}{\partial t} - \eta \frac{\partial \Psi}{\partial x}\right) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} + \left(|\Psi|^2 + e|\Phi|^2\right)\Psi = 0.
\end{aligned} \tag{3}$$

where α is a real parameters.

Rest of the paper is organized as follows: In Section II, the differential transform method is produced. Section III is devoted to the numerical tests of the method on the problems related to the coupled KdV, coupled Burgers equations and Coupled Schrödinger equation. In Section IV, the results are concluded.

II. BASIC DEFINITIONS

With reference to the articles [15]–[24], the basic definitions of differential transformation are introduced as follows:

A. One-dimensional differential transform

The transformation of the k-th derivative of a function in one variable is as follows:

Definition 2.1: If $u(t) \in R$ can be expressed as a Taylor series about fixed point t_0, then $u(t)$ can be represented as

$$u(t) = \sum_{k=0}^{\infty} \frac{u^{(k)}(t_0)}{k!}(t - t_0)^k. \tag{4}$$

If $u_n(t)$ is the n-partial sums of a Taylor series Eq. (4), then

$$u_n(t) = \sum_{k=0}^{n} \frac{u^{(k)}(t_0)}{k!}(t - t_0)^k + R_n(t). \tag{5}$$

where $u_n(t)$ is called the n-th Taylor polynomial for $u(t)$ about t_0 and $R_n(t)$ is remainder term.

If $U(k)$ is defined as

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}, \tag{6}$$

where $k = 0, 1, .., \infty$ then Eq. (4) reduce to

$$u(t) = \sum_{k=0}^{\infty} U(k)(t - t_0)^k. \tag{7}$$

and the n-partial sums of a Taylor series Eq. (5) reduce to

$$u_n(t) = \sum_{k=0}^{n} U(k)(t - t_0)^k + R_n(t). \tag{8}$$

The $U(k)$ defined in Eq. (8), is called the differential transform of function $u(t)$.

For simplicity assume that $t_0 = 0$, then the Eq. (8) reduce to

$$u_n(t) = \sum_{k=0}^{n} U(k)t^k + R_n(t). \tag{9}$$

From the above definitions, it can be found that the concept of the one-dimensional differential transform is derived from the Taylor series expansion.

Remark 2.1: In this paper, the symbol $\star$ is used to denote the differential transform version of multiplication.

The relationships (6)–(9) give us the following theorems.

Theorem 2.1: Assume that $W(k)$, $U(k)$ and $V(k)$, are the differential transforms of the functions $w(t)$, $u(t)$ and $v(t)$, respectively, then

(i) If $w(t) = u(t) \pm v(t)$, then $W(k) = U(k) \pm V(k)$.
(ii) If $w(t) = \lambda u(t)$, then $W(k) = \lambda U(k)$.
(iii) If $w(t) = \frac{d^m u(t)}{dt^m}$, then $W(k) = \frac{(k+m)!}{k!}U(k+m)$.
(iv) If $w(t) = u(t)v(t)$, then

$$W(k) = U(k) \star V(k) = \sum_{l=0}^{k} U(l)V(k - l).$$

(v) If $w(x) = x^m$ then

$$W(k) = \delta(k - m) = \begin{cases} 1 & k = m, \\ 0 & \text{otherwise} \end{cases}$$

(vi) If $w(t) = \exp(\lambda t)$, then $W(k) = \frac{\lambda^k}{k!}$.
(vii) If $w(t) = \sin(\alpha t + \beta)$, then $W(k) = \frac{\alpha^k}{k!}\sin(\frac{k\pi}{2} + \beta)$.
(viii) If $w(t) = \cos(\alpha t + \beta)$, then $W(k) = \frac{\alpha^k}{k!}\cos(\frac{k\pi}{2} + \beta)$.

Proof: See ([21], [22], [23], and their references). ∎

B. Two-dimensional differential transform

Consider a function of two variables $w(x, t)$, and suppose that it can be represented as a product of two single-variable function, i.e., $w(x, t) = f(x)g(t)$. On the basis of the properties of the one-dimensional differential transform, the function $w(x, t)$ can be represented as

$$w(x, t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i, j)x^i t^j \tag{10}$$

where $W(i, j) = F(i)G(j)$ is called the spectrum of $w(x, t)$.

The basic definitions and operations for two-dimensional differential transform are introduced as follows:

Definition 2.2: If $w(x, t)$ is analytic and continuously differentiable with respect to time t in the domain of interest, then

$$W(k, h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}w(x, t)\right]_{\substack{x=x_0 \\ t=t_0}}, \tag{11}$$

where the spectrum function $W(k, h)$ is the transformed function, which is also called T-function in brief.

In this paper, (lower case) $w(x, t)$ represents the original function while (upper case) $W(k, h)$ stands for the transformed function (T-function).

The differential inverse transform of $W(k, h)$ is defined as:

$$w(x, t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k, h)(x - x_0)^k(t - t_0)^h. \tag{12}$$

Combining Eq. (11) and Eq. (12), it can be obtained that

$$w(x, t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}w(x, t)\right]_{\substack{x=x_0 \\ t=t_0}}(x - x_0)^k(t - t_0)^h.$$

When (x_0, t_0) are taken as $(0, 0)$, then Eq. (12) can be expressed as

$$w(x, t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k, h)x^k t^h. \tag{13}$$

In real applications, the function $w(x,t)$ is represented by a finite series of Eq. (13) can be written as

$$w(x,t) = \sum_{k=0}^{n}\sum_{h=0}^{m} W(k,h)x^k t^h + R_{nm}(x,t), \qquad (14)$$

and Eq. (13) implies that

$$R_{nm}(x,t) = \sum_{k=n+1}^{\infty}\sum_{h=m+1}^{\infty} W(k,h)x^k t^h,$$

is negligibly small. Usually, the values of n and m are decided by convergency of the series coefficients.

From the above definitions, it can be found that the concept of the two-dimensional differential transform is derived from the two-dimensional Taylor series expansion. With Eq. (11) and Eq. (12), the fundamental mathematical operations performed using the two-dimensional differential transform be readily obtained and these are listed in Theorem 2.2. (See [2], [3], [5], [20]).

Theorem 2.2: Assume that $W(k,h)$, $U(k,h)$ and $V(k,h)$, are the differential transforms of the functions $w(x,t)$, $u(x,t)$ and $v(x,t)$, respectively, then

(i) If $w(x,t) = u(x,t) \pm v(x,t)$, then

$$W(k,h) = U(k,h) \pm V(k,h).$$

(ii) If $w(x,t) = \lambda u(x,t)$, then $W(k,h) = \lambda U(k,h)$.

(iii) If $w(x,t) = \frac{\partial^{r+s}}{\partial x^r \partial t^s} u(x,t)$, then

$$W(k,h) = \frac{(k+r)!(h+s)!}{k!h!} U(k+r,h+s).$$

(iv) If $w(x,t) = u(x,t)v(x,t)$, then

$$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s).$$

(v) If $w(x,t) = x^m t^n$ then

$$W(k,h) = \delta(k-m,h-n) = \begin{cases} 1 & k=m, h=n \\ 0 & \text{otherwise} \end{cases}$$

(vi) If $w(x,t) = \frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial t}v(x,t)$, then

$$W(k,h) = \frac{\partial}{\partial x}u(x,t) \star \frac{\partial}{\partial t}v(x,t)$$
$$= \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(h-s+1)U(k-r+1,s)V(r,h-s+1).$$

Proof: See ([2], [3], [5], [20], and their references). ∎

III. APPLICATIONS

This section is devoted to computational results. We applied the method presented in this paper and solved the three famous coupled partial differential differential equation with physical interest namely, the coupled Korteweg-de Vries(KdV) equations, the coupled Burgers equations and coupled nonlinear Schrödinger equation. In these examples, we first obtain a recurrence systems for the differential transform of nonlinear equation and solve it by programming in MATLAB environment. These examples are chosen such that there exist exact solutions for them.

A. Coupled KdV equations

Consider the coupled KdV equations,

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial^3 u}{\partial x^3} + 6\alpha u\frac{\partial u}{\partial x} - 2\gamma v\frac{\partial v}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} + \beta\frac{\partial^3 v}{\partial x^3} + 3\beta u\frac{\partial v}{\partial x} = 0, \qquad (15)$$

with the initial conditions

$$u(x,0) = f(x), \qquad v(x,0) = g(x),$$

using the operations of Theorem 2.2, we get the differential transform of Eq. (15) as follow

$$(h+1)U(k,h+1) + \alpha\frac{(k+3)!}{k!}U(k+3,h)$$
$$+ 6\alpha\, u \star u_x|_{\substack{x=k\\t=h}} - 2\gamma\, v \star v_x|_{\substack{x=k\\t=h}} = 0,$$
$$(h+1)V(k,h+1) + \beta\frac{(k+3)!}{k!}V(k+3,h)$$
$$+ 3\beta u \star v_x|_{\substack{x=k\\t=h}} = 0, \qquad (16)$$

where $U(k,h)$, and $V(k,h)$ are the differential transformations of $u(x,t)$, and $v(x,t)$ respectively. suppose that $x_0 = t_0 = 0$, in Definition 2.2, then from initial conditions, we have

$$\sum_{k=0}^{\infty} U(k,0)x^k = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k,$$
$$\sum_{k=0}^{\infty} V(k,0)x^k = \sum_{k=0}^{\infty}\frac{g^{(k)}(0)}{k!}x^k. \qquad (17)$$

By recurrence Eq.(16), we obtain

$$U(k,h+1) = \frac{1}{(h+1)}\Big\{ -\alpha\frac{(k+3)!}{k!}U(k+3,h)$$
$$- 6\alpha\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s)$$
$$+ 2\gamma\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)V(r,h-s)V(k-r+1,s)\Big\}, \quad (18)$$
$$V(k,h+1) = \frac{1}{(h+1)}\Big\{ -\beta\frac{(k+3)!}{k!}V(k+3,h)$$
$$- 3\beta\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)V(k-r+1,s)\Big\}.$$

Example 3.1: Consider the nonlinear coupled KdV equations (15), with $\gamma = 3$, and $\alpha = \beta$ [4], [10], [11], [14]

$$\frac{\partial u}{\partial t} + \alpha\frac{\partial^3 u}{\partial x^3} + 6\alpha u\frac{\partial u}{\partial x} - 6 v\frac{\partial v}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} + \alpha\frac{\partial^3 v}{\partial x^3} + 3\alpha u\frac{\partial v}{\partial x} = 0, \qquad (19)$$

and the initial conditions

$$\begin{cases} u(x,0) = \frac{\lambda}{\alpha}\text{sech}^2\Big[\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}\,x\Big], \\ v(x,0) = \frac{\sqrt{2}}{2}\sqrt{\frac{1}{\alpha}}\lambda\text{sech}^2\Big[\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}\,x\Big], \end{cases} \qquad (20)$$

where α and λ, being arbitrary constants.

According to relationship (17) and subject to initial condition (20), we get

$$
\begin{cases}
\sum_{\ell=0}^{k} \left(\left(\frac{1}{\ell!} + \frac{(-1)^{\ell}}{\ell!} \right) \left(\frac{\lambda}{\alpha} \right)^{\frac{\ell}{2}} + 2\delta(\ell) \right) U(k-\ell, 0) \\
\qquad - \frac{4\lambda}{\alpha} \delta(k) = 0, \\
\sum_{\ell=0}^{k} \left(\left(\frac{1}{\ell!} + \frac{(-1)^{\ell}}{\ell!} \right) \left(\frac{\lambda}{\alpha} \right)^{\frac{\ell}{2}} + 2\delta(\ell) \right) V(k-\ell, 0) \\
\qquad - 2\lambda \sqrt{\frac{2}{\alpha}} \delta(k) = 0,
\end{cases}
\tag{21}
$$

using the differential transformation operations (18), we get

$U(0,1) = -6\alpha U(3,0) - 6\alpha U(0,0)U(1,0) + 6V(0,0)V(1,0) = 0,$

$V(0,1) = -6\alpha V(3,0) - 3\alpha U(0,0)V(1,0) = 0,$

$U(1,1) = -24\alpha U(4,0) - 12\alpha U(0,0)U(2,0) - 6\alpha U(1,0)^2$
$\qquad + 12V(0,0)V(2,0) + 6V(1,0)^2 = \frac{\lambda^3}{2\alpha^2},$

$V(1,1) = -24\alpha V(4,0) - 6\alpha U(0,0)V(2,0) - 3\alpha U(1,0)V(1,0)$
$\qquad = \frac{\sqrt{2}\sqrt{\frac{1}{\alpha}}\lambda^3}{4\alpha},$

$U(2,1) = -60\alpha U(5,0) - 18\alpha U(0,0)U(3,0) - 18\alpha U(1,0)U(2,0)$
$\qquad + 18V(0,0)V(3,0) + 18V(1,0)V(2,0) = 0,$

$V(2,1) = -60\alpha V(5,0) - 9\alpha U(0,0)V(3,0) - 6\alpha U(1,0)V(2,0)$
$\qquad - 3\alpha U(2,0)V(1,0) = 0,$

$U(3,1) = -120\alpha U(6,0) - 24\alpha U(0,0)U(4,0) - 24\alpha U(1,0)U(3,0)$
$\qquad - 12\alpha U(2,0)^2 + 24V(0,0)V(4,0) + 24V(1,0)V(3,0)$
$\qquad + 12V(2,0)^2 = -\frac{\lambda^4}{6\alpha^3},$

$V(3,1) = -120\alpha V(6,0) - 12\alpha U(0,0)V(4,0) - 9\alpha U(1,0)V(3,0)$
$\qquad - 6\alpha U(2,0)V(2,0) - 3\alpha U(3,0)V(1,0) = -\frac{\sqrt{2}\sqrt{\frac{1}{\alpha}}\lambda^4}{12\alpha^2},$

In the same manner, the rest of components were obtained using the MAPLE Package.

According to the inverse differential transform method (12), we can conclude that

$$
u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h \approx \frac{\lambda}{\alpha} - \frac{\lambda^2}{4\alpha^2}x^2 + \frac{\lambda^3}{2\alpha^2}xt - \frac{\lambda^4}{4\alpha^2}t^2
$$
$$
+ \frac{\lambda^3}{24\alpha^3}x^4 - \frac{\lambda^4}{6\alpha^3}x^3 t + \frac{\lambda^5}{4\alpha^3}x^2 t^2 - \frac{\lambda^6}{6\alpha^3}xt^3 + \frac{\lambda^7}{24\alpha^3}t^4 + ...,
$$
$$
v(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V(k,h)x^k t^h \approx \frac{\sqrt{2}}{2}\sqrt{\frac{1}{\alpha}}\lambda - \frac{\sqrt{2}\sqrt{\frac{1}{\alpha}}\lambda^2}{8\alpha}x^2
$$
$$
+ \frac{\sqrt{2}\sqrt{\frac{1}{\alpha}}\lambda^3}{4\alpha}xt - \frac{\sqrt{2}\sqrt{\frac{1}{\alpha}}\lambda^4}{8\alpha}t^2 + \frac{\sqrt{2}\sqrt{\frac{1}{\alpha}}\lambda^3}{48\alpha^2}x^4 + ...,
$$

which is the same as the Taylors expansion of the exact solutions

$$
u(x,t) = \frac{\lambda}{\alpha}\text{sech}^2\left[\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}(x - \lambda t)\right],
$$
$$
v(x,t) = \frac{\sqrt{2}}{2}\sqrt{\frac{1}{\alpha}}\lambda\text{sech}^2\left[\frac{1}{2}\sqrt{\frac{\lambda}{\alpha}}(x - \lambda t)\right].
$$

B. Coupled Burgers equations

Consider the (1+1)–coupled Burgers equations,

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} + \alpha\frac{\partial(uv)}{\partial x} = 0, \\
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + \beta\frac{\partial(uv)}{\partial x} = 0,
\end{aligned}
\tag{22}
$$
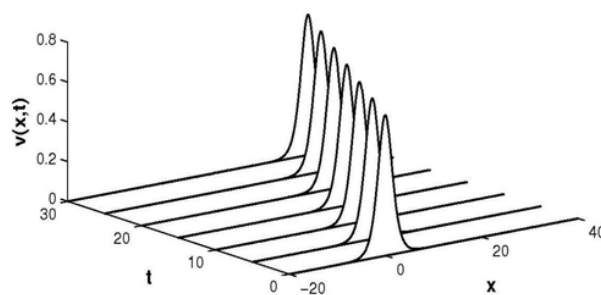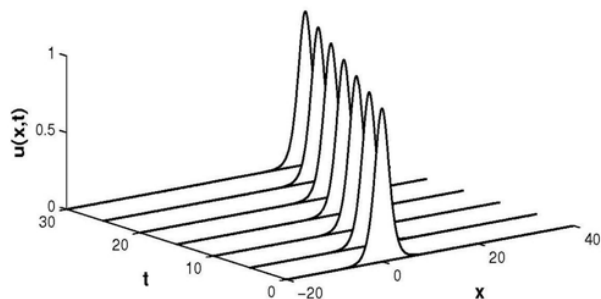

Fig. 1. The exact solutions of the amplitude $u(x,t)$ and $v(x,t)$, versus the coordinate x,t of example. 3.1 with $\lambda = 1$, and $\alpha = 1$.

with the initial conditions

$$
u(x,0) = f(x), \qquad v(x,0) = g(x),
$$

using the operations of Theorem 2.2, we get the differential transform of Eq. (22) as follow

$$
\begin{aligned}
(h+1)U(k,h+1) - \frac{(k+2)!}{k!}U(k+2,h) - 2u \star \frac{\partial u}{\partial x}\big|_{\substack{x=k \\ t=h}} \\
+ \alpha v \star \frac{\partial u}{\partial x}\big|_{\substack{x=k \\ t=h}} + \alpha u \star \frac{\partial v}{\partial x}\big|_{\substack{x=k \\ t=h}} = 0, \\
(h+1)V(k,h+1) - \frac{(k+2)!}{k!}V(k+2,h) - 2v \star \frac{\partial v}{\partial x}\big|_{\substack{x=k \\ t=h}} \\
+ \beta v \star \frac{\partial u}{\partial x}\big|_{\substack{x=k \\ t=h}} + \beta u \star \frac{\partial v}{\partial x}\big|_{\substack{x=k \\ t=h}} = 0,
\end{aligned}
\tag{23}
$$

where $U(k,h)$, and $V(k,h)$ are the differential transformations of $u(x,t)$, and $v(x,t)$ respectively. suppose that $x_0 = t_0 = 0$, in Definition 2.2, then from initial conditions, we have

$$
\begin{aligned}
\sum_{k=0}^{\infty} U(k,0)x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k, \\
\sum_{k=0}^{\infty} V(k,0)x^k = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}x^k.
\end{aligned}
\tag{24}
$$

By recurrence Eq.(33), we get

$$U(k, h+1) = \frac{1}{(h+1)}\left\{ \frac{(k+2)!}{k!}U(k+2, h) \right.$$
$$+ 2\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r, h-s)U(k-r+1, s)$$
$$- \alpha\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)V(r, h-s)U(k-r+1, s)$$
$$\left. - \alpha\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r, h-s)V(k-r+1, s) \right\},$$
$$V(k, h+1) = \frac{1}{(h+1)}\left\{ \frac{(k+2)!}{k!}V(k+2, h) \right. \tag{25}$$
$$+ 2\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)V(r, h-s)V(k-r+1, s)$$
$$- \beta\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)V(r, h-s)U(k-r+1, s)$$
$$\left. - \beta\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r, h-s)V(k-r+1, s) \right\}.$$

Example 3.2: Consider the (1+1)–coupled Burgers equations [5], [9],

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} + \frac{5}{2}\frac{\partial(uv)}{\partial x} = 0,$$
$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + \frac{5}{2}\frac{\partial(uv)}{\partial x} = 0, \tag{26}$$

with the initial conditions

$$u(x, 0) = v(x, 0) = \lambda\left(1 - \tanh(\frac{3}{2}\lambda x)\right), \tag{27}$$

where λ is an arbitrary constant.

By using relationship (24) and initial conditions (27), we get

$$\sum_{k=0}^{\infty}U(k, 0)x^k = \sum_{k=0}^{\infty}V(k, 0)x^k = \lambda - \frac{3\lambda^2}{2}x + \frac{9\lambda^4}{8}x^3$$
$$- \frac{81\lambda^6}{80}x^5 + \frac{4131\lambda^8}{4480}x^7 - \frac{7533\lambda^{10}}{8960}x^9 + \dots . \tag{28}$$

then from recurrence Eq. (25), we get

$$U(0, 1) = V(0, 1) = \left\{2U(2, 0) + 2U(0, 0)U(1, 0) - \frac{5}{2}V(0, 0)U(1, 0)\right.$$
$$\left. - \frac{5}{2}U(0, 0)V(1, 0)\right\} = \frac{9}{2}\lambda^3,$$
$$U(1, 1) = V(1, 1) = \left\{6U(3, 0) + 4U(0, 0)U(2, 0) + 2U(1, 0)^2 \right.$$
$$\left. - 5V(0, 0)U(2, 0) - 5V(1, 0)U(1, 0) - 5U(0, 0)V(2, 0)\right\} = 0,$$
$$U(2, 1) = V(2, 1) = \left\{12U(4, 0) + 6U(0, 0)U(3, 0) + 6U(1, 0)U(2, 0) \right.$$
$$- \frac{15}{2}V(0, 0)U(3, 0) - \frac{15}{2}V(1, 0)U(2, 0) - \frac{15}{2}V(2, 0)U(1, 0)$$
$$\left. - \frac{15}{2}U(0, 0)V(3, 0)\right\} = -\frac{81}{8}\lambda^5,$$
$$U(3, 1) = V(3, 1) = \left\{20U(5, 0) + 8U(0, 0)U(4, 0) + 8U(1, 0)U(3, 0) \right.$$
$$+ 4U(2, 0)^2 - 10V(0, 0)U(4, 0) - 10V(1, 0)U(3, 0)$$
$$\left. - 10V(2, 0)U(2, 0) - 10V(3, 0)U(1, 0) - 10U(0, 0)V(4, 0)\right\} = 0,$$
$$U(4, 1) = V(4, 1) = \left\{30U(6, 0) + 10U(0, 0)U(5, 0) + 10U(1, 0)U(4, 0) \right.$$
$$+ 10U(2, 0)U(3, 0) - \frac{25}{2}V(0, 0)U(5, 0) - \frac{25}{2}V(1, 0)U(4, 0)$$
$$- \frac{25}{2}V(2, 0)U(3, 0) - \frac{25}{2}V(3, 0)U(2, 0) - \frac{25}{2}V(4, 0)U(1, 0)$$
$$\left. - \frac{25}{2}U(0, 0)V(5, 0)\right\} = \frac{243}{16}\lambda^7,$$

In the same manner, the rest of components can be obtained using the recurrence relation (25).

Substituted the obtained quantities in inverse differential transform Eq. (12), the approximation solution in a series form of Example 3.2 is:

$$u(x, t) = v(x, t) \simeq \lambda - \frac{3\lambda^2}{2}x + \frac{9\lambda^3}{2}t + \frac{9\lambda^4}{8}x^3 - \frac{81\lambda^5}{8}x^2 t$$
$$+ \frac{243\lambda^6}{8}x t^2 - \frac{243\lambda^7}{8}t^3 - \frac{81\lambda^6}{80}x^5 + \dots,$$

which is the same as the Taylors expansion of the exact solutions

$$u(x, t) = v(x, t) = \lambda\left[1 - \tanh\left(\frac{3}{2}\lambda(x - 3\lambda t)\right)\right].$$

and is exactly the same as the results obtained by VIM [9].

C. Coupled Schrödinger equation

Now, consider the Coupled Schrödinger equation (2),

$$i\left(\frac{\partial\Phi}{\partial t} + \eta\frac{\partial\Phi}{\partial x}\right) + \frac{1}{2}\frac{\partial^2\Phi}{\partial x^2} + \left(|\Phi|^2 + e|\Psi|^2\right)\Phi = 0,$$
$$i\left(\frac{\partial\Psi}{\partial t} - \eta\frac{\partial\Psi}{\partial x}\right) + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + \left(|\Psi|^2 + e|\Phi|^2\right)\Psi = 0. \tag{29}$$

subject to initial conditions

$$\Phi(x, 0) = \varphi(x), \qquad \Psi(x, 0) = \psi(x), \tag{30}$$

where $\phi(x)$, and $\varphi(x)$, are complex functions.

For our numerical work, we decompose the complex functions Φ and Ψ into their real and imaginary parts by writing [2], [3]

$$\Phi(x, t) = u_1(x, t) + iv_1(x, t),$$
$$\Psi(x, t) = u_2(x, t) + iv_2(x, t), \tag{31}$$

where u_j, $(j = 1, 2)$ are real functions. Therefore the coupled equation given in Eq. (29), can be written in a matrix-vector form as

$$\frac{\partial\theta}{\partial t} + \eta A\frac{\partial\theta}{\partial x} + \frac{1}{2}B\frac{\partial^2\theta}{\partial x^2} + F(\theta)\theta = 0, \tag{32}$$

where

$$\theta = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}, F(\theta) = \begin{pmatrix} 0 & \mathcal{Z}_1 & 0 & 0 \\ -\mathcal{Z}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_2 \\ 0 & 0 & -\mathcal{Z}_2 & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and

$$\begin{aligned} \mathcal{Z}_1 &= (u_1^2 + v_1^2) + e(u_2^2 + v_2^2), \\ \mathcal{Z}_2 &= (u_2^2 + v_2^2) + e(u_1^2 + v_1^2), \end{aligned} \tag{33}$$

Then the differential transform of system given in Eq. (32), will be

$$\begin{aligned} (h+1)\Theta(k, h+1) &+ \eta(k+1)A\Theta(k+1, h) \\ &+ \frac{1}{2}\frac{(k+2)!}{k!}B\Theta(k+2, h) \\ &+ \sum_{r=0}^{k}\sum_{s=0}^{h} F(\Theta(r, h-s))\Theta(k-r, s) = 0, \end{aligned} \tag{34}$$

where $\Theta = [U_1, V_1, U_2, V_2]^T$, and $F(\Theta)$ are the differential transform of $\theta = [u_1, v_1, u_2, v_2]^T$, and $F(\theta)$, respectively.

Eq. (34), can be rewritten as follows:

$$\begin{cases} (h+1)U_1(k, h+1) + \eta(k+1)U_1(k+1, h) \\ \quad + \frac{1}{2}\frac{(k+2)!}{k!}V_1(k+2, h) + \mathcal{Z}_1 \star v_1|_{\substack{x=k \\ t=h}} = 0, \\ (h+1)V_1(k, h+1) + \eta(k+1)V_1(k+1, h) \\ \quad - \frac{1}{2}\frac{(k+2)!}{k!}U_1(k+2, h) - \mathcal{Z}_1 \star u_1|_{\substack{x=k \\ t=h}} = 0, \\ (h+1)U_2(k, h+1) - \eta(k+1)U_2(k+1, h) \\ \quad + \frac{1}{2}\frac{(k+2)!}{k!}V_2(k+2, h) + \mathcal{Z}_2 \star v_2|_{\substack{x=k \\ t=h}} = 0, \\ (h+1)V_2(k, h+1) - \eta(k+1)V_2(k+1, h) \\ \quad - \frac{1}{2}\frac{(k+2)!}{k!}U_2(k+2, h) - \mathcal{Z}_2 \star u_2|_{\substack{x=k \\ t=h}} = 0. \end{cases} \tag{35}$$

where $U_j(k, h)$ and $V_j(k, h)$, are the differential transformation of $u_j(x, t)$ and $v_j(x, t)$, respectively for $j = 1, 2$.

In order to obtain the unknowns of $U_j(k, h)$, $V_j(k, h)$, $k, h = 0, 1, 2, ..., (j = 1, 2)$ we must construct and solve the above equations, and substitute in Eq. (14) to obtain the series form of exact solutions.

Example 3.3: Consider the coupled Schrödinger equations (29), when $e = \frac{2}{3}$, $\alpha = 1$, $\nu = 1$, and $\eta = \frac{1}{2}$: [2], [3]

$$\begin{aligned} i\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\frac{\partial \Phi}{\partial x}\right) + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2} + \left(|\Phi|^2 + \frac{2}{3}|\Psi|^2\right)\Phi = 0, \\ i\left(\frac{\partial \Psi}{\partial t} - \frac{1}{2}\frac{\partial \Psi}{\partial x}\right) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2} + \left(|\Psi|^2 + \frac{2}{3}|\Phi|^2\right)\Psi = 0. \end{aligned} \tag{36}$$

subject to the initial conditions

$$\begin{aligned} \Phi(x, 0) &= \frac{1}{5}\sqrt{30}\operatorname{sech}(\sqrt{2}\,x)\exp\left(\frac{i}{2}x\right), \\ \Psi(x, 0) &= \frac{1}{5}\sqrt{30}\operatorname{sech}(\sqrt{2}\,x)\exp\left(\frac{3i}{2}x\right). \end{aligned} \tag{37}$$

From the initial conditions (37), we get

$$\begin{aligned} u_1(x, 0) &= \frac{1}{5}\sqrt{30}\operatorname{sech}(\sqrt{2}\,x)\cos\left(\frac{1}{2}x\right), \\ v_1(x, 0) &= \frac{1}{5}\sqrt{30}\operatorname{sech}(\sqrt{2}\,x)\sin\left(\frac{1}{2}x\right), \\ u_2(x, 0) &= \frac{1}{5}\sqrt{30}\operatorname{sech}(\sqrt{2}\,x)\cos\left(\frac{3}{2}x\right), \\ v_2(x, 0) &= \frac{1}{5}\sqrt{30}\operatorname{sech}(\sqrt{2}\,x)\sin\left(\frac{3}{2}x\right). \end{aligned} \tag{38}$$

The differentia transform version of initial conditions (38), can be obtained from following recurrence equations

$$\begin{aligned} \sum_{\ell=0}^{k}\left(\frac{1}{\ell!} + \frac{(-1)^\ell}{\ell!}\right)2^{\frac{\ell}{2}-1}U_1(k-\ell, 0) &= \frac{\sqrt{30}}{5}\frac{(\frac{1}{2})^k}{k!}\cos\left(\frac{k\pi}{2}\right), \\ \sum_{\ell=0}^{k}\left(\frac{1}{\ell!} + \frac{(-1)^\ell}{\ell!}\right)2^{\frac{\ell}{2}-1}V_1(k-\ell, 0) &= \frac{\sqrt{30}}{5}\frac{(\frac{1}{2})^k}{k!}\sin\left(\frac{k\pi}{2}\right), \\ \sum_{\ell=0}^{k}\left(\frac{1}{\ell!} + \frac{(-1)^\ell}{\ell!}\right)2^{\frac{\ell}{2}-1}U_2(k-\ell, 0) &= \frac{\sqrt{30}}{5}\frac{(\frac{3}{2})^k}{k!}\cos\left(\frac{k\pi}{2}\right), \\ \sum_{\ell=0}^{k}\left(\frac{1}{\ell!} + \frac{(-1)^\ell}{\ell!}\right)2^{\frac{\ell}{2}-1}V_2(k-\ell, 0) &= \frac{\sqrt{30}}{5}\frac{(\frac{3}{2})^k}{k!}\sin\left(\frac{k\pi}{2}\right), \end{aligned} \tag{39}$$

For $n, m \leq 4$, using differential transform version of initial conditions recurrence equations (36) with the recurrence equations (35), gives all values of $U_1(:, :), V_1(:, :), U_2(:, :)$, and $V_2(:, :)$, which implies
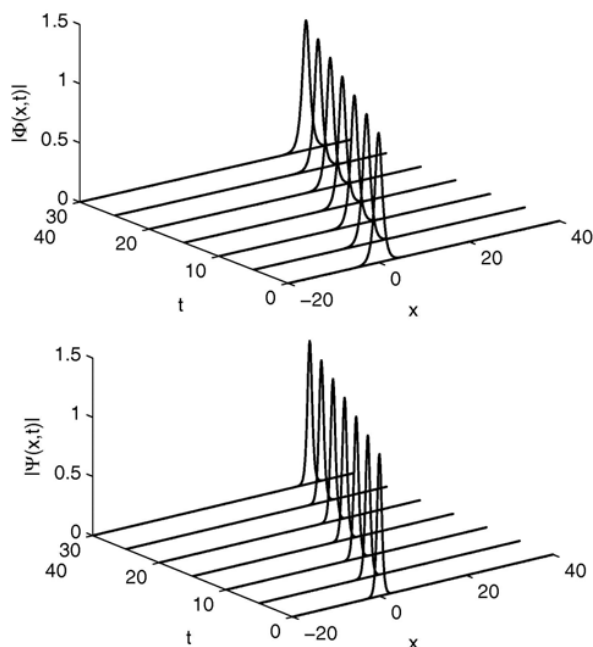

Fig. 2. The modulus of the amplitude $|\Phi|$ and $|\Psi|$, versus the coordinate x, t of example. 3.3 with $\nu = 1$, $\alpha = 1$, $\eta = \frac{1}{2}$, and $e = \frac{2}{3}$.

$$\sum_{k=0}^{4}\sum_{h=0}^{4}U_1(k,h)x^k\,t^h = \sqrt{30}\Big(\frac{1}{5}-\frac{9}{40}x^2+\frac{27}{80}tx-\frac{153}{640}t^2$$
$$+\frac{123}{640}x^4-\frac{417}{640}tx^3+\frac{4833}{5120}t^2x^2-\frac{1389}{2048}t^3x+\frac{6783}{32768}t^4$$
$$-\frac{32487}{16384}t^2x^4+\frac{214227}{81920}t^3x^3-\frac{2672979}{1310720}t^4x^2\Big),$$

$$\sum_{k=0}^{4}\sum_{h=0}^{4}V_1(k,h)x^k\,t^h = \sqrt{30}\Big(\frac{1}{10}x+\frac{1}{8}t-\frac{5}{48}x^3+\frac{19}{320}tx^2$$
$$+\frac{167}{1280}t^2x-\frac{409}{3072}t^3-\frac{3403}{15360}tx^4+\frac{2353}{30720}t^2x^3$$
$$+\frac{6817}{24576}t^3x^2-\frac{64771}{196608}t^4x\Big),$$

(40)

$$\sum_{k=0}^{4}\sum_{h=0}^{4}U_2(k,h)x^k\,t^h = \sqrt{30}\Big(\frac{1}{5}-\frac{17}{40}x^2-\frac{13}{80}tx-\frac{353}{640}t^2$$
$$+\frac{833}{1920}x^4-\frac{659}{1920}tx^3+\frac{4337}{5120}t^2x^2-\frac{2935}{6144}t^3x+\frac{61069}{98304}t^4$$
$$-\frac{81901}{49152}t^2x^4+\frac{1435763}{737280}t^3x^3-\frac{8095697}{3932160}t^4x^2\Big),$$

$$\sum_{k=0}^{4}\sum_{h=0}^{4}V_2(k,h)x^k\,t^h = \sqrt{30}\Big(\frac{3}{10}x+\frac{3}{8}t-\frac{33}{80}x^3-\frac{63}{320}tx^2$$
$$-\frac{99}{1280}t^2x-\frac{609}{1024}t^3-\frac{2107}{5120}tx^4+\frac{2497}{10240}t^2x^3$$
$$+\frac{5809}{8192}t^3x^2-\frac{49651}{65536}t^4x\Big),$$

The closed form of solutions (36) are

$$\Phi(x,y) = u_1(x,t) + i\,v_1(x,t)$$
$$= \frac{1}{5}\sqrt{30}\,\operatorname{sech}\big(\sqrt{2}\,(x-t)\big)\exp\big(\frac{i}{2}(x+\frac{5}{4}t)\big),$$
$$\Psi(x,y) = u_2(x,t) + i\,v_2(x,t)$$
$$= \frac{1}{5}\sqrt{30}\,\operatorname{sech}\big(\sqrt{2}\,(x-t)\big)\exp\big(\frac{3i}{2}(x+\frac{5}{4}t)\big).$$

which is the same as the Taylors expansion of the exact solutions.

IV. CONCLUSION

In this paper, we have shown that the differential transformation method can be used successfully for solving the three famous coupled partial differential differential equation with physical interest namely, the coupled Korteweg-de Vries(KdV) equations, the coupled Burgers equations and coupled nonlinear Schrödinger equation. This method is simple and easy to use and solves the problem without any need for discretizing the variables. Therefore, this method can be applied to many complicated linear and non-linear PDEs and system of PDEs and does not require linearization, discretization or perturbation.
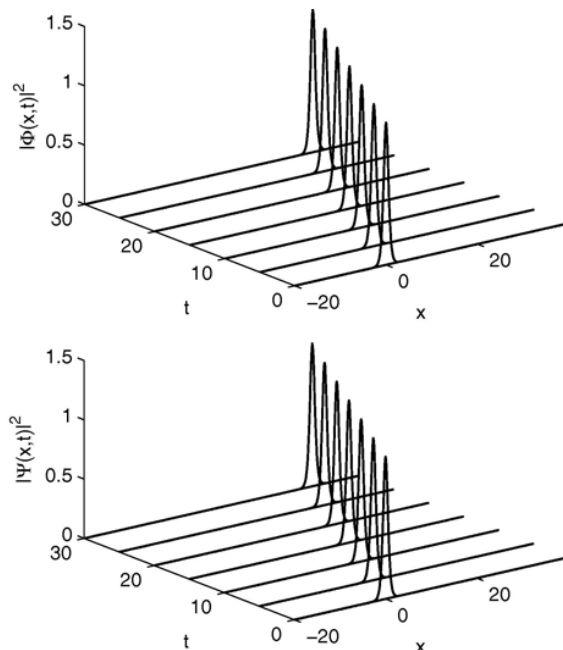
Fig. 3. Square of the modulus of the amplitude $|\Phi|^2$ and $|\Psi^2|$, versus the coordinate x, t of example. 3.3 with $\nu = 1$, $\alpha = 1$, $\eta = \frac{1}{2}$, and $e = \frac{2}{3}$.

REFERENCES

[1] A. Borhanifar, Reza Abazari, *An unconditionally stable parallel difference scheme for Telegraph equation*, Math. Prob. Eng, vol. 2009, Article ID 969610, 17pages, 2009. http://doi:10.1155/2009/969610.

[2] A. Borhanifar, Reza Abazari, *Exact solutions for non-linear Schrödinger equations by differential transformation method*, J. Appl. Math. Comput, http://doi:10.1007/s12190-009-0338-2.

[3] A. Borhanifar, Reza Abazari, *Numerical study of nonlinear Schrödinger and coupled Schrödinger equations by differential transformation method*, Opt. Commun, http://doi:10.1016/j.optcom.2010.01.046.

[4] Hirota. R, Satsuma. J, *Soliton solutions of a coupled Kortewege-de Vries equation*, Phys. Lett. A, 85 (1981), 407-408.

[5] Reza Abazari, A. Borhanifar, *Numerical study of Burgers' and coupled Burgers' equations by differential transformation method*, Comput. Math. Appl, http://doi:10.1016/j.camwa.2010.01.039.

[6] Chen. R, Wu. Z, *Solving partial differential equation by using multi-quadric quasi-interpolation*, Appl.Math.Comput, 186 (2007), 1502-1510.

[7] Xu. Y, Shu. C-W, *Local discontinuous Galerkin methods for the Kuramoto- Sivashinsky equations and the Ito-type coupled equations*, Comput. Methods. Appl. Mech, 195 (2006), 3430-3447.

[8] Kaya. D, Inan. EI, *Exact and numerical traveling wave solutions for nonlinear coupled equations using symbolic computation*, Appl. Math. Comput, 151 (2004), 775-787.

[9] J. Biazar, H. Aminikhah, *Exact and numerical solutions for non–linear Burgers' equation by VIM*, Math. Comput. Modelling, 49 (2009) 1394–1400.

[10] Zayed EME, Zedan HA, Gepreel KA, *On the solitary wave solutions for nonlinear Hirota Satsuma coupled-KdV of equations*, Chaos. Solitons. Fractals, 22 (2004), 285-303.

[11] Caom DB, Yan JR, Zang Y, *Exact solutions for a new coupled MKdV equations and a coupled KdV equations*, J. Phys. Lett. A, 297 (2002), 68-74.

[12] Zhang JL, Wang ML, Feng ZD, *The improved F-expansion method and its applications*, Phys. Lett. A, 350 (2006), 103-109.

[13] Khater AH, Temsah RS, Hassan MM, *A Chebyshev spectral collocation method for solving Burgers type equations*, J. Appl. Math. Comput, http://doi:10.1016/j.cam.2007.11.007.

[14] Ganji DD, Rafei M, *Solitary wave equations for a generalized Hirotasat-suma coupled-KdV equation by homotopy perturbation method*, Phys. Lett. A, 356 (2006), 131-137.

[15] J.K. Zhou, Differential Transformation and its Application for Electrical Circuits, Huazhong University Press, Wuhan, China, 1986.

[16] M. Mossa Al-sawalha, M.S.M. Noorani , A numeric–analytic method for approximating the chaotic Chen system, Chaos. Soliton. Fract. 42 (2009) 1784-1791.

[17] F. Ayaz, Application of differential transform method to differential–algebraic equations, Appl. Math. Comput. 152 (2004) 649–657.

[18] A. Arikoglu, I. Ozkol, Solution of difference equations by using differential transform method, Appl. Math. Comput. 174 (2006) 1216–1228.

[19] A. Arikoglu, I. Ozkol, Solution of differential-difference equations by using differential transform method, Appl. Math. Comput. 181 (2006) 153–162.

[20] Figen Kangalgil, Fatma Ayaz, Solitary wave solutions for the KdV and mKdV equations by differential transform method, Chaos. Soliton. Fract. 41 (2009) 464–472.

[21] A. Arikoglu, I. Ozkol, Solution of fractional differential equations by using differential transform method, Chaos. Soliton. Fract. 34 (2007) 1473–1481.

[22] Nemat Abazari, Reza Abazari, Solution of nonlinear second-order pantograph equations via differential transform method, International Conference on Computer and Applied Mathematics, October 28-30, 2009, Venice, Italy (Available at http://www.waset.org/journals/waset/v58).

[23] A. Arikoglu, I. Ozkol, Solution of boundary value problems for integro–differential equations by using differential transform method, Appl. Math. Comput. 168 (2005) 1145–1158.

[24] Reza Abazari, Solution of Riccati types matrix differential equations using matrix differential transform method, J. Appl. Math. & Informatics. 27 (2009), pp. 1133–1143.

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