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Numerical Solving of General Fuzzy Linear Systems by Huang's Method

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***Abstract*—**In this paper the Huang's method for solving a $m\times n$ fuzzy linear system when, $m\leq n$, is considered. The method in detail is discussed and illustrated by solving some numerical examples.

***Keywords*—**Fuzzy number; fuzzy linear systems; Huang's method.

I. Introduction

SYSTEMS of linear equations are important for studying and solving a large proportion of the problems in many topics in applied mathematics. Usually in many applications some of the parameters in our problems are represented by fuzzy number rather than crisp, and hence it is important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them.

Friedman et al.[1] proposed a general model for solving an $n\times n$ fuzzy linear system, whose coefficients matrix is crisp and right-hand side column is an arbitrary fuzzy number vector, by the embedding approach. Asady et al.[2], who merely discuss the full row rank system, use the same method to solve the $m\times n$ fuzzy linear system for $m\leq n$ .Zheng and wang[3,4] discuss the solution of the general $m\times n$ consistent and inconsistent fuzzy linear system.

In this paper, we investigate the $m\times n$ fuzzy linear system whose coefficients matrix is crisp and right-hand side column is a fuzzy number vector.

We first replace the original $m\times n$ fuzzy linear system by a $(2m)\times(2n)$ crisp function linear system. And then the fuzzy least squares solutions to the system are discussed by using the Huang's method.

In section II, we introduce some basic definitions and results on fuzzy linear systems. In section III, we use Huang's method for solving a $m\times n$ fuzzy linear system for $m\leq n$ Numerical examples are given section IV, and conclusions in section V.

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II. Preliminaries

We represent an arbitrary fuzzy number in parametric from [5,6] by an ordered pair of functions $(\underline{u}(r),\overline{u}(r)), 0\leq r\leq 1$, which satisfy the following requirements:

1) $\underline{u}(r)$ is a left-continuous non-decreasing function over[0,1].
2) $\overline{u}(r)$ is a left-continuous non-increasing function over [0,1].
3) $\underline{u}(r)\leq \overline{u}(r) \quad ,0\leq r\leq 1$.

A crisp number α is simply represented by $\underline{u}(r)=\overline{u}(r)$, $0\leq r\leq 1$.

By appropriate definitions the fuzzy numbers space $\{\underline{u}(r),\ \overline{u}(r)\}$ becomes a convex cone, E^1.Using the extension principle [7], the addition and the scalar multiplication of fuzzy numbers are defined by

$$(u+v)(x)=\sup_{x=s+t}\min\{u(s),v(t)\}$$
$$(ku)(x)=u(x/k);\quad k\neq 0$$

For $u,v\in E^1.\ k\in\Re$. Equivalently, for arbitrary $u=(\underline{u},\overline{u}), v=(\underline{v},\overline{v})$ and $k\in\Re$, we may define the addition and the scalar multiplication as

$$\underline{(u+v)}(r)=\underline{u}(r)+\underline{v}(r)$$
$$\overline{(u,v)}(r)=\overline{u}(r)+\overline{v}(r)$$
$$(\underline{ku})(r)=(k\underline{u})(r),\qquad (\overline{ku})(r)=k\overline{u}(r),\ k\geq 0,$$
$$(\underline{ku})(r)=(k\overline{u})(r)\ \text{(r)},\qquad (\overline{ku})(r)=k\underline{u}(r),\ k<0$$

A. Definition

The $m\times n$ linear system

$$\begin{cases} a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=y_1\\ a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=y_2\\ \quad\quad .\\ \quad\quad .\\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=y_m \end{cases} \tag{1}$$

where the coefficient matrix $A = (a_{ij})$ is a crisp $m \times n$ matrix and $y_i \quad i = 1,2,3,...,m$ are fuzzy numbers , is called a fuzzy linear system (FLS).

Let $x_j = (\underline{x}_j(r), \overline{x}_j(r)), j = 1,2,....,n$ and $y_i = (\underline{y}_i(r), \overline{y}_i(r)), i = 1,2,3,...,m$ are fuzzy numbers. Then FLS (1) can be represented in the form of the following function linear system :

$$\begin{cases} \underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y}_i, \end{cases} \quad i = 1,2,3,...,m \tag{2}$$

In particular, if $a_{ij} \geq 0, 1 \leq j \leq n$ for some i, then

$$\sum_{j=1}^{n} a_{ij} \underline{x}_j = \underline{y}_i, \qquad \sum_{j=1}^{n} a_{ij} \overline{x}_j = \overline{y}_i, \tag{3}$$

B. Definition

A fuzzy number vector ($x_1, x_2, ..., x_n$) given by

$$x_i = (\underline{x}_i(r), \overline{x}_i(r)), \qquad 1 \leq i \leq n, \qquad 0 \leq r \leq 1$$

is called a solution of the fuzzy system if it satisfies (2).

To solve (1), following [1], we may assume a $(2m) \times (2n)$ matrix $s = (s_{ij})$ is determined as follows:

$$\begin{aligned} &a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij}, \qquad s_{m+i,n+j} = a_{ij}, \\ &a_{ij} < 0 \Rightarrow s_{i,n+j} = -a_{ij}, \qquad s_{m+ij} = -a_{ij}, \end{aligned} \tag{4}$$

and any s_{ij} which is not determined by (4) is zero. In this case, the system (2) is extended to the following crisp block form

$$SX = Y \rightarrow \begin{pmatrix} S_1 \geq 0 \, S_2 \geq 0 \\ S_2 \geq 0 \, S_1 \geq 0 \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ -\overline{Y} \end{pmatrix} \tag{5}$$

Where

$$S = \begin{pmatrix} S_1 \geq 0 \, S_2 \geq 0 \\ S_2 \geq 0 \, S_1 \geq 0 \end{pmatrix} \tag{6}$$

S_1 and S_2 are $m \times n$ matrices, and

$$\underline{X} = \begin{pmatrix} \underline{x}_1(r) \\ . \\ . \\ \underline{x}_n(r) \end{pmatrix}, \qquad \overline{X} = \begin{pmatrix} \overline{x}_1(r) \\ . \\ . \\ \overline{x}_n(r) \end{pmatrix},$$

$$\underline{Y} = \begin{pmatrix} \underline{y}_1(r) \\ . \\ . \\ \underline{y}_m(r) \end{pmatrix}, \qquad \overline{Y} = \begin{pmatrix} \overline{y}_1(r) \\ . \\ . \\ \overline{y}_m(r) \end{pmatrix}$$

When $m = n$, the fuzzy system is studied by Friedman et al.[1] and the linear system of (5) can be uniquely solved for *x* if and only if the matrix S is nonsingular. For the general $m \times n$ FLS, Asady et al.[2] have considered the existence and expression of the solution to the system (5) is the case S is of full row rank. An approximate solution that is often used is the least square solution of (5), defined as a vector *x* minimizing the Euclidean norm of (Y-SX).

C. Definition

Let $X = \left\{ (\underline{x}_i(r), \overline{x}_i(r)), i = 1,2,...,n \right\}$ denotes a least squares solution of $SX = Y$. The fuzzy number vector $U = \left\{ (\underline{u}_i(r), \overline{u}_i(r)), i = 1,2,...,n \right\}$ defined by

$$\underline{u}_i(r) = \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}$$
$$\overline{u}_i(r) = \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1), \overline{x}_i(1)\}$$

is called a fuzzy least squares solution of $SX = Y$. If $(\underline{x}_i(r), \overline{x}_i(r)), 1 \leq i \leq n$ are all fuzzy numbers, then $\underline{u}_i(r) = \underline{x}\ (i)(r), \quad \overline{u}_i(r) = \overline{x}_i(r), \; 1 \leq i \leq n$ and U is called a strong fuzzy least squares .Otherwise, U is a weak fuzzy least squares solution.

III. Huang's Method for Solving a FLS

Let $S^t = [s_1, s_2, ..., s_{2m}]$, the Huang's algorithm is defined by the following procedure:

Huang's Algorithm:

(A) Let $X_1 \in \Re^{2n}$ be arbitrary. Let $H_1 = I \in \mathbb{R}^{(2n)\times(2n)}$. set $i = 1$ and go to step (B).

(B) Compute the search vector p_i by $p_i = H_i s_i$ and go to step (C).

(C) Update the approximation of the solution by

$$X_{i+1} = X_i - \alpha_i p_i$$

Where the step size α_i is given by

$$\alpha_i = (s_i^t X_i - y_i) / s_i^t p_i$$

If $i = 2m$ stop (X_{2m+1} solves the system); otherwise go to step (D).

(D) Update the matrix H_i by

$$H_{i+1} = H_i - (p_i p_i^t) / s_i^t p_i$$

Increment the index I by one and go to step (B).

A. Theorem [7]

Consider the Huang's algorithm with the following choice X_1 :

$$X_1 = \sum_{j=1}^{k} \beta_j s_j$$

With $k \prec 2m$ and the β_j some scalars, then for $i \geq k$, X_{i+1} is the vector with minimal Euclidean norm.

If $X_1 = \beta s_1$ the previous theorem implies that the whole sequence of iterates $X_2, ..., X_{2m+1}$ generated by the Huang's algorithm, consists of minimal Euclidean norm solution of the associated subsystems. The next theorem shows that the condition $X_1 = \beta s_1$ is also necessary for the whole sequence of solutions to be of minimal Euclidean norm.

B. Theorem [7]

The vector X_2 generated by the Huang's algorithm is the minimal Euclidean norm vector if and only $X_1 = \beta s_1$ for some arbitrary scalar β. For arbitrary starting point X_1, X_{i+1} is the solution of the first *i* equations of minimal Euclidean norm [7].

IV. Numerical Examples

In the following examples, we consider $X_1 = S_1$.

A. Example

And $Y = (r,-2,-2+r,1+r)^t$.

The system has a solution:

$$X = X_5 = \begin{pmatrix} -4/3+4r/3 \\ -1/3-2r/3 \\ -5/3+5r/3 \\ 4/3-r/3 \end{pmatrix}$$

i.e.

$$x_1 = (-4/3+4r/3, 5/3-5r/3), x_2 = (-1/3-2r/3, -4/3+r/3)$$

The fact that x_2 is not fuzzy number and the fuzzy in this case a weak solution given by

$$u_1 = (-4/3+4r/3, 5/3-5r/3), u_2 = (-4/3+r/3, -1/3-2r/3)$$

B. Example

Consider the 2×2 fuzzy linear system:

$$x_1 - x_2 = (r, 2-r).$$

The extended 4×2 matrix S^t is

$$S^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

And $Y = (r,,-2+r)^t$.

The system has a strong fuzzy least squares solution:

$$X = X_3 = \begin{pmatrix} r/2 \\ -1+r/2 \\ -1+r/2 \\ r/2 \end{pmatrix}$$

i.e.

$$x_1 = (r/2, 1-r/2), x_2 = (-1+r/2, -r/2)$$

C. Example

Consider the 2×3 fuzzy linear system:

$$\begin{cases} x_1 + x_2 + x_3 = (r, 2-r), \\ 2x_1 + x_2 - x_3 = (0, 1-r). \end{cases}$$

The extended 6×4 matrix S^t is:

$$S^t = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

And $Y = (r, 0, -2+r, -1+r)^t$

Then one least squares solution of the extended system is:

$$X = X_5 = \begin{pmatrix} 11/14-9r/14 \\ -3/7+5r/7 \\ -5/14+13r/14 \\ 3/14-5r/14 \\ -15/14+11r/14 \\ -8/7+4r/7 \end{pmatrix}$$

i.e.

$$\begin{cases} x_1 = (11/14-9r/14, -3/14+5r/14), \\ x_2 = (-3/7+5r/7, 15/14-11r/14), \\ x_3 = (-5/14+13r/14, 8/7-4r/7). \end{cases}$$

Which implies that x_1 is not fuzzy number, therefore, the corresponding fuzzy solution is a weak fuzzy least squares solution given by:

$$\begin{cases} u_1 = (-3/14+5r/14, 11/14-9r/14), \\ u_2 = (-3/7+5r/7, 15/14-11r/14), \\ u_3 = (-5/14+13r/14, 8/7-4r/7). \end{cases}$$

V. Conclusion

In this paper consider the $m\times n$ fuzzy linear systems. By the embedding method, the original system is converted to a $(2m)\times(2n)$ crisp linear system .then by Huang's method solving the crisp system; we obtain the least squares solution and the minimal Euclidean norm least squares solution to the system.

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