

# Numerical solution of Hammerstein integral equations by using quasi-interpolation

M. Zarebnia and S. Khani

**Abstract**—In this paper first, a numerical method based on quasi-interpolation for solving nonlinear Fredholm integral equations of the Hammerstein-type is presented. Then, we approximate the solution of Hammerstein integral equations by Nystrom's method. Also, we compare the methods with some numerical examples.

**Keywords**—Hammerstein integral equations, quasi-interpolation, Nystrom's method.

## I. INTRODUCTION

**T**HE problem of finding numerical solution for Fredholm integral equations of the second kinds is one of the oldest problems in the applied mathematics and many computational methods are introduced in this filed [2], [3], [4]. Hammerstein integral equations are defined as follows:

$$y(x) = f(x) + \lambda \int_{-1}^1 k(x,t)G(y(t))dt, \quad (1)$$

where, the function  $f(x)$  is known,  $k(x,t)$  is the kernel function which is known, continuous and  $G(y(t))$  is known nonlinear function, the aim is to find the unknown function  $y(x)$  which is solution of equation (1). Previously, some kinds of Fredholm integral equations had been solved numerically, by different methods that are indicated below. Borzabadi et al. [1], introduced a numerical method for a class of nonlinear Fredholm integral equations of the second kind. In [5], Javidi et al. solved nonlinear Fredholm integral equations by using modified homotopy perturbation method.

The method of quasi-interpolation was 1991 introduced by Maz'ya [7] and became popular under the name approximate approximations. In the following years many applications of this method were presented by Maz'ya and Schmidt which are collected in the tex book [9].

We know that quasi-interpolations are defined by

$$\mu_{h,D} u(x) = \sum_{m=-\infty}^{\infty} \frac{u(mh)e^{-\frac{(x-mh)^2}{Dh^2}}}{\sqrt{\pi D}}, \quad (2)$$

where the function  $u$  is twice continuously differentiable with bounded derivatives [8], [10]. The Taylor expansion of  $u$  at the point  $mh$  has the form

$$u(mh) = u(x) + u'(x)(mh - x) + u''(x_m) \frac{(mh - x)^2}{2},$$

for some  $x_m$  between  $x$  and  $mh$ . We apply this method to solve the equation (1) and reduce it to system of equation.

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The outline of the paper is as follows. First, in Section 2 we review some of the main properties of quasi function and quasi-interpolation that are necessary for the formulation of the discrete system. In Section 3, we illustrate how the quasi-interpolation and Nystrom's method may be used to replace Eq. (1) by an explicit system of nonlinear algebraic equations, which is solved by Newtons method. In Section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical schemes by considering some numerical examples.

## II. APPROXIMATION OF INTEGRAL OPERATORS

Let us consider the integral operator

$$T : C([-1, 1]) \rightarrow C([-1, 1]), \quad (TG u)(s) := \int_{-1}^1 k(s,t)G u(t) dt, \quad (3)$$

where  $k : [-1, 1]^2 \rightarrow \mathbf{R}$  is continuous kernel.

We approximate  $TG(u(s))$  with the trapezoidal rule in point  $mh$ . Let  $N \in \mathbf{N}$  and  $h = \frac{1}{N}$ . We obtain

$$\begin{aligned} TG u(s) &= \int_{-1}^1 k(s,t)G(u(t)) dt \\ &\approx \sum_{m=-N}^N \frac{h}{2} k(s, mh)G(u(mh))(2 - \delta_{|m|N}), \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

We suppose the operator  $T_N : C([-1, 1]) \rightarrow C([-1, 1])$  as follow:

$$T_N G u(s) := \sum_{m=-N}^N \frac{h}{2} k(s, mh)G(u(mh))(2 - \delta_{|m|N}). \quad (4)$$

We obtain

$$TG u(s) \approx T_N G u(s).$$

**definition 1.** The error function is defined by

$$erf(a, b) := \frac{2}{\sqrt{\pi}} \int_a^b e^{-t^2} dt,$$

where  $a, b \in \mathbf{R}$  and  $a \leq b$ .

The quasi-interpolation for  $d > 0$  is defined as follows:

$$G u_{d,N} : [-1, 1] \rightarrow \mathbf{R}, \quad G u_{d,N} := \sum_{m=-N}^N G u(mh) \frac{e^{-\frac{(t-mh)^2}{dh^2}}}{\sqrt{\pi d}}.$$

By replacing  $Gu$  with the quasi-interpolation,  $TGu(s)$  is obtained as:

$$TGu(s) = \int_{-1}^1 k(s,t)Gu(t) dt \approx \int_{-1}^1 k(s,t)Gu_{d,N}(t) dt$$

$$= \sum_{m=-N}^N Gu(mh) \int_{-1}^1 k(s,t) \frac{e^{-\frac{(t-mh)^2}{dh^2}}}{\sqrt{\pi d}} dt.$$

We obtain estimate of the nonlinear integral operator by replacing  $k(s,t)$  with  $k(s,mh)$  as follows:

$$TGu(s) \approx \sum_{m=-N}^N Gu(mh) \int_{-1}^1 k(s,t) \frac{e^{-\frac{(t-m)^2}{dh^2}}}{\sqrt{\pi d}} dt$$

$$\approx \sum_{m=-N}^N k(s,mh)Gu(mh) \int_{-1}^1 \frac{e^{-\frac{(t-m)^2}{dh^2}}}{\sqrt{\pi d}} dt$$

$$= \sum_{m=-N}^N k(s,mh)Gu(mh) \frac{h}{\sqrt{\pi}} \int_{\frac{m-N}{\sqrt{d}}}^{\frac{m+N}{\sqrt{d}}} e^{-t^2} dt$$

$$= \sum_{m=-N}^N k(s,mh)Gu(mh) \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right).$$

We define the operator

$$T_{d,N} : C([-1, 1]) \rightarrow C([-1, 1])$$

by

$$(T_{d,N}Gu)(s) := \sum_{m=-N}^N \frac{h}{2} k(s,mh)Gu(mh) \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right),$$

we have

$$(TGu)(s) \approx (T_{d,N}Gu)(s).$$

**Lemma 1.** It holds:

$$\lim_{d \rightarrow 0} \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right) = 2 - \delta_{|m|N}.$$

**proof.** We have

$$\lim_{d \rightarrow 0} \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right) = \lim_{d \rightarrow 0} \frac{2}{\sqrt{\pi}} \int_{\frac{m-N}{\sqrt{d}}}^{\frac{m+N}{\sqrt{d}}} e^{-t^2} dt$$

and

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = 1,$$

then we obtain

$$\lim_{d \rightarrow 0} \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right) = \begin{cases} 1, & m = -N, \\ 2, & |m| \neq N, \\ 1. & m = N. \end{cases}$$

therefore

$$\lim_{d \rightarrow 0} \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right) = 2 - \delta_{|m|N}.$$

**Lemma 2.** It holds:

$$\lim_{d \rightarrow 0} T_{d,N} = T_N$$

**proof.** Let  $Gu \in C([-1, 1])$

and

$$M := \|k\|_\infty = \sup_{(s,t) \in [-1,1]^2} |k(s,t)|.$$

We obtain

$$\|T_{d,N}Gu - T_NGu\|_\infty = \sup_{s \in [-1,1]} |(T_{d,N}Gu)(s) - (T_NGu)(s)|$$

$$\approx \sum_{m=-N}^N \frac{h}{2} M \|Gu\|_\infty \left| \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right) - (2 - \delta_{|m|N}) \right|,$$

therefore from the above relation and lemma 1, we conclude that

$$\|T_{d,N} - T_N\| = \sup_{\|Gu\|_\infty=1} \|T_{d,N}Gu - T_NGu\|_\infty$$

$$\leq \sum_{m=-N}^N \frac{h}{2} M \|Gu\|_\infty \left| \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right) - (2 - \delta_{|m|N}) \right| = 0.$$

### III. APPLICATION TO HAMMERSTEIN INTEGRAL EQUATIONS

Consider the Hammerstein integral equations

$$X - TGX = b, \tag{6}$$

where  $b \in C[-1, 1]$  and  $T$  is defined in (3).

*A. Approximation with quasi-interpolation*

We estimate solution of the nonlinear integral equation (6) using quasi-interpolation. By substituting  $T$  with  $T_N$  from (5), we have

$$X_{d,N} - T_{d,N}GX_{d,N} = b, \tag{7}$$

similarly we obtain the nonlinear system in the point  $jh$  as following:

$$u_d(jh) - \sum_{m=-N}^N \frac{h}{2} k(jh,mh) \operatorname{erf} \left( \frac{m-N}{\sqrt{d}}, \frac{m+N}{\sqrt{d}} \right) Gu_{d,m} = b(jh), \tag{8}$$

where  $u_{d,m} = X_{d,N}(mh)$  are then approximate values for  $u(mh)$ .

*B. Approximation with Nystrom's method*

In this section we obtain approximate of integral equation by Nystrom's method [6].

Let  $u$  be the solution of (6) and  $s \in [-1, 1]$ , we'll have

$$u(s) - (TGu)(s) = b(s).$$

If we employ the trapezoidal rule for the quadrature procedure and approximate equation

$$X_N - T_NGX_N = b, \tag{9}$$

with  $T_N$  is defined in (4), we can obtain the nonlinear system of equation in the points  $jh$ .

$$u(jh) - \sum_{m=-N}^N \frac{h}{2} k(jh, mh) Gu_m (2 - \delta_{|m|N}) = b(jh), \quad (10)$$

that  $j = \{-N, \dots, N\}$  and the values  $u_m = X_N(mh)$  are then approximation for  $u(mh)$ .

IV. NUMERICAL EXAMPLES

In this section, we use the above proposed methods in examples with detailed explanations. we compare the results of numerical solution this method with the solution of the Nystrom's method.

**Example 1** Consider the following nonlinear Fredholm integral equation:

$$u(s) = e^s - \frac{2(2 + e^6)s}{9e^3} + \int_{-1}^1 stu^3(t) dt,$$

with exact solution  $u(s) = e^s$ . Table I shows the solution  $u_d$  of the nonlinear system (8) with  $d = 0.1$  and  $d = 0.01$  and the solution  $u$  of the nonlinear system (10) for  $N = 4$ .

**TABLE I**  
COMPARISON OF NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH  $N = 4$ ,  $d = 0.1$  AND  $d = 0.01$ .

$u$	$u_{0.1}$	$u_{0.01}$
0.4628229775674136	0.4628215310881807	0.4628215310881807
0.5435742050379936	0.5435731201785685	0.5435731201785685
0.6540024279106195	0.654001704671003	0.654001704671003
0.8025366671703977	0.8025363055505895	0.8025363055505895
1.0000000000000000	1.0000000000000000	1.0000000000000000
1.2602895325887487	1.2602898942085567	1.2602898942085567
1.6012495025021423	1.6012502257417587	1.6012502257417587
2.0457923643156954	2.0457934491751204	2.0457934491751204
2.6233382920630737	2.6233397385423065	2.6233397385423065

**Example 2** As the second example consider the following nonlinear integral equation:

$$u(s) = s^2 - \frac{4}{15}s - 1 + \int_{-1}^1 \frac{1}{4} s(t-1)u^2(t) dt,$$

with exact solution  $u(s) = s^2 - 1$ . The Table II illustrate the numerical results for  $N = 4$  the solution  $u_d$  of the nonlinear system (8) with  $d = 0.1$  and  $d = 0.01$  and the solution  $u$  of the nonlinear system (10).

**TABLE II**  
COMPARISON OF NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH  $N = 4$ ,  $d = 0.1$  AND  $d = 0.01$ .

$u$	$u_{0.1}$	$u_{0.01}$
0.515512338823494	0.515512294219369	0.515512294219369
-0.0508657458823796	-0.050865779335473	-0.050865779335473
-0.492243830588253	-0.492243852890315	-0.492243852890315
-0.808621915294126	-0.808621926445158	-0.808621926445158
-1.0000000000000000	-1.0000000000000000	-1.0000000000000000
-1.066378084705874	-1.066378073554842	-1.066378073554842
-1.00775616941175	-1.007756147109684	-1.007756147109684
-0.824134254117620	-0.824134220664527	-0.824134220664527
-0.515512338823494	-0.515512294219369	-0.515512294219369

**Example 3** Consider the nonlinear Fredholm integral equation

$$u(s) = \sin 2\pi s + \int_{-1}^1 t \sin(2\pi s) u^2(t) dt,$$

with exact solution  $u(s) = \sin 2\pi s$ . The following table shows for  $N = 4$  the solution of the nonlinear system (8) with  $d = 0.1$  and  $d = 0.01$  and the solution of the nonlinear system (10). The obtained solutions of Nystrom's and quasi-interpolation methods are exact for this example.

**TABLE III**  
ABSOLUTE ERRORS ON THE NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH  $N = 4$ ,  $d = 0.1$  AND  $d = 0.01$ .

$ u_{ex} - u $	$ u_{ex} - u_{0.1} $	$ u_{ex} - u_{0.01} $
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000
0.0000000000000000	0.0000000000000000	0.0000000000000000

**Example 4** Consider the Hammerstein integral equation

$$u(s) = s^2 - \frac{56}{15}s + 1 + \int_{-1}^1 (s-t)u^2(t) dt.$$

In this example the exact solution of the nonlinear integral equation is  $u(s) = s^2 + 1$ . Errors results for the nonlinear system (8) and the nonlinear system (10) with  $N = 7$ ,  $d = 0.1$  and  $d = 0.01$  are given in Table IV.

**TABLE IV**

ABSOLUTE ERRORS ON THE NYSTRÖM'S METHOD AND QUASI-INTERPOLATION METHOD WITH  $N = 7, d = 0.1$  AND  $d = 0.01$ .

$ u_{ex} - U $	$ u_{ex} - u_{0.1} $	$ u_{ex} - u_{0.01} $
0.006850882568	0.006850048150	0.006850048150
0.006541928759	0.006541131773	0.006541131773
0.006232974951	0.006232215395	0.006232215395
0.005924021143	0.005923299017	0.005923299017
0.005615067335	0.005614382640	0.005614382640
0.005306113526	0.005305466262	0.005305466262
0.004997159718	0.004996549884	0.004996549884
0.004688205910	0.004687633507	0.004687633507
0.004379252102	0.004378717129	0.004378717129
0.004070298293	0.004069800751	0.004069800751
0.003761344485	0.003760884374	0.003760884374
0.003451967996	0.003451967996	0.003451967996
0.003143051619	0.003143051619	0.003143051619
0.002834483060	0.002834135241	0.002834135241
0.002525529252	0.002525218863	0.002525218863

**Example 5** Consider the nonlinear Fredholm integral equation

$$u(s) = s - \frac{\pi}{4} + \frac{1}{2} \int_{-1}^1 \frac{1}{1 + u^2(t)} dt$$

with exact solution  $u(s) = s$ . The following table shows the errors nonlinear systems (8) and (10) for  $N = 4, d = 0.1$  and  $d = 0.01$ .

**TABLE V**

ABSOLUTE ERRORS ON THE NYSTRÖM'S METHOD AND QUASI-INTERPOLATION METHOD WITH  $N = 4, d = 0.1$  AND  $d = 0.01$ .

$ u_{ex} - U $	$ u_{ex} - u_{0.1} $	$ u_{ex} - u_{0.01} $
0.00260574350092058	0.00260636384788038	0.00260636384788038
0.00260574350092069	0.00260636384788027	0.00260636384788027
0.00260574350092069	0.00260636384788027	0.00260636384788027
0.00260574350092063	0.00260636384788027	0.00260636384788027
0.00260574350092065	0.00260636384788027	0.00260636384788027
0.00260574350092066	0.00260636384788027	0.00260636384788027
0.00260574350092069	0.00260636384788032	0.00260636384788032
0.00260574350092069	0.00260636384788027	0.00260636384788027
0.00260574350092069	0.00260636384788027	0.00260636384788027

V. CONCLUSION

In this paper, we approximate solution of Hammerstein integral equation (1) by using quasi-interpolation. We show that the approximation of the nonlinear integral equation, gained with this method, lead to the same numerical results as Nystrom's method with the trapezoidal rule. Also approximate solutions of quasi-interpolation method and Nystrom's method are convergence to exact solution.

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