Numerical Applications of Tikhonov Regularization for the Fourier Multiplier Operators

Fethi Soltani, Adel Almarashi, Idir Mechai

Abstract—Tikhonov regularization and reproducing kernels are the most popular approaches to solve ill-posed problems in computational mathematics and applications. And the Fourier multiplier operators are an essential tool to extend some known linear transforms in Euclidean Fourier analysis, as: Weierstrass transform, Poisson integral, Hilbert transform, Riesz transforms, Bochner-Riesz mean operators, partial Fourier integral, Riesz potential, Bessel potential, etc. Using the theory of reproducing kernels, we construct a simple and efficient representations for some class of Fourier multiplier operators T_m on the Paley-Wiener space H_h. In addition, we give an error estimate formula for the approximation and obtain some convergence results as the parameters and the independent variables approaches zero. Furthermore, using numerical quadrature integration rules to compute single and multiple integrals, we give numerical examples and we write explicitly the extremal function and the corresponding Fourier multiplier operators.

Keywords—Fourier multiplier operators, Gauss-Kronrod method of integration, Paley-Wiener space, Tikhonov regularization.

I. Introduction

Tikhonov regularization is the most widely used method for regularization of ill-posed problems. It has applications to various operator equations for numerical analysis and to many inverse problems [2], [6], [9], [10], [12]. In particular, a simple and efficient representation can obtained by using the theory of reproducing kernels to both mathematical and numerical theories for bounded linear operators in Hilbert spaces [3], [13], [14].

We first consider the space $\mathbb{R}^n$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. We denote by μ the measure on $\mathbb{R}^n$ given by $\mathrm{d}\mu(y) := (2\pi)^{-n/2}\mathrm{d}y$. Furthermore, we denote the space of measurable functions f on $\mathbb{R}^n$ by $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$, such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \Big(\int_{\mathbb{R}^n} |f(y)|^p \mathrm{d}\mu(y)\Big)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\mathbb{R}^n)} := \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} |f(y)| < \infty.$$

Next, we define the Fourier transform for a given function $f \in L^1(\mathbb{R}^n)$ as

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}^n} e^{-i\langle x,y\rangle} f(y)\mathrm{d}\mu(y), \quad x \in \mathbb{R}^n,$$

and the Fourier multiplier operators T_m are defined for $f \in L^2(\mathbb{R}^n)$ by

$$T_m f := \mathcal{F}^{-1}(m\mathcal{F}(f)),$$

where m is a function in $L^\infty(\mathbb{R}^n)$. These operators have attracted the interest of several authors because it provides an essential tool to extend some known linear transforms in Euclidean Fourier analysis [5], [6], [8], [11], like: Weierstrass transform, Poisson integral, Hilbert transform, Riesz transforms, Bochner-Riesz mean operators, partial Fourier integral, Riesz potential, Bessel potential, etc.

Following the ideas of Matsuura et al. [2], Saitoh [5], [7] and Yamada et al. [15], and using the theory of reproducing kernels [1], [4], we give best approximation of the Fourier multiplier operator T_m on the Paley-Wiener space H_h. More precisely, for all $\eta > 0$, $g \in L^2(\mathbb{R}^n)$, the infimum

$$\inf_{f\in H_h}\Big\{\eta\|f\|^2_{H_h} + \|g - T_m f\|^2_{L^2(\mathbb{R}^n)}\Big\},$$

is attained at one function $F^*_{\eta,g}$, called the extremal function, and given by

$$F^*_{\eta,g}(y) = \int_{\mathbb{R}^n} e^{i\langle y,z\rangle}\frac{\chi_h(z)\overline{m(z)}\mathcal{F}(g)(z)}{\eta + |m(z)|^2}\mathrm{d}\mu(z).$$

The extremal function $F^*_{\eta,g}$ satisfies the following properties.

(i) $\|F^*_{\eta,g}\|_{H_h} \le \dfrac{1}{2\sqrt{\eta}}\|g\|_{L^2(\mathbb{R}^n)}$.

(ii) $\lim_{\eta\to 0+} \|T_m F^*_{\eta,g} - g\|_{L^2(\mathbb{R}^n)} = 0$

(iii) $\lim_{\eta\to 0+} \|F^*_{\eta,T_m f} - f\|_{H_h} = 0$.

We also give numerical experiments for some problems and write explicitly the computed formulas for the extremal function and the corresponding Fourier multiplier operators. The results are presented as plots for different values of h and t.

This paper is organized as follows. In Section II, we define and study the Fourier multiplier operators T_m on the Paley-Wiener spaces H_h. Furthermore,we give an application of the theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the operators T_m on the Paley-Wiener spaces H_h. Section III is devoted to present some numerical computation results to validate the theory. Finally, in Section IV, we summarize the obtained results and describe future work.

II. Tikhonov Regularization on Paley-Wiener Space

The Fourier transform $\mathcal{F}$ satisfies the following properties:
(i) $L^1 - L^\infty$-boundedness: For all $f \in L^1(\mathbb{R}^n)$, $\mathcal{F}(f) \in L^\infty(\mathbb{R}^n)$ and

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}.$$

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(ii) Inversion theorem: Let $f \in L^1(\mathbb{R}^n)$, such that $\mathcal{F}(f) \in L^1(\mathbb{R}^n)$. Then

$$f(x) = \mathcal{F}(\mathcal{F}(f))(-x), \quad \text{a.e.} \quad x \in \mathbb{R}^n.$$

(iii) Plancherel theorem: The Fourier transform $\mathcal{F}$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}^n)$ onto itself. In particular,

$$\|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

Let $h > 0$ and χ_h the function defined by

$$\chi_h(z) := \prod_{i=1}^{n} \chi_{(-1/h,1/h)}(z_i), \quad z = (z_1, ..., z_n) \in \mathbb{R}^n,$$

where $\chi_{(-1/h,1/h)}$ is the characteristic function on the interval $(-1/h, 1/h)$.

We define the Paley-Wiener space H_h, as

$$H_h := \mathcal{F}^{-1}(\chi_h L^2(\mathbb{R}^n)).$$

The space H_h satisfies

$$H_h \subset L^2(\mathbb{R}^n), \quad \mathcal{F}(H_h) \subset L^1 \cap L^2(\mathbb{R}^n).$$

We see that any element $f \in H_h$ is represented uniquely by a function $F \in L^2(\mathbb{R}^n)$ in the form

$$f = \mathcal{F}^{-1}(\chi_h F).$$

The space H_h provided with the norm

$$\|f\|_{H_h} = \|F\|_{L^2(\mathbb{R}^n)}.$$

For a given function m in $L^\infty(\mathbb{R}^n)$, we define the Fourier multiplier operators T_m for $f \in L^2(\mathbb{R}^n)$ as

$$T_m f := \mathcal{F}^{-1}(m\mathcal{F}(f)),$$

which are a bounded linear operators from H_h into $L^2(\mathbb{R}^n)$, and we have

$$\|T_m f\|_{L^2(\mathbb{R}^n)} \leq \|m\|_{L^\infty(\mathbb{R}^n)} \|f\|_{H_h}.$$

As application on multiplier operators, we give the following examples:

1) Let m be the function defined for $t > 0$ by

$$m(z) := e^{-t\ell(z)}, \quad \ell(z) = \sum_{j=1}^{n} |z_j|, \quad z = (z_1, ..., z_n),$$

then

$$T_m f(y) = \int_{\mathbb{R}^n} e^{i\langle z,y\rangle} e^{-t\ell(z)} \mathcal{F}(f)(z)\, \mathrm{d}\mu(z).$$

2) For m defined for $t > 0$ as

$$m(z) := \prod_{j=1}^{n} \frac{1}{(t|z_j| + 1)}, \quad z = (z_1, ..., z_n),$$

thus

$$T_m f(y) = \int_{\mathbb{R}^n} e^{i\langle z,y\rangle} \prod_{j=1}^{n} \frac{1}{(t|z_j| + 1)} \mathcal{F}(f)(z)\, \mathrm{d}\mu(z).$$

We denote by $\langle .,. \rangle_{\eta,H_h}$ for $\eta > 0$, the inner product defined on the space H_h by

$$\langle f, g\rangle_{\eta,H_h} := \eta\langle f, g\rangle_{H_h} + \langle T_m f, T_m g\rangle_{L^2(\mathbb{R}^n)},$$

and the norm $\|f\|_{\eta,H_h} := \sqrt{\langle f, f\rangle_{\eta,H_h}}$.

Let $\eta > 0$ and $m \in L^\infty(\mathbb{R}^n)$. The space $(H_h, \langle .,. \rangle_{\eta,H_h})$ has the reproducing kernel

$$K_h(x, y) = \int_{\mathbb{R}^n} \frac{\chi_h(z) e^{i\langle x-y,z\rangle}}{\eta + |m(z)|^2} \mathrm{d}\mu(z), \tag{1}$$

that is

(i) For all $y \in \mathbb{R}^n$, the function $x \to K_h(x, y)$ belongs to H_h.

(ii) The reproducing property: For all $f \in H_h$ and $y \in \mathbb{R}^n$,

$$\langle f, K_h(., y)\rangle_{\eta,H_h} = f(y).$$

Next, by using the theory of extremal function and reproducing kernel of Hilbert space [4], [5], [6], [7] we establish the extremal function associated to the Fourier multiplier operators T_m.

Theorem 1. *Let $m \in L^\infty(\mathbb{R}^n)$. For any $g \in L^2(\mathbb{R}^n)$ and for any $\eta > 0$, there exists a unique function $F^*_{\eta,g}$, where the infimum*

$$\inf_{f \in H_h} \left\{ \eta\|f\|^2_{H_h} + \|g - T_m f\|^2_{L^2(\mathbb{R}^n)} \right\}$$

is attained. Moreover, the extremal function $F^*_{\eta,g}$ is given by

$$F^*_{\eta,g}(y) = \langle g, T_m(K_h(., y))\rangle_{L^2(\mathbb{R}^n)},$$

where K_h is the kernel given by (2.1).

Corollary 1. *Let $\eta > 0$ and $g \in L^2(\mathbb{R}^n)$. The extremal function $F^*_{\eta,g}$ satisfies*

(i) $F^*_{\eta,g}(y) = \displaystyle\int_{\mathbb{R}^n} g(x) \left[\int_{\mathbb{R}^n} \frac{\chi_h(z)\overline{m(z)} e^{-i\langle x-y,z\rangle}}{\eta + |m(z)|^2} \mathrm{d}\mu(z) \right] \mathrm{d}\mu(x).$

(ii) $|F^*_{\eta,g}(y)| \leq \dfrac{2^{(n-4)/4}}{\pi^{n/4} h^{n/2} \sqrt{\eta}} \|g\|_{L^2(\mathbb{R}^n)}.$

(iii) $F^*_{\eta,g}(y) = \displaystyle\int_{\mathbb{R}^n} e^{i\langle y,z\rangle} \frac{\chi_h(z)\overline{m(z)}\mathcal{F}(g)(z)}{\eta + |m(z)|^2} \mathrm{d}\mu(z).$

(iv) $\mathcal{F}(F^*_{\eta,g})(z) = \dfrac{\chi_h(z)\overline{m(z)}\mathcal{F}(g)(z)}{\eta + |m(z)|^2}.$

(v) $\|F^*_{\eta,g}\|_{H_h} \leq \dfrac{1}{2\sqrt{\eta}} \|g\|_{L^2(\mathbb{R}^n)}$

Theorem 2. *Let $\eta > 0$. For every $g \in L^2(\mathbb{R}^n)$, we have*

(i) $T_m F^*_{\eta,g}(y) = \displaystyle\int_{\mathbb{R}^n} e^{i\langle y,z\rangle} \frac{\chi_h(z)|m(z)|^2\mathcal{F}(g)(z)}{\eta + |m(z)|^2} \mathrm{d}\mu(z).$

(ii) $\mathcal{F}(T_m F^*_{\eta,g})(z) = \dfrac{\chi_h(z)|m(z)|^2\mathcal{F}(g)(z)}{\eta + |m(z)|^2}.$

(iii) $T_m F^*_{\eta,g}(y) = F^*_{\eta,T_m g}(y).$

(iv) $\lim\limits_{\eta\to 0^+} \|T_m F^*_{\eta,g} - g\|_{L^2(\mathbb{R}^n)} = 0.$

Corollary 2. *Let $\eta > 0$. For every $f \in H_h$, we have*

(i) $\lim\limits_{\eta\to 0^+} \|F^*_{\eta,T_m f} - f\|_{L^\infty(\mathbb{R}^n)} = 0.$

(ii) $\lim\limits_{\eta\to 0^+} \|F^*_{\eta,T_m f} - f\|_{H_h} = 0.$

Remark 1. Let $m \in L^\infty(\mathbb{R}^n)$ with $m \neq 0$; and let $g \in L^2(\mathbb{R}^n)$. From the dominated convergence theorem we have

$$F^*_{0,g}(y) = \int_{\mathbb{R}^n} g(x) \left[\int_{\mathbb{R}^n} \frac{\chi_h(z) e^{-i\langle x-y,z\rangle}}{m(z)} \mathrm{d}\mu(z) \right] \mathrm{d}\mu(x).$$

As application of the external functions, we give the following examples.

Example 1. Let $\eta > 0$, and $g(x) := \prod_{j=1}^{n} \chi_{(-1,1)}(x_j)$. If $m(z) := e^{-t\ell(z)}$, $t > 0$, then

$$F^*_{\eta,g}(y) = \int_{(-1,1)^n} \left(\int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{\prod_{j=1}^n e^{-i(x_j-y_j)z_j}}{\eta e^{t\ell(z)} + e^{-t\ell(z)}} \mathrm{d}\mu(z) \right) \mathrm{d}\mu(x),$$

where

$$\ell(z) = \sum_{j=1}^{n} |z_j|, \quad z = (z_1, ..., z_n).$$

Since $m(z)$ is an even function, then $F^*_{\eta,g}(y)$ can be simplified as

$$F^*_{\eta,g}(y) = \int_{(-1,1)^n} \left(\int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{\prod_{i=1}^n \cos(x_j z_j - y_j z_j)}{\eta e^{t\ell(z)} + e^{-t\ell(z)}} \mathrm{d}\mu(z) \right) \mathrm{d}\mu(x),$$
$$= \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{(2\pi)^{-n/2}\left(\prod_{j=1}^n \int_{-1}^1 \cos(x_i z_i - y_j z_j)\mathrm{d}x_j\right)}{\eta e^{t\ell(z)} + e^{-t\ell(z)}} \mathrm{d}\mu(z).$$

Then

$$F^*_{\eta,g}(y) = \left(\frac{2}{\pi}\right)^{n/2} \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{\prod_{j=1}^n \frac{\sin(z_j)\cos(y_j z_j)}{z_j}}{\eta e^{t\ell(z)} + e^{-t\ell(z)}} \mathrm{d}\mu(z). \tag{2}$$

Next, taking $\eta \to 0$ yields

$$F^*_{0,g}(y) = \pi^{-n} \prod_{j=1}^{n} \left(\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{e^{t|z_j|} \sin(z_i)\cos(y_j z_j)}{z_j} \mathrm{d}z_j \right). \tag{3}$$

Similarly, the Fourier multiplier operator $T_m F^*_{\eta,g}(y)$ can be written as

$$T_m F^*_{\eta,g}(y) = \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{e^{i\langle y,z\rangle}\left(\int_{(-1,1)^n} e^{-i\langle z,x\rangle}\mathrm{d}\mu(x)\right)}{\eta e^{2t\ell(z)}+1} \mathrm{d}\mu(z),$$
$$= \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{e^{i\langle y,z\rangle}\left((2\pi)^{-n/2}\prod_{j=1}^n \int_{-1}^1 e^{-iz_j x_j}\mathrm{d}x_j\right)}{\eta e^{2t\ell(z)}+1} \mathrm{d}\mu(z),$$
$$= \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{e^{i\langle y,z\rangle}\left((2\pi)^{-n/2}\prod_{j=1}^n \int_{-1}^1 \cos(z_j x_j)\mathrm{d}x_j\right)}{\eta e^{2t\ell(z)}+1} \mathrm{d}\mu(z).$$

Thus

$$T_m F^*_{\eta,g}(y) = \left(\frac{2}{\pi}\right)^{n/2} \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{\prod_{j=1}^n \frac{\sin(z_j)\cos(z_j y_j)}{z_j}}{\eta e^{2t\ell(z)} + 1} \mathrm{d}\mu(z). \tag{4}$$

Setting $\eta \to 0$ implies

$$T_m F^*_{0,g}(y) = \frac{1}{\pi^n} \prod_{j=1}^{n} \left(\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{\sin(z_j)\cos(z_j y_j)}{z_j} \mathrm{d}z_j \right). \tag{5}$$

Example 2. Let $\eta > 0$, and $g(x) := \prod_{j=1}^{n} \chi_{(-1,1)}(x_j)$. If $m(z) := \prod_{j=1}^{n} \frac{1}{(t|z_j|+1)}$, $t > 0$, then as in the Example 1, we obtain:

$$F^*_{\eta,g}(y) = \int_{(-1,1)^n} \left(\int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{\prod_{j=1}^n (t|z_j|+1) e^{-i(x_j-y_j)z_j}}{\eta \prod_{j=1}^n (t|z_j|+1)^2+1} \mathrm{d}\mu(z) \right) \mathrm{d}\mu(x).$$

Thus

$$F^*_{\eta,g}(y) = \left(\frac{2}{\pi}\right)^{n/2} \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{\prod_{j=1}^n (t|z_j|+1)\frac{\sin(z_j)\cos(y_j z_j)}{z_j}}{\eta \prod_{j=1}^n (t|z_j|+1)^2+1} \mathrm{d}\mu(z). \tag{6}$$

and

$$F^*_{0,g}(y) = \frac{1}{\pi^n} \prod_{j=1}^{n} \left(\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{(t|z_j|+1)\sin(z_j)\cos(y_j z_j)}{z_j} \mathrm{d}z_j \right). \tag{7}$$

On the other hand we have

$$T_m F^*_{\eta,g}(y) = \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{e^{i\langle y,z\rangle}\left(\int_{(-1,1)^n} e^{-i\langle z,x\rangle}\mathrm{d}\mu(x)\right)}{\eta\prod_{j=1}^n (t|z_j|+1)^2+1} \mathrm{d}\mu(z),$$

and therefore

$$T_m F^*_{\eta,g}(y) = \left(\frac{2}{\pi}\right)^{n/2} \int_{(\frac{-1}{h},\frac{1}{h})^n} \frac{\prod_{j=1}^n \frac{\sin(z_j)\cos(z_j y_j)}{z_j}}{\eta\prod_{j=1}^n (t|z_j|+1)^2+1} \mathrm{d}\mu(z). \tag{8}$$

III. Numerical Results

In this section, we use the Gauss–Kronrod method to integrate numerically and plot $F^*_{\eta,g}(y)$ and $T_m F^*_{\eta,g}(y)$ given in the the Examples 1 and 2, for $n = 2$ and different values of t and h. In the Example 1, the integrals (2), (3), (4), (5) become, respectively:

$$F^*_{\eta,g}(y) = \frac{1}{\pi^2}\int_{\frac{-1}{h}}^{\frac{1}{h}}\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{\prod_{j=1}^2 \frac{1}{z_j}\sin(z_j)\cos(y_j z_j)}{\eta e^{t(|z_1|+|z_2|)}+e^{-t(|z_1|+|z_2|)}} \mathrm{d}z_1\mathrm{d}z_2 \tag{9}$$

$$F^*_{0,g}(y) = \frac{1}{\pi^2}\prod_{j=1}^{2}\left(\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{e^{t|u|}\sin(u)\cos(y_j u)}{u}\mathrm{d}u\right), \tag{10}$$

$$T_m F^*_{\eta,g}(y) = \frac{1}{\pi^2}\int_{\frac{-1}{h}}^{\frac{1}{h}}\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{\prod_{j=1}^2 \frac{1}{z_j}\sin(z_j)\cos(y_j z_j)}{\eta e^{2t(|z_1|+|z_2|)}+1} \mathrm{d}z_1\mathrm{d}z_2, \tag{11}$$

$$T_m F^*_{0,g}(y) = \frac{1}{\pi^2}\prod_{j=1}^{2}\left(\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{\sin(u)\cos(y_j u)}{u}\mathrm{d}u\right). \tag{12}$$

Similarly, for the example 2. we have

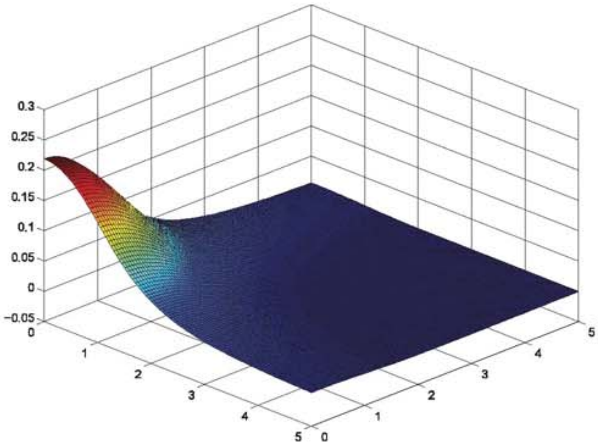
$$F^*_{\eta,g}(y) = \frac{1}{\pi^2}\int_{\frac{-1}{h}}^{\frac{1}{h}}\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{\prod_{j=1}^2 \frac{(t|z_j|+1)\sin(z_j)\cos(y_j z_j)}{z_j}}{\eta(t|z_1|+1)^2(t|z_2|+1)^2+1} \mathrm{d}z_1\mathrm{d}z_2, \tag{13}$$

$$F^*_{0,g}(y) = \frac{1}{\pi^2}\prod_{j=1}^{2}\left(\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{(t|u|+1)\sin(u)\cos(y_j u)}{u}\mathrm{d}u\right), \tag{14}$$

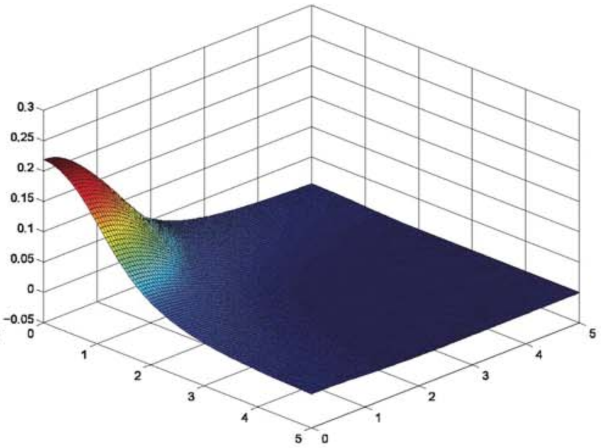
$$T_m F^*_{\eta,g}(y) = \frac{1}{\pi^2}\int_{\frac{-1}{h}}^{\frac{1}{h}}\int_{\frac{-1}{h}}^{\frac{1}{h}} \frac{\prod_{j=1}^2 \frac{\sin(z_j)\cos(y_j z_j)}{z_j}}{\eta(t|z_1|+1)^2(t|z_2|+1)^2+1} \mathrm{d}z_1\mathrm{d}z_2. \tag{15}$$

IV. Conclusion

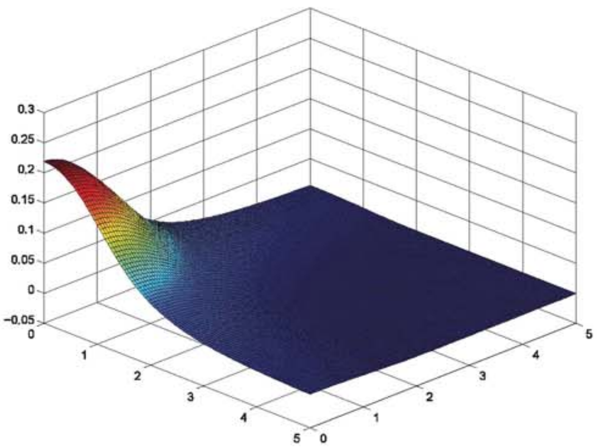
We investigated the Tikhonov regularization method, and we constructed a simple and efficient representations for some class of Fourier multiplier operators. We gave an error estimates formulas for the approximation and we obtained some convergence as the variable $\eta \to 0^+$. Finally, we tested the obtained results numerically by using numerical quadrature integration rules to compute the single and double integrals corresponding to the extremal function and the Fourier multiplier operators. The same results obtained in



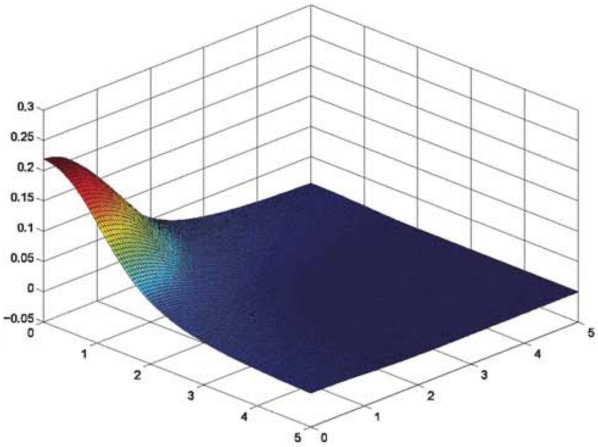
(a) $h = 1/400$



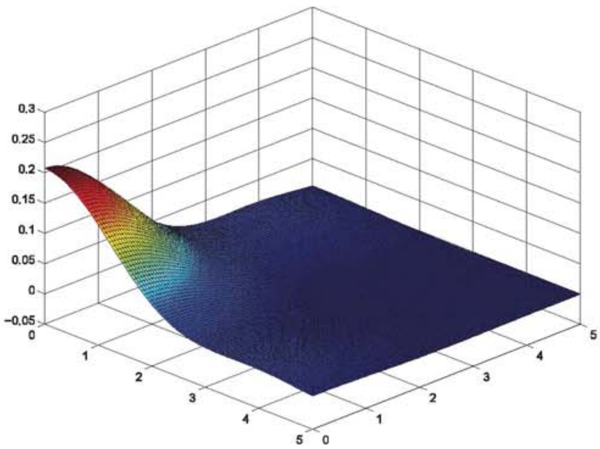
(b) $h = 1/150$



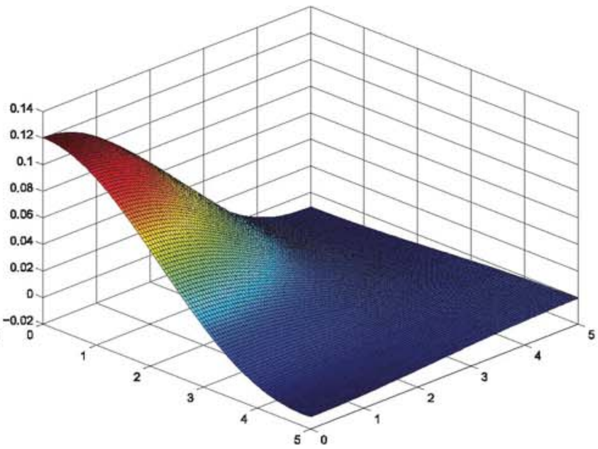
(c) $h = 1/60$



(d) $h = 1/7$

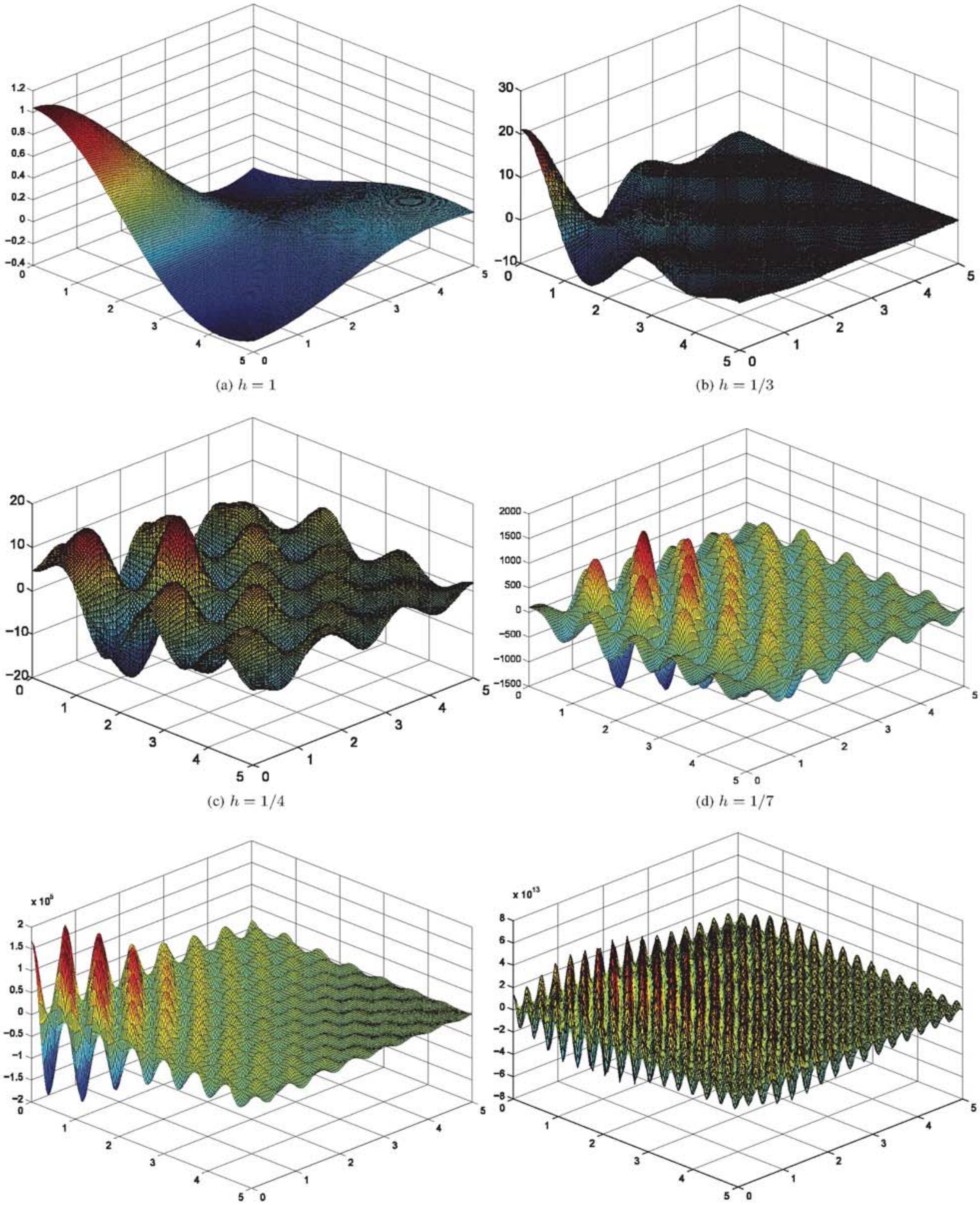


(e) $h = 1/2$



(f) $h = 1$

Fig. 1 Extremal function $F^*_{1,g}(y)$ given by (9) for $t = 1$

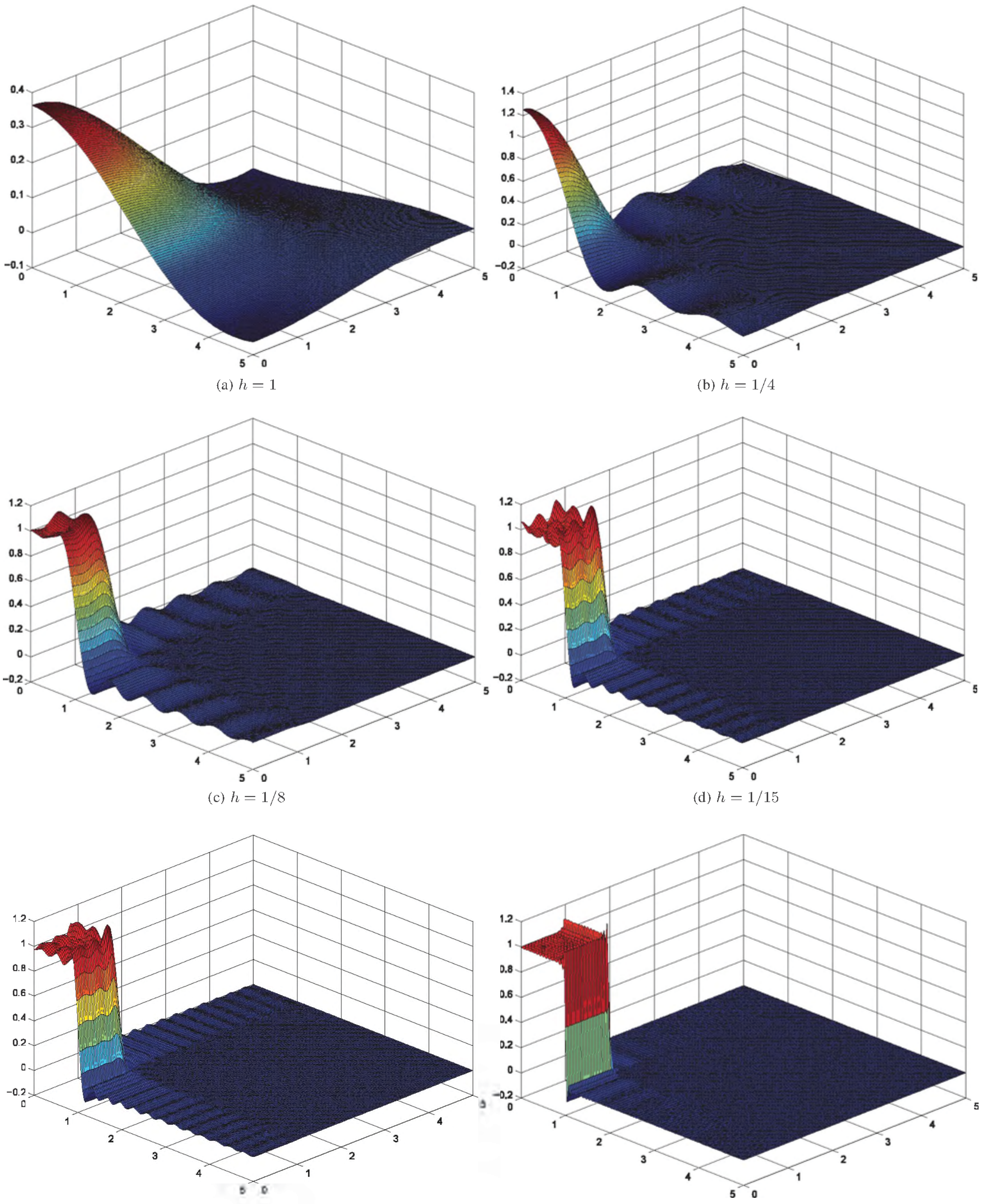


(a) $h = 1$ (b) $h = 1/3$

(c) $h = 1/4$ (d) $h = 1/7$

(e) $h = 1/9$ (f) $h = 1/20$

Fig. 2 Extremal function $F^{*}_{0,g}(y)$ given by (10) for $t = 1$

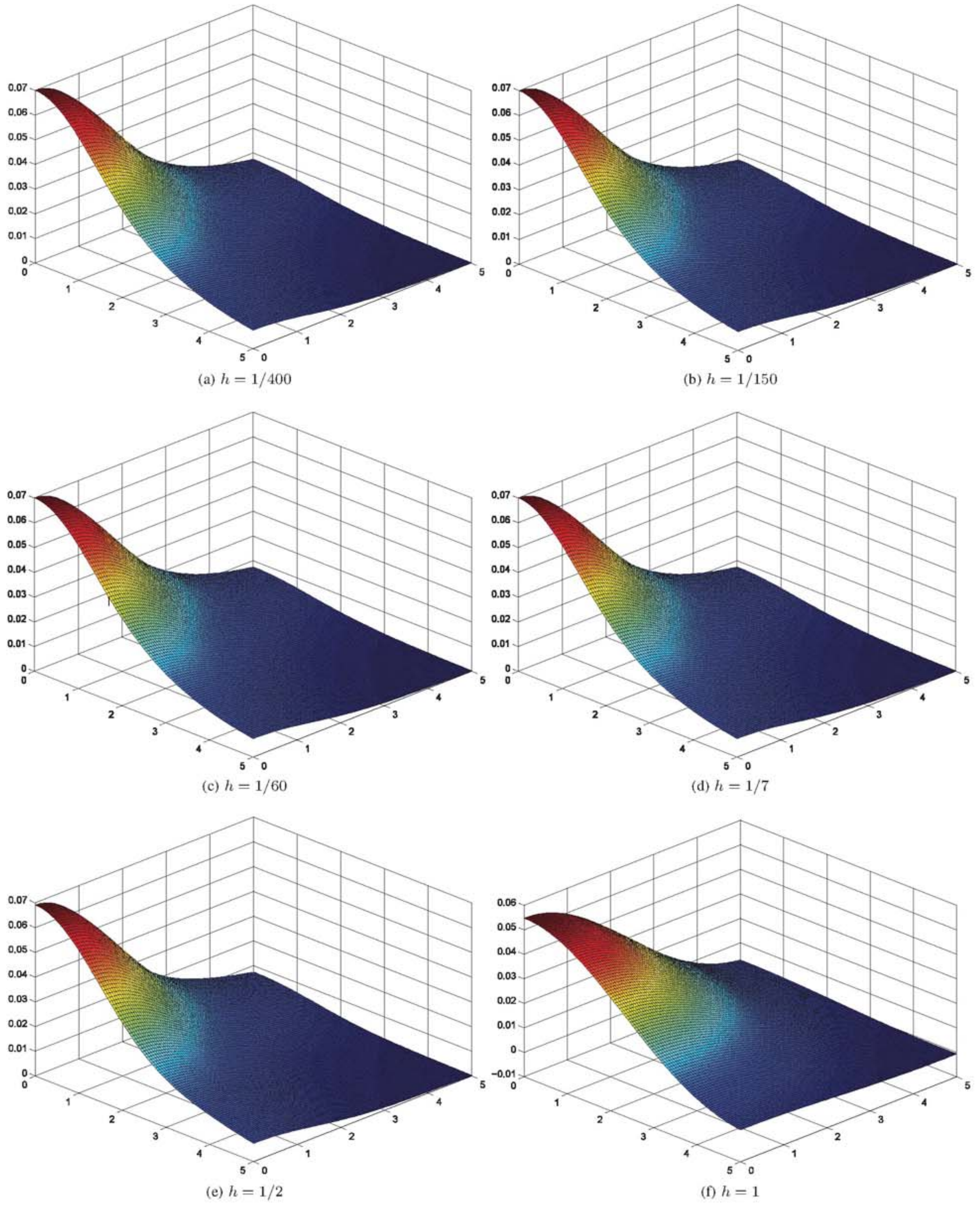


(a) $h = 1$ (b) $h = 1/4$

(c) $h = 1/8$ (d) $h = 1/15$

(e) $h = 1/20$ (f) $h = 1/100$

Fig. 3 Extremal function $F^*_{0,g}(y)$ given by (10) for $t = 10^{-7}$

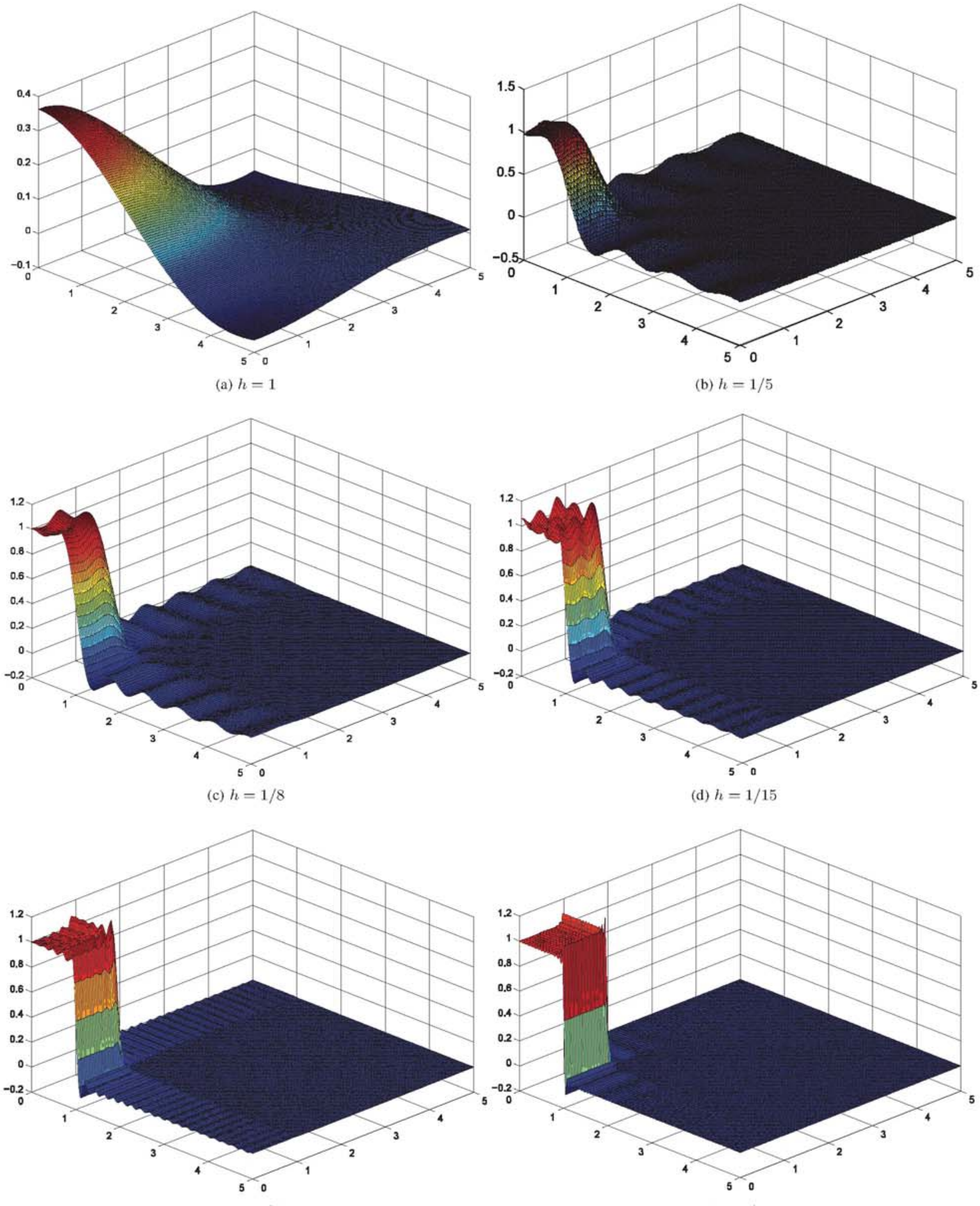


(a) $h = 1/400$ (b) $h = 1/150$

(c) $h = 1/60$ (d) $h = 1/7$

(e) $h = 1/2$ (f) $h = 1$

Fig. 4 Fourier multiplier operators $T_m F^{*}_{1,g}(y)$ given by (11)

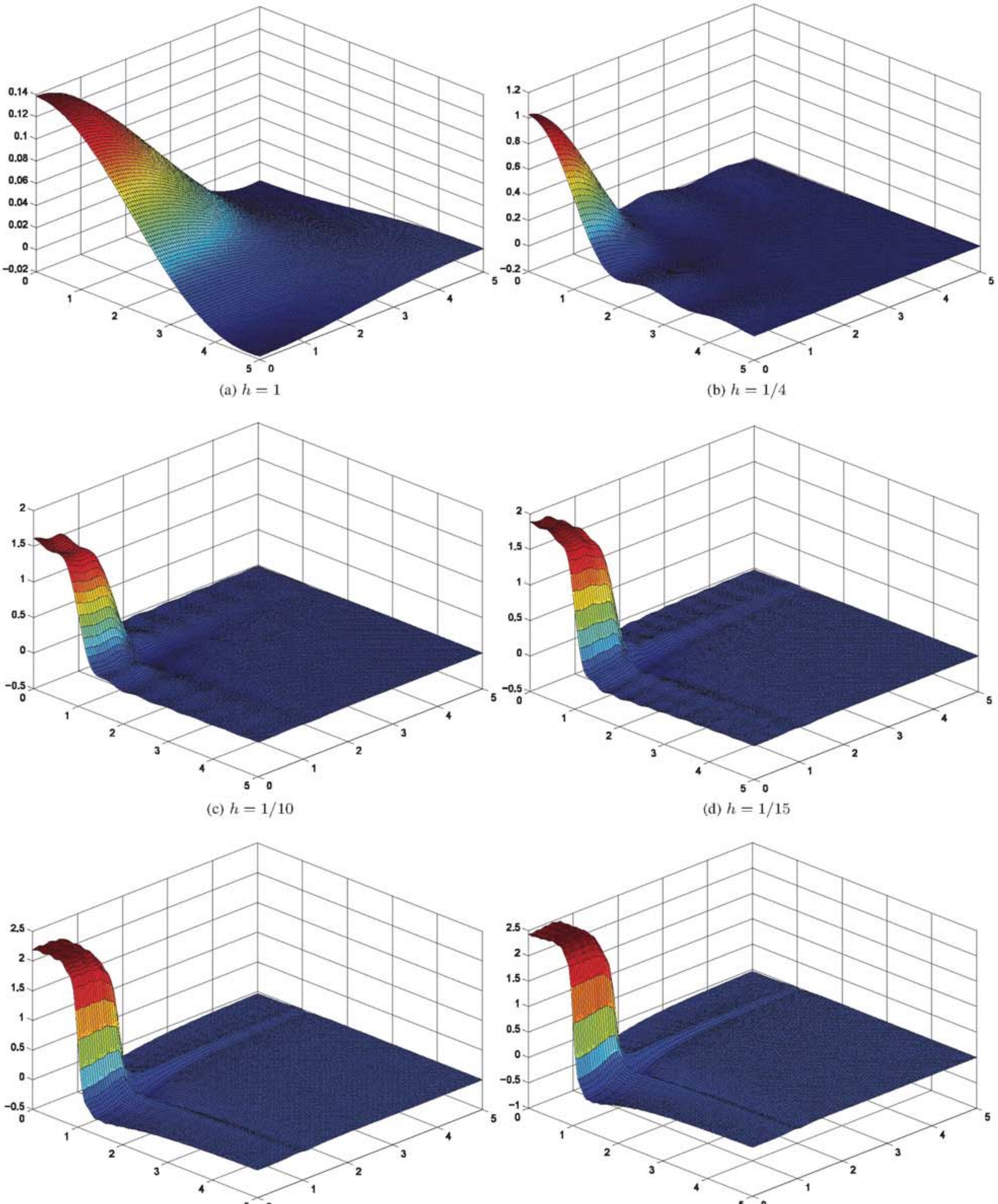


(a) $h = 1$ (b) $h = 1/5$

(c) $h = 1/8$ (d) $h = 1/15$

(e) $h = 1/30$ (f) $h = 1/100$

Fig. 5 Fourier multiplier operators $T_m F^{*}_{0,g}(y)$ given by (12)

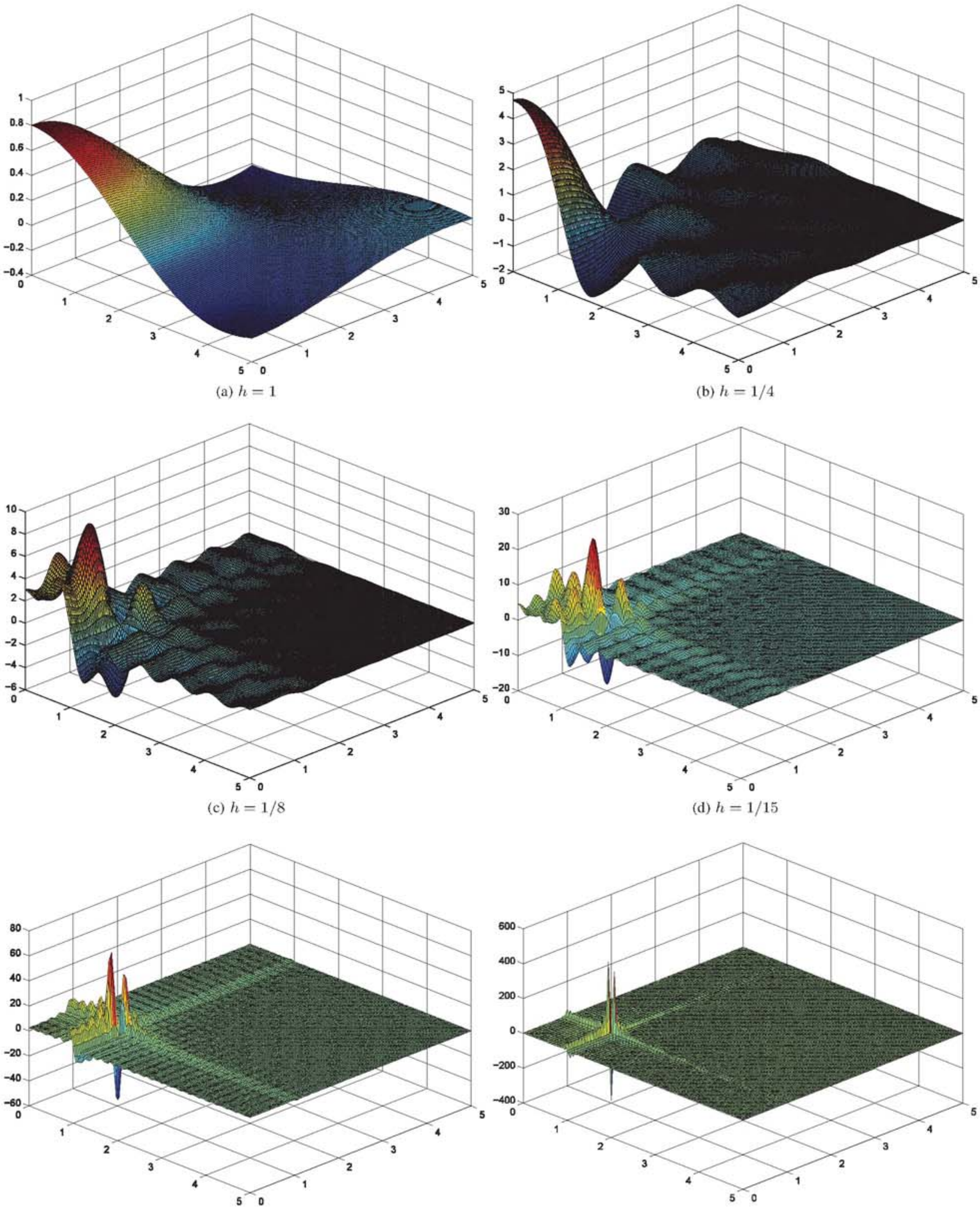


(a) $h = 1$ (b) $h = 1/4$

(c) $h = 1/10$ (d) $h = 1/15$

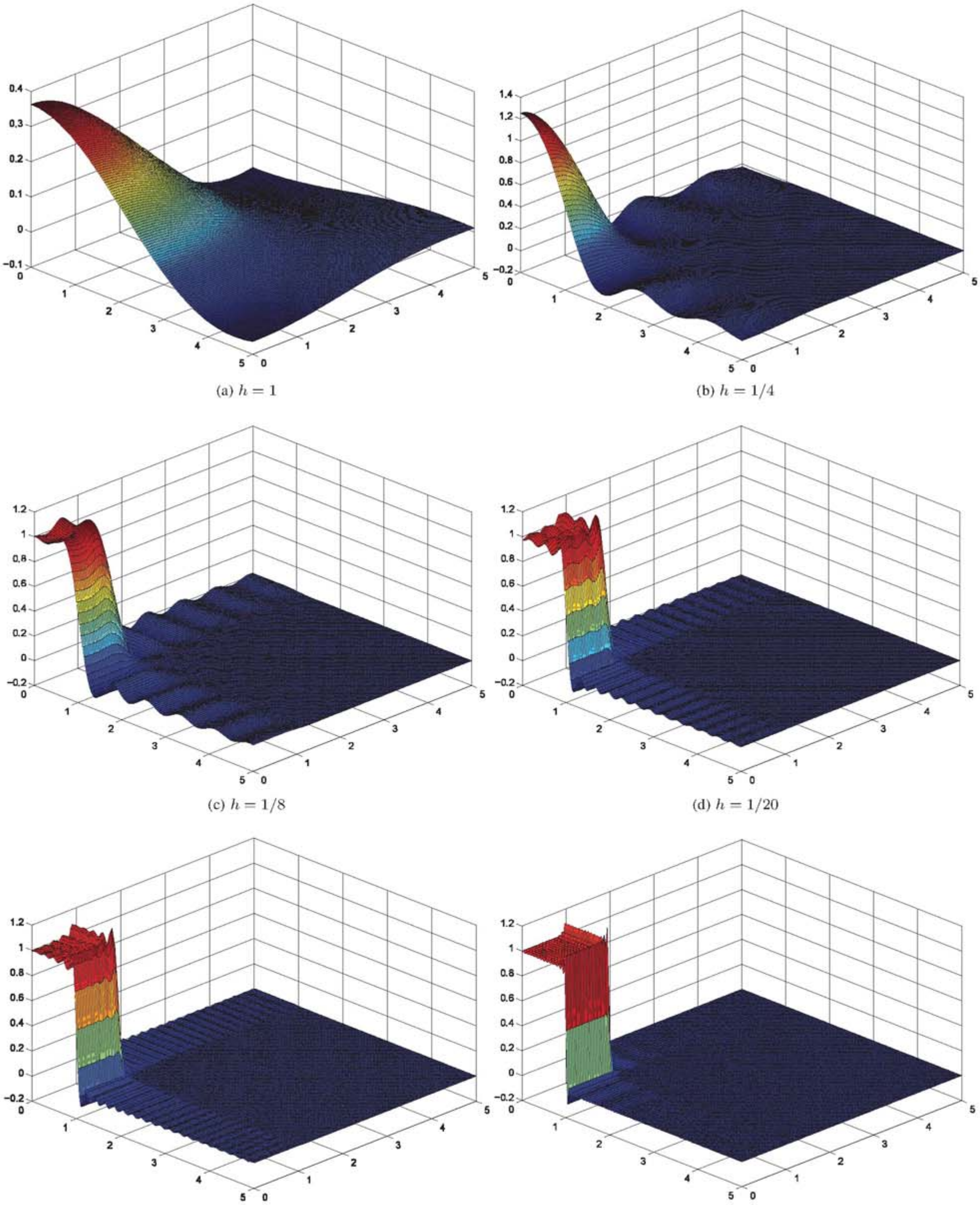
(e) $h = 1/25$ (f) $h = 1/35$

Fig. 6 Extremal function $F^{*}_{1,g}(y)$ given by (13) for $t = 1$



(a) $h = 1$ (b) $h = 1/4$ (c) $h = 1/8$ (d) $h = 1/15$ (e) $h = 1/30$ (f) $h = 1/100$

Fig. 7 Extremal function $F^{*}_{0,g}(y)$ given by (14) for $t = 1$

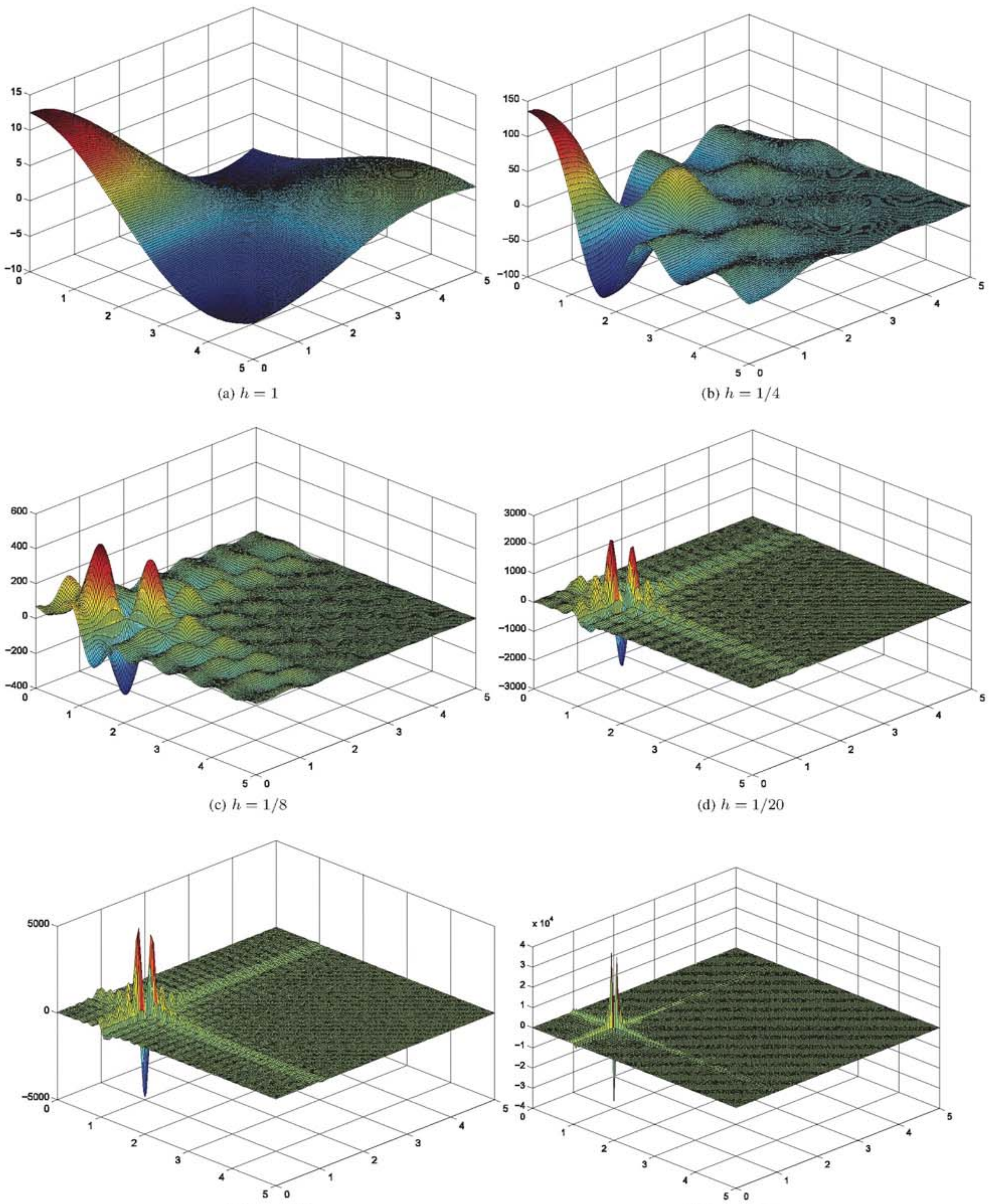


(a) $h = 1$ (b) $h = 1/4$

(c) $h = 1/8$ (d) $h = 1/20$

(e) $h = 1/30$ (f) $h = 1/100$

Fig. 8 Extremal function $F^*_{0,g}(y)$ given by (14) for $t = 10^{-7}$

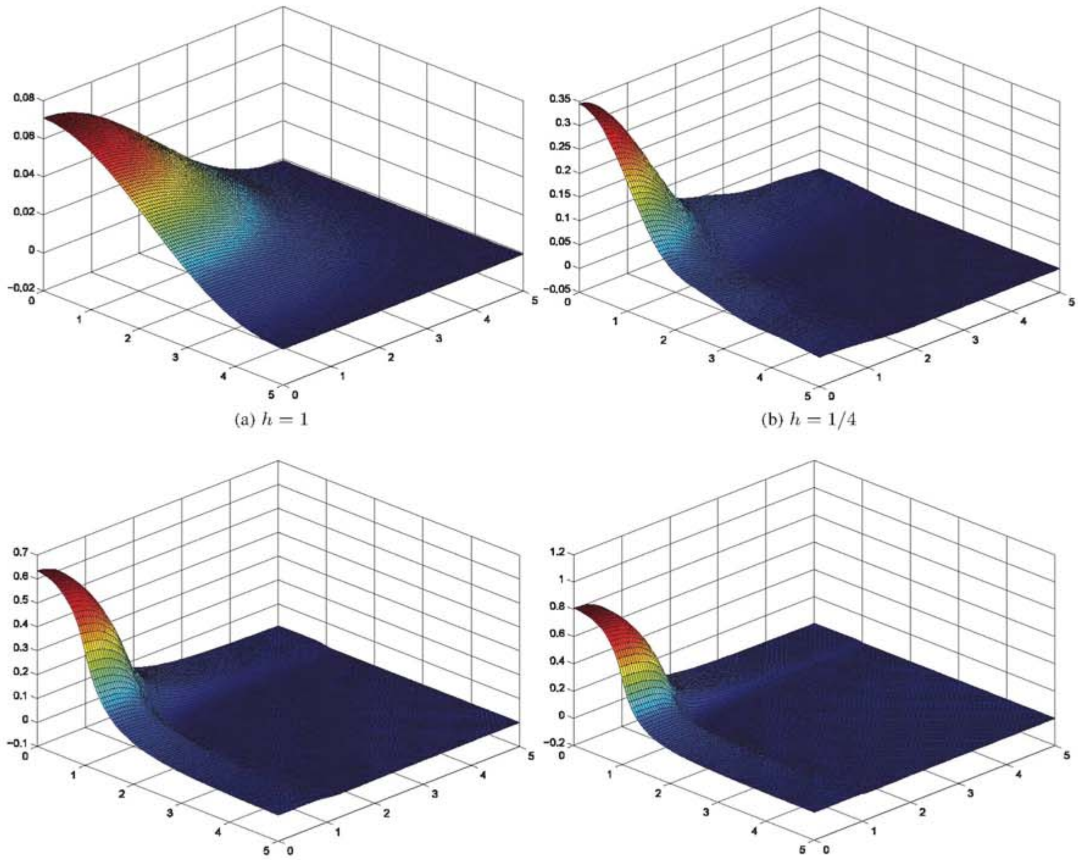


(a) $h = 1$ (b) $h = 1/4$

(c) $h = 1/8$ (d) $h = 1/20$

(e) $h = 1/30$ (f) $h = 1/100$

Fig. 9 Extremal function $F^{*}_{0,g}(y)$ given by (14) for $t = 10$



(a) $h = 1$ (b) $h = 1/4$

(c) $h = 1/15$ (d) $h = 1/35$

Fig. 10 Fourier multiplier operators $T_m F^{*}_{0,g}(y)$ given by (15) for $t = 1$

the case of the Fourier transform can be expanded for different transformations such as: Hartley transform, Hankel transform,and Dunkl transform.

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