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Notes on Fractional k-Covered Graphs

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Abstract—A graph G is fractional k-covered if for each edge e of G, there exists a fractional k-factor h, such that $h(e) = 1$. If $k = 2$, then a fractional k-covered graph is called a fractional 2-covered graph. The binding number $bind(G)$ is defined as follows,

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

In this paper, it is proved that G is fractional 2-covered if $\delta(G) \geq 4$ and $bind(G) > \frac{5}{3}$.

Keywords—graph, binding number, fractional k-factor, fractional k-covered graph.

I. Introduction

MANY physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorizations in networks are very useful in combinatorial design, network design, circuit layout, and so on. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term "graph" instead of "network".

We investigate the fractional factor problem in graphs, which can be considered as a relaxations of the well-known cardinality matching problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

We consider only finite simple graph G with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$, the minimum vertex degree of $V(G)$ is denoted by $\delta(G)$. For any $S \subseteq V(G)$, we denote by $N_G(S)$ the neighborhood set of S in G, by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S,T)$ the number of edges joining S and T. We write $i(G)$ for the number of isolated vertices in G. The binding number of G is defined by Woodall [1] as

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. A fractional (g,f)-factor is a function h that assigns to each edge of a graph G a number in [0,1], so that for each vertex x of G we have $g(x) \leq d_G^h(x) \leq f(x)$, where $d_G^h(x) = \sum_{e \ni x} h(e)$(the sum is taken over all edges incident to x) is a fractional degree of x in G. If $g(x) = f(x) = k$ for each $x \in V(G)$, then a fractional (g,f)-factor is a fractional k-factor. If $k = 2$, then a fractional k-factor is a fractional 2-factor. A graph G is fractional k-covered if for each edge e of G, there exists a fractional k-factor h, such that $h(e) = 1$. If $k = 2$, then a fractional k-covered graph is called a fractional 2-covered graph. The other terminologies and notations not given in this paper can be found in [2,3].

Zhou Sizhong [4–7] showed some sufficient conditions for graphs to have factors. Liu Guizhen, et al. [8] studied the fractional (g,f)-factors of graphs. Zhou Sizhong [9–12] gave some sufficient conditions for graphs to have fractional factors. Yan Guiying, et al. [13,14] obtained some sufficient conditions for graphs to be fractional k-covered graphs. In this paper, we give a sufficient condition for a graph to be a fractional k-covered graph.

Anstee [15] obtained a necessary and sufficient condition for a graph to have a fractional (g,f)-factor.

Lemma 1.1. ([15]). Let G be a graph, g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a fractional (g,f)-factor if and only if for any $S \subseteq V(G)$,

$$g(T) - d_{G-S}(T) \leq f(S),$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$.

For any $S \subseteq V(G)$, let $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$, denote

$$\delta_G(S,T) = d_{G-S}(T) - g(T) + f(S).$$

Li obtained a necessary and sufficient condition for a graph to be fractional (g,f)-covered.

Lemma 1.2. ([13]). Let G be a graph, g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G is fractional (g,f)-covered

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if and only if $\delta_G(S,T) \geq \varepsilon(S,T)$ for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$.
Where $\varepsilon(S,T)$ is defined as follows,

(1) $\varepsilon(S,T) = 2$, if S is not independent.

(2) $\varepsilon(S,T) = 1$, if S is independent and $e_G(S, V(G) \setminus (S \cup T)) \geq 1$, or there exists an edge $e = uv$, such that $u \in S, v \in T$ and $d_{G-S}(v) = g(v)$.

(3) $\varepsilon(S,T) = 0$, if neither (1) nor (2) holds.

If $f(x) = g(x) = k$ for all $x \in V(G)$, we have

Lemma 1.3. ([13]). Let G be a graph and $k > 0$ be an integer. Then G is fractional k-covered if and only if for all $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$,

$$\sum_{j=0}^{k-1}(k-j)p_j(G-S) \leq k|S| - \varepsilon(S,T),$$

where $p_j(G-S)$ denote the number of vertices in $G-S$ with degree j.

II. The Main Result and Its Proof

Theorem 1. Let G be a graph, $\delta(G) \geq 4$. If $bind(G) > \frac{5}{3}$, then graph G is fractional 2-covered.

Proof. Suppose that $bind(G) > \frac{5}{3}$ and G is not fractional 2-covered. By Lemma 1.3, there exists $S_0 \subseteq V(G)$ such that

$$2p_0(G-S_0) + p_1(G-S_0) > 2|S_0| - \varepsilon(S_0,T). \quad (1)$$

Let $T_0 = \{x : x \in V(G) \setminus S_0, d_{G-S_0}(x) = 0\}$, $T_1 = \{x : x \in V(G) \setminus S_0, d_{G-S_0}(x) = 1\}$. We have $|T_0| = p_0(G-S_0) = i(G-S_0)$ and $|T_1| = p_1(G-S_0)$. The proof splits into three cases.

Case 1. $T_1 = \emptyset$

Subcase 1.1. $S_0 = \emptyset$. We have $\varepsilon(S_0,T) = 0$.

Since $\delta(G) \geq 4$ and $S_0 = \emptyset$, we get

$$p_0(G-S_0) = 0.$$

By (1), we obtain

$$0 = 2p_0(G-S_0) + p_1(G-S_0) > 2|S_0| - \varepsilon(S_0,T) = 0,$$

which is contradicted.

Subcase 1.2. $|S_0| = 1$. We have $\varepsilon(S_0,T) \leq 1$.

Since $\delta(G) \geq 4$ and $|S_0| = 1$, we obtain

$$p_0(G-S_0) = 0.$$

It follows from (1) that

$$0 = 2p_0(G-S_0) + p_1(G-S_0) > 2|S_0| - \varepsilon(S_0,T) \geq 2-1 = 1,$$

a contradiction.

Subcase 1.3. $2 \leq |S_0| \leq 3$. We get $\varepsilon(S_0,T) \leq 2$.

Since $\delta(G) \geq 4$ and $2 \leq |S_0| \leq 3$, thus

$$p_0(G-S_0) = 0.$$

According to (1), we have

$$0 = 2p_0(G-S_0) + p_1(G-S_0) > 2|S_0| - \varepsilon(S_0,T) \geq 4-2 = 2,$$

a contradiction.

Subcase 1.4. $|S_0| \geq 4$. We get $\varepsilon(S_0,T) \leq 2$.

By (1), we have

$$\begin{aligned} 2p_0(G-S_0) &= 2p_0(G-S_0) + p_1(G-S_0) \\ &> 2|S_0| - \varepsilon(S_0,T) \geq 6. \end{aligned}$$

So

$$p_0(G-S_0) > 3.$$

We write X for the set of isolated vertices of $G-S_0$, it is easily seen that $4 \leq |X| = p_0(G-S_0)$ and $|N_G(X)| \leq |S_0|$. Thus

$$\frac{5}{3} < bind(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S_0|}{p_0(G-S_0)}.$$

So

$$2|S_0| > \frac{10}{3}p_0(G-S_0).$$

It follows from (1) that

$$\begin{aligned} \frac{10}{3}p_0(G-S_0) &< 2|S_0| \\ &< 2p_0(G-S_0) + p_1(G-S_0) + \varepsilon(S_0,T) \\ &\leq 2p_0(G-S_0) + 2. \end{aligned}$$

Thus

$$p_0(G-S_0) < \frac{3}{2},$$

which contradicts $p_0(G-S_0) > 3$.

Case 2. $T_1 \neq \emptyset$ and $T_1 = N_{G-S_0}(T_1)$

By the definition of T_1, we have $|T_1| = 2r$ (r be positive-integer).

Subcase 2.1. $2 \leq |T_1| \leq 4$

Since $\delta(G) \geq 4$, we get

$$|S_0| \geq 3.$$

Subcase 2.1.1. $p_0(G-S_0) = 0$

By (1), we get

$$\begin{aligned} 4 &\geq |T_1| = p_1(G-S_0) \\ &= 2p_0(G-S_0) + p_1(G-S_0) \\ &> 2|S_0| - \varepsilon(S_0,T) \geq 6-2 = 4, \end{aligned}$$

a contradiction.

Subcase 2.1.2. $p_0(G-S_0) \geq 1$

Let $X = T_0 \cup T_1$, then $|N_G(X)| \leq |S_0| + p_1(G-S_0)$. Thus

$$\frac{5}{3} < bind(G) \leq \frac{|N_G(X)|}{|X|} \leq \frac{|S_0| + p_1(G-S_0)}{p_0(G-S_0) + p_1(G-S_0)}.$$

So

$$\begin{aligned} 2|S_0| &> \frac{10}{3}p_0(G-S_0) + \frac{4}{3}p_1(G-S_0) \\ &= 2p_0(G-S_0) + p_1(G-S_0) + \frac{4}{3}p_0(G-S_0) \\ &\quad + \frac{1}{3}p_1(G-S_0) \\ &\geq 2p_0(G-S_0) + p_1(G-S_0) + 2, \end{aligned}$$

which contradicts (1).

Subcase 2.2. $|T_1| \geq 6$

For any $u \in T_1$, let $X = T_0 \cup (T_1 - u)$, then $|X| = p_0(G-S_0)+p_1(G-S_0)-1$, $|N_G(X)| \le |S_0|+p_1(G-S_0)-1$, we have

$$\frac{5}{3} < bind(G) \le \frac{|N_G(X)|}{|X|} \le \frac{|S_0| + p_1(G-S_0) - 1}{p_0(G-S_0) + p_1(G-S_0) - 1}.$$

So

$$3|S_0| \ge 5p_0(G-S_0) + 2p_1(G-S_0) - 1,$$

that is,

$$\begin{aligned} 2|S_0| &\ge \frac{10}{3}p_0(G-S_0) + \frac{4}{3}p_1(G-S_0) - \frac{2}{3} \\ &= 2p_0(G-S_0) + p_1(G-S_0) + \frac{4}{3}p_0(G-S_0) \\ &\quad + \frac{1}{3}p_1(G-S_0) - \frac{2}{3} \\ &\ge 2p_0(G-S_0) + p_1(G-S_0) + 2 - \frac{2}{3}. \end{aligned}$$

By the integrity of $|S_0|$, $p_0(G-S_0)$ and $p_1(G-S_0)$, we have

$$\begin{aligned} 2|S_0| &\ge 2p_0(G-S_0) + p_1(G-S_0) + 2 \\ &\ge 2p_0(G-S_0) + p_1(G-S_0) + \varepsilon(S_0, T), \end{aligned}$$

this contradicts (1).

Case 3. $T_1 \ne \emptyset$ and $T_1 \ne N_{G-S_0}(T_1)$

There exists $u \in N_{G-S_0}(T_1) \setminus T_1$, such that $d_{G-S_0}(u) \ge 2$. Let r be the edge number in $[T_1]_{G-S_0}$, we write $X = S_0 \cup (N_{G-S_0}(T_1) - T_1 - u)$. By the definition of T_1, we can easily obtain

$$|X| \le |S_0| + p_1(G-S_0) - 2r - 1$$

and

$$\begin{aligned} i(G-X) &\ge i(G-S_0) + (p_1(G-S_0) - 2r) + r \\ &= P_0(G-S_0) + p_1(G-S_0) - r \\ &\ge p_0(G-S_0) + \frac{1}{2}p_1(G-S_0) > 0. \end{aligned}$$

We write Y for the set of isolated vertices of $G-X$, then $|Y| = i(G-X)$ and $|N_G(Y)| \le |X| \le |S_0| + p_1(G-S_0) - 2r - 1$. Thus

$$\begin{aligned} \frac{5}{3} &< bind(G) \le \frac{|N_G(Y)|}{|Y|} \\ &\le \frac{|S_0| + p_1(G-S_0) - 2r - 1}{p_0(G-S_0) + p_1(G-S_0) - r} \\ &\le \frac{|S_0| + p_1(G-S_0) - 1}{p_0(G-S_0) + p_1(G-S_0)}. \end{aligned}$$

So

$$3|S_0| \ge 5p_0(G-S_0) + 2p_1(G-S_0) + 3,$$

that is,

$$\begin{aligned} 2|S_0| &\ge \frac{10}{3}p_0(G-S_0) + \frac{4}{3}p_1(G-S_0) + 2 \\ &\ge 2p_0(G-S_0) + p_1(G-S_0) + \varepsilon(S_0, T), \end{aligned}$$

which contradicts (1).

From all the cases above, we deduced the contradiction. Hence, G is fractional 2-covered.

Remark. In the proof of Theorem 1, it is required that $bind(G) > \frac{5}{3}$. But I do not know whether the condition can be placed by $bind(G) \ge \frac{5}{3}$.

Finally we present the following problem.

Problem 1. Find the relationship between binding number and fractional k-covered graph, where k is any positive integer.

Problem 2. Find the relationship between binding number and fractional (g, f)-covered graph, where g and f is two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$.

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