

# Nonlinear Stability of Convection in a Thermally Modulated Anisotropic Porous Medium

M. Meenasaranya, S. Saravanan

**Abstract**—Conditions corresponding to the unconditional stability of convection in a mechanically anisotropic fluid saturated porous medium of infinite horizontal extent are determined. The medium is heated from below and its bounding surfaces are subjected to temperature modulation which consists of a steady part and a time periodic oscillating part. The Brinkman model is employed in the momentum equation with the Boussinesq approximation. The stability region is found for arbitrary values of modulational frequency and amplitude using the energy method. Higher order numerical computations are carried out to find critical boundaries and subcritical instability regions more accurately.

**Keywords**—Convection, porous medium, anisotropy, temperature modulation, nonlinear stability.

## I. INTRODUCTION

THE study of convective flow in modulated fluid layers or fluid saturated porous layers has received considerable attention of researchers because of its occurrence in many physical applications. For example in the solidification process, high quality crystals are produced in the presence of microgravity conditions. It is known that such crystals can be produced on the ground itself by imposing appropriate modulation to the boundary temperature [1]. The problem of determining thermal modulation effect on the Benard problem was first studied by Venezian [2]. A similar system was considered by Rosenblat and Herbert [3] with a thermally modulated lower wall and an isothermal upper wall to obtain asymptotic solutions for arbitrary values of modulation amplitude. This work was further analysed by Rosenblat and Tanaka [4] considering rigid walls and by Yih and Li [5] considering thermal modulation at both walls.

Homsy [6] made a nonlinear stability analysis of a fluid layer heated from below under gravity and surface temperature modulations. An experimental cum analytical study was done by Finucane and Kelly [7] to determine the onset condition for convection in a thermally modulated fluid layer. Later Caltagirone [8] determined the instability region for a thermally modulated densely packed porous medium using the Galerkin method. Chhoun et al. [9] carried out an experimental investigation of the same problem. The linear stability analysis of a sparsely packed porous medium with surface temperature modulation was further extended by Rudraiah et al. [10] for a viscoelastic fluid, by Malashetty et al. [11] for stress free boundaries and by Bhaduria [12] for rigid boundaries. Recently Singh et al. [13] made an energy analysis for a

rotating fluid layer by taking into account sinusoidal surface temperature modulation and Coriolis force.

All practical porous media exhibit a kind of anisotropy in them. Anisotropy, in general, refers to a property which depends on directions. Many stability analyses have been carried out for a horizontal anisotropic porous media. Castinel and Combarous [14] were the first to determine the onset of convection in a horizontal fluid saturated porous layer with anisotropic permeability. Their analysis was extended by Epherre [15] for anisotropic thermal diffusivity. These works and similar later ones were reviewed by Storesletten [16]. Malashetty and Basavaraja [17] made a linear stability analysis of double diffusive convection in a horizontal anisotropic porous medium subjected to thermal modulation effect.

Linear and nonlinear stabilities of a horizontal anisotropic fluid saturated porous medium with permeability depending on the vertical direction were studied by Capone et al. ([18], [19]). They considered the effects of different factors such as internal heat generation, double diffusion and mass throughflow. Saravanan and Sivakumar [20] determined the instability region corresponding to a gravity modulated fluid saturated anisotropic porous medium heated from below or from above. A weakly nonlinear stability analysis was made by Siddheshwar et al. [21] to determine the effect of variable viscosity in a gravity modulated, horizontal and anisotropic porous medium. In addition, Bhaduria and Kiran [22] found weakly nonlinear stability results for a thermally modulated, horizontal and anisotropic porous medium saturated with a viscoelastic fluid.

It is known from the literature that nonlinear stability results are available for thermally modulated fluid layers ([6], [13]) whereas only linear and weakly nonlinear stability analyses have been carried out for thermally modulated porous media with anisotropy ([17], [22]). Hence an attempt is made here to determine the effect of surface temperature modulation on convection in an anisotropic fluid saturated porous medium using nonlinear analysis. In this study we consider a more general Brinkman model.

## II. MATHEMATICAL FORMULATION

We consider a three dimensional, infinite and horizontal porous medium of height  $d$ , placed between two walls  $z = 0$  and  $z = d$ . It is saturated by a Newtonian, incompressible and viscous fluid with  $x, y$  coordinates in the horizontal directions. The gravity is acting downwards. The medium is heated from below and its wall temperatures are subjected to time periodic modulations. The density of the fluid is assumed to change with temperature, following Boussinesq approximation.

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The dimensional governing equations for the considered system take the form

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{q} - \frac{\nu}{\bar{k}} \vec{q} + \alpha T g_0 \vec{k} \quad (1)$$

$$\nabla \cdot \vec{q} = 0 \quad (2)$$

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla) T = \chi \nabla^2 T \quad (3)$$

with the boundary conditions

$$w = \frac{\partial^2 w}{\partial z^2} = 0 \quad \text{at } z = 0, d \quad (4)$$

$$T = T_r + \Delta T(1 + a \cos(\Omega t))/2 \quad \text{at } z = 0 \quad (5)$$

$$T = T_r - \Delta T(1 - a \cos(\Omega t + \phi))/2 \quad \text{at } z = d \quad (6)$$

where  $\vec{q} = (u, v, w)$ ,  $t$ ,  $\rho$ ,  $p$ ,  $\nu$ ,  $\alpha$ ,  $g_0 \vec{k}$ ,  $\chi$ ,  $T$ ,  $T_r$ ,  $\Delta T$ ,  $a$ ,  $\Omega$  and  $\phi$  represent the velocity, time, density, pressure, kinematic viscosity, coefficient of thermal expansion, reference gravity, thermal diffusivity, temperature, reference temperature, temperature difference between the walls, modulational amplitude, modulational frequency and phase angle respectively. The permeability of the medium is taken in the form

$$\bar{k} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_x & 0 \\ 0 & 0 & k_z \end{bmatrix}$$

and this assumption makes the system an anisotropic one.

We try to find solutions for the equations (1)-(3) of the form  $\vec{q} = \vec{q}_b = (0, 0, 0)$ ,  $T = T_b(z, t)$  and  $p = p_b(z, t)$ . This leads to

$$T_b(z, t) = T_r + \frac{\Delta T}{2d}(d - 2z) + a \operatorname{Re}[e^{-i\Omega t} \{a(\beta)e^{\beta z/d} + a(-\beta)e^{-\beta z/d}\}]$$

with  $a(\beta) = \frac{\Delta T(e^{-i\phi} - e^{-\beta})}{(2(e^\beta - e^{-\beta}))}$  and  $\beta = (1 - i)\sqrt{\Omega d^2/(2\chi)}$ . The hydrostatic pressure depends on the buoyancy force. Applying the perturbation  $(\vec{q}', \theta, p')$  to velocity, temperature and pressure in the above basic state solution and using the reference values  $d, \chi/d, d^2/\chi, \rho\chi\nu/d^2, \Delta T$  and  $\chi/d^2$  for length, velocity, time, pressure, temperature and modulation frequency result in the following equations for perturbations (primes are removed):

$$\frac{1}{Pr} \left[ \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} \right] = -\nabla p + \nabla^2 \vec{q} - \frac{1}{Da} \vec{q}_a + Ra \theta \vec{k} \quad (7)$$

$$\nabla \cdot \vec{q} = 0 \quad (8)$$

$$\frac{\partial \theta}{\partial t} = \nabla^2 \theta - w \frac{\partial T_b}{\partial z} - (\vec{q} \cdot \nabla) \theta \quad (9)$$

where

$$\partial T_b / \partial z = -1 + af(z, t)$$

with  $f(z, t) = \operatorname{Re}[e^{-i\omega t} \{A(\beta)e^{\beta z} + A(-\beta)e^{-\beta z}\}]$ ,  $A(\beta) = \beta(e^{-i\phi} - e^{-\beta})/(2(e^\beta - e^{-\beta}))$ ,  $\beta = (1 - i)\sqrt{\omega/2}$  and  $\omega$  denoting the non-dimensional modulational frequency. Here  $Pr = \nu/\chi$  represents the Prandtl number,  $Da = k_z/d^2$  the Darcy number,  $k_r = k_x/k_z$  the anisotropy parameter,  $\vec{q}_a = (q_1/k_r, q_2/k_r, q_3)$  the anisotropy modified velocity and  $Ra = \alpha g \Delta T d^3/(\nu\chi)$  the Rayleigh number.

### III. ENERGY ANALYSIS

In order to analyse stability of the basic state identified earlier, we define energy evolution function in the form

$$E = \frac{\langle \vec{q}^2 \rangle}{2Pr} + \frac{\langle \Phi^2 \rangle}{2}$$

where  $\Phi = \sqrt{Ra\lambda} \theta$ . Here  $\lambda$  represents the coupling parameter and  $\langle \cdot \rangle$  denotes the  $L^2$ -inner product over the volume  $V$  of porous medium. Employing (7)-(9) in the above function results in

$$\frac{dE}{dt} = \sqrt{Ra} I(t) - D \quad (10)$$

with

$$I(t) = \left\langle \left( \frac{1}{\sqrt{\lambda}} - \sqrt{\lambda} \frac{\partial T_b}{\partial z} \right) w \Phi \right\rangle,$$

$$D = \langle \nabla \vec{q} : \nabla \vec{q} \rangle + \frac{1}{Da} \langle \vec{q} \cdot \vec{q}_a \rangle + \langle |\nabla \Phi|^2 \rangle.$$

The equation (10) together with the isoperimetric inequality  $D \geq \zeta^2 E$ , where  $\zeta$  is a constant, imply

$$\begin{aligned} \frac{dE}{dt} &\leq \zeta^2 E \left( -1 + \frac{\sqrt{Ra}}{\rho} \right) \\ &\leq \zeta^2 E \left( -1 + \frac{\sqrt{Ra}}{Ra_N} \right) \end{aligned}$$

Here  $\frac{1}{\rho} = \max \frac{I(t)}{H \frac{D}{D}}$ ,  $H$  the space of admissible functions and  $Ra_N = \min_{t \in [0, 2\pi/\omega]} \rho$ . We notice that the minimization is done over a period.

If  $\sqrt{Ra} \leq Ra_N$  we notice exponential decay of  $E$  for all disturbances irrespective of their magnitudes. Thus we end up with the following Euler-Lagrange equations for the variational problem under consideration:

$$\nabla^2 \vec{q} - \frac{1}{Da} \vec{q}_a + \frac{\rho}{2} F(z, t) \vec{\Phi} = \nabla \psi \quad (11)$$

$$\nabla^2 \Phi + \frac{\rho}{2} F(z, t) w = 0 \quad (12)$$

where  $F(z, t) = (1 + \lambda - \lambda af(z, t))/\sqrt{\lambda}$  and  $\psi$  denotes a multiplier. After curling the equation (11) twice and introducing the normal mode expansions

$$w(x, y, z, t) = w(z, t) e^{i(\alpha_x x + \alpha_y y)}$$

$$\Phi(x, y, z, t) = \Phi(z, t) e^{i(\alpha_x x + \alpha_y y)},$$

one can arrive at

$$(D^2 - \alpha^2)^2 w - \frac{1}{Da} \left( \frac{1}{k_r} D^2 - \alpha^2 \right) w = \frac{\rho}{2} \alpha^2 F(z, t) \Phi \quad (13)$$

$$(D^2 - \alpha^2) \Phi = -\frac{\rho}{2} F(z, t) w \quad (14)$$

where  $D \equiv \partial/\partial z$  and  $\alpha^2 = \alpha_x^2 + \alpha_y^2$  is the wave number. These equations need to be solved with relevant conditions. Stress free conditions are considered at the walls.

These equations (13) & (14) are solved using the standard Galerkin method which is a simple and powerful method. We choose approximations to the perturbed quantities in the form

$$w(z, t) = \sum_{m=1}^n A_m(t) \sin(m\pi z)$$

$$\Phi(z, t) = \sum_{m=1}^n B_m(t) \sin(m\pi z)$$

where  $A_m$  and  $B_m$  are coefficients to be found. We substitute these in (13) & (14) and multiply them by each trial function and then integrate them over  $V$ . This results in an eigenvalue problem with  $\rho$  as the eigenvalue, depending on the parameters  $t, \alpha$  and  $\lambda$ . By fixing numerical values to these parameters, we obtain an eigenvalue problem of order  $2n$ . The eigenvalues are all found to be real and the eigenvalue nearest to zero is assigned for  $\rho$ . Consequently, the nonlinear stability limit is defined via the following optimization

$$Ra_N = \min_t \min_{\alpha} \max_{\lambda} \rho(\lambda, t, \alpha). \quad (15)$$

#### IV. RESULTS AND DISCUSSION

An anisotropic fluid saturated porous medium heated from below with thermally modulated walls is examined using nonlinear analysis. Figs. 2-5 provide the variation of critical Darcy-Rayleigh number, denoted by  $R_{N,cr} (= Ra_N^2 * Da)$ , against thermal modulation frequency  $\omega$  for different values of  $k_r$  and  $Da$ . The region lying below the continuous curve corresponds to the global stability region and represents conduction state for respective parameters. The critical wave number ( $\alpha_{N,cr}$ ), defined as the value of  $\alpha$  at which  $Ra_N$  is attained, is given in Table I for corresponding parameters. We assign  $k_r = 0.1, 10$  [20] and  $Da = 10^{-4}$  and  $10^{-1}$  to represent the Darcy and Brinkman models respectively [23].

It was sufficient to vary the coupling parameter and time in the intervals (0,5) and (0,10) respectively to determine the critical values. In order to maintain the error percentage of the results between those of  $n$  and  $n+1$  trial functions well within 1%, we fixed  $n = 10$  uniformly throughout the computations.

TABLE I: UNMODULATED RESULTS WITH  $a = 0$

Da = 1				
$k_r$	$R_{L,cr}$	$\alpha_{L,cr}$	$R_{N,cr}$	$\alpha_{N,cr}$
0.1	939.61	2.58	939.61	2.58
0.5	730.30	2.30	730.30	2.30
$k_r = 0.1$				
Da	$R_{L,cr}$	$\alpha_{L,cr}$	$R_{N,cr}$	$\alpha_{N,cr}$
0.001	173.15	5.50	173.15	5.50
1.0	939.61	2.58	939.61	2.58
10.0	6881.98	2.29	6881.98	2.29

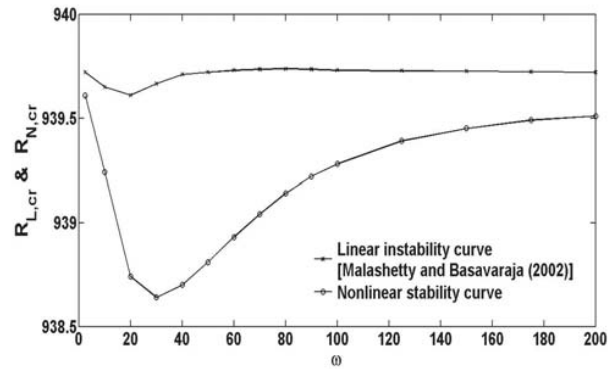


Fig. 1 (a)  $R_{N,cr}$  &  $R_{L,cr}$  against  $\omega$  with  $a = 0.1$ ,  $Da = 1, k_r = 0.1$  for symmetric modulation

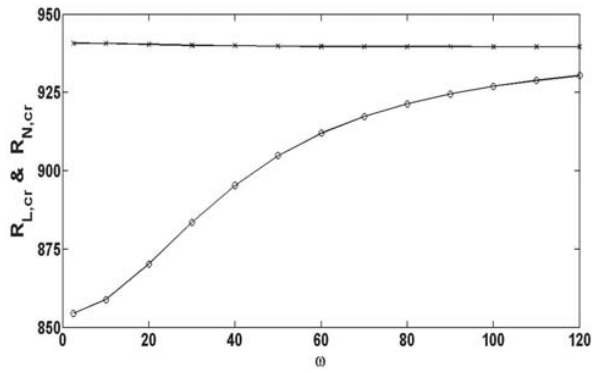


Fig. 1 (b)  $R_{N,cr}$  &  $R_{L,cr}$  against  $\omega$  with  $a = 0.1$ ,  $Da = 1, k_r = 0.1$  for asymmetric modulation

The entries in Table I for the unmodulated system and the curves in Fig. 1 for the modulated system clearly show that the nonlinear boundaries lie below the linear ones as anticipated. The linear results,  $R_{L,cr}$  and  $\alpha_{L,cr}$  taken for comparison from the literature, were obtained using the perturbation method [17]. The region between  $R_{L,cr}$  and  $R_{N,cr}$  represent possible subcritical instability for the system under consideration and a similar result was obtained by Homsy [6] for a fluid layer.

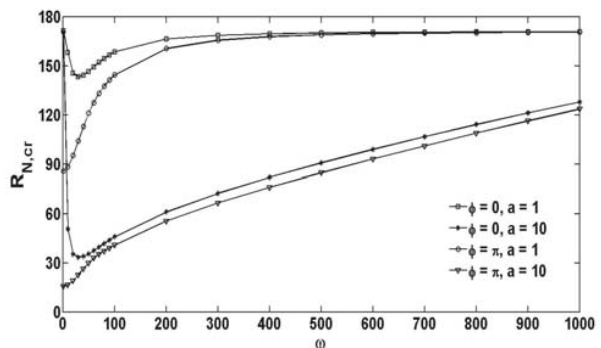
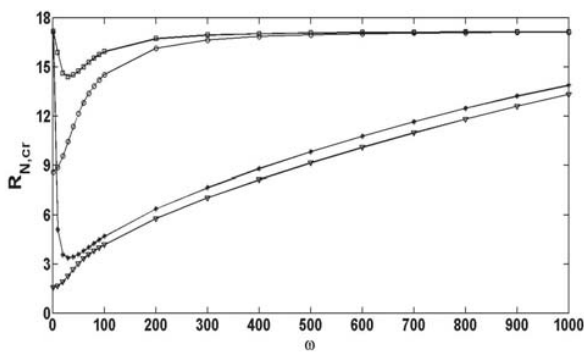
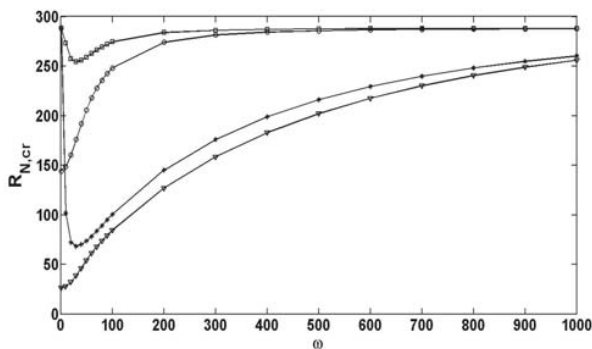
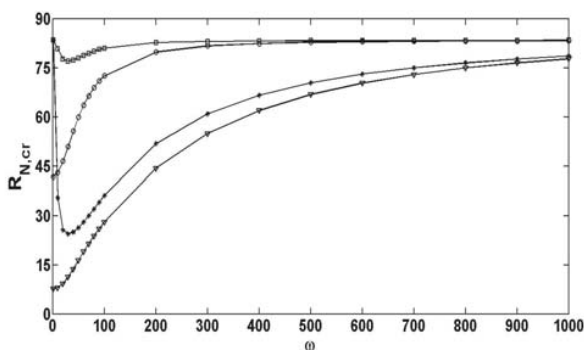


Fig. 2  $R_{N,cr}$  against  $\omega$  with  $Da = 0.0001, k_r = 0.1$

Fig. 3  $R_{N,cr}$  against  $\omega$  with  $Da = 0.0001$ ,  $k_r = 10$ Fig. 4  $R_{N,cr}$  against  $\omega$  with  $Da = 0.1$ ,  $k_r = 0.1$ Fig. 5  $R_{N,cr}$  against  $\omega$  with  $Da = 0.1$ ,  $k_r = 10$ 

When  $\phi = 0$ , an increase in the modulatory frequency destabilizes the system. The maximum destabilization is found to occur at  $\omega = 30$ . The system is stabilized against frequency beyond  $\omega = 30$ . On the other hand when  $\phi = \pi$ , the response is unique. An increase in the frequency causes a stabilizing effect. The fluid particles in the system do not respond to external modulation for sufficiently high frequencies. Thus  $R_{N,cr}$  reaches the unmodulated result as  $\omega \rightarrow \infty$ . Moreover we find that the onset of convection is advanced as the modulatory amplitude is increased. Also an increase in the anisotropy parameter is found to destabilize the system. This behaviour is in consistent with that of [17].

One may also notice from the critical boundaries that the amount of heat required for the onset of buoyancy driven

convection is higher in the case of the Brinkman model. This may be due to the existence of boundary layer near the walls with thickness proportional to the square root of Darcy value. The fluid particles in the boundary layer effectively withstand increased resistance which may otherwise cause deformation. However the stability curves are qualitatively similar for both Darcy and Brinkman models. In general the unmodulated results are attained little faster for the symmetric modulation compared to the asymmetric one.

## V. CONCLUSION

Nonlinear stability limits corresponding to convection in a thermally modulated and anisotropic porous medium are obtained using the energy method. The modulational frequency produces two different effects in the case of symmetric modulation whereas it always has a stabilizing effect in the case of asymmetric modulation. However an increase in the modulatory amplitude and anisotropy parameter always produce destabilizing effect. Thus the stability of the system can be increased or decreased by choosing appropriate amplitude and frequency, depending on the extent of anisotropy.

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