International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:9, 2012

New Laguerre's Type Method for Solving of a Polynomial Equations Systems

Oleksandr Poliakov, Yevgen Pashkov, Marina Kolesova, Olena Chepenyuk, Mykhaylo Kalinin, Vadym Kramar

***Abstract*—**In this paper we present a substantiation of a new Laguerre's type iterative method for solving of a nonlinear polynomial equations systems with real coefficients. The problems of its implementation, including relating to the structural choice of initial approximations, were considered. Test examples demonstrate the effectiveness of the method at the solving of many practical problems solving.

***Keywords*—**Iterative method, Laguerre's method, Newton's method, polynomial equation, system of equations

I. Introduction

It is well known that the problem of roots finding of a nonlinear equations and their systems attracted the attention of researchers for several centuries, so a variety of methods of its solution were developed and published in the scientific literature. Despite this, it remains one of the most important tasks of computational mathematics, due to the necessity of solving a large number of applications, whose models are presented by systems of nonlinear equations.

A special case of nonlinear systems are nonlinear systems of polynomial equations, which solving algorithms substantiated and investigated in most detail. Nevertheless, a general method for such systems solving, which could be considered universal for most practical problems, has not been developed yet. This is a motivation to search for new algorithms that are adapted, at least for typical applications.

For example, mathematical models of many problems in kinematics and dynamics of multilink mechanisms with a finite number of degrees of freedom, in the numerical solution of which the most commonly used algorithms of Newton-Raphson or derivatives [1] - [7] etc., can be reduced to the systems of a nonlinear polynomial equations. But it is known that these algorithms in some cases may not be effective enough.

O. Poliakov is with the Sevastopol National Technical University, Laboratory of Biomechanics, Sevastopol, 99053 Ukraine (Corresponding Author Work Phone: +38-0692-435-161; e-mail: alex_polyakov@inbox.ru).
Y. Pashkov is with the Sevastopol National Technical University, Sevastopol, 99053 Ukraine.
M. Kolesova is with the Sevastopol National Technical University, Sevastopol, 99053, Ukraine.
O. Chepenyuk is with the Sevastopol National Technical University, Sevastopol, 99053 Ukraine.
M. Kalinin is with the Sevastopol National Technical University, Laboratory of biomechanics, Sevastopol, 99053 Ukraine.
V. Kramar is with the Sevastopol National Technical University, Sevastopol, 99053 Ukraine.

The main aim of this work is to develop a unified solution algorithm of polynomial equations and systems, effective in solving the problems of analysis and synthesis of multilink mechanisms with an absolute rigid or elastic links.

In [8] is presented a process of withdrawal of the iterative formula for the ε-estimates search of the real roots of polynomial equations of finite degree. It turns out that a constructive approach used for it, based on the continuation by the parameter [9], leads to one of the Laguerre's type Hansen-Patrick family formulas [10], which can be written as follows:

$$r_{k+1} = r_k + \frac{p_n(r_k)}{(n-1)\cdot\dfrac{p_n(r_k)}{r_k} - p_n^{'}(r_k)}$$

or

$$r_{k+1} = r_k \frac{\sum_{i=0}^{n}(n-i)\cdot a_{n-i}\cdot r_k^i}{\sum_{i=0}^{n}(n-i-1)\cdot a_{n-i}\cdot r_k^i}\ , \qquad (1)$$

where a_i, $i=\overline{0,n}$ - the real coefficients of an n- degree polynomial equation $p_n(x)=0$; r_k- ε- estimation of the real root of the equation by k- iteration. As shown by O. Tikhonov, a formula of the form (1) is most effective compared with others in selecting a relatively large $|r|$ [11]. However, the results of a set of numerical experiments and solutions of the practical problems demonstrate its effectiveness in other cases [12].

Later it was shown that generalization of (1) for the system of nonlinear polynomial equations is possible. In [13] has been proved a new iterative method for finding the ε- estimates vector of the real roots of a finite system of polynomial equations with real coefficients, and also a comparative analysis with the Newton's method is presented. This paper presents a generalization of the method and discusses its practical implementation.

II. Iterative Laguerre's Type Method for Solving of a Polynomial Equations Systems

A. Substantiation of the method

In the derivation of the iterative formula (1) were used an LRP-polynomials of the form

$$p_n(\nu,x) = \sum_{i=0}^{n} a_{n-i}x^i - \nu\sum_{i=0}^{n}(n-i)a_{n-i}x^i = p_n(x) - \nu g_{n-1}(x),$$

where ν - is a some real parameter [14]. The value of ν changes during the iterative process and is determined from the condition $p_n(\nu, r_k) = 0$, i.e. as

$$\nu_{k+1} = \frac{p_n(r_k)}{g_{n-1}(r_k)}.$$

Thus, if the iterative process converges, then $\lim_{k\to\infty} p_n(\nu_k, x) = p_n(0, x) = p_n(x)$ and, therefore, at some iteration number K when $\nu_K \le \varepsilon$ we get r_K - ε - estimation of the real roots of the equation $p_n(x) = 0$ [14].

Now consider a multivariate polynomial

$$P_S^i(x_1,\dots,x_m) = \sum_{k_1=0}^{n_1} \dots \sum_{k_m=0}^{n_m} a_{n_1-k_1,\dots,n_m-k_m} x_1^{k_1} \cdot \dots \cdot x_m^{k_m}, \quad (2)$$

from the polynomial ring $R[x_1,\dots,x_m]$ with m variables over the field $\mathbb{R}$: $a_{n_1-k_1,\dots,n_m-k_m} \in \mathbb{R}$; $n_1,\dots,n_m \in \mathbb{N}$, where $S = \deg(P_S^i)$ - the degree of the polynomial, and the corresponding polynomial equation

$$P_S^i(x_1,\dots,x_m) = 0. \quad (3)$$

If any m- dimensional vector $R^0 = (r_1^0,\dots,r_m^0)^T$, $r_1^0,\dots,r_m^0 \in \mathbb{R}$ is not a solution of equation (3), then, assume that the solution is m- dimensional vector

$$R^1 = \left(\frac{r_1^0}{1-\nu_1^0},\dots,\frac{r_m^0}{1-\nu_m^0}\right)^T, \quad (4)$$

where { $\nu_j^0 \in \mathbb{R}$: $\nu_j^0 \ne 1$ }, $j = 1,\dots,m$ - is the components of the errors vector $E^0 = (\nu_1^0,\dots,\nu_m^0)^T$. Then

$$\mathrm{P}_1^i = P_S^i\left(\frac{r_1^0}{1-\nu_1^0},\dots,\frac{r_m^0}{1-\nu_m^0}\right) \le \delta_0 \approx 0,$$

that allows us to write

$$P_S^i(\nu_1^0,\dots,\nu_m^0, r_1^0,\dots,r_m^0) = P_S^i(r_1^0,\dots,r_m^0) - G_{S-1}^i E^0 \approx 0, \quad (5)$$

where $G_{S-1}^i = (g_1^i,\dots,g_m^i)$ - a row vector whose components are values of polynomials g_j^i, $j = 1,\dots,m$, of degrees $s_j = \deg(g_j^i) \le S-1$ in the point R^0. Polynomials g_j^i are obtained in a natural way as a coefficients at ν_j^0 in the polynomial (2) after the substitution of the components of (4), if we neglect the multiplications $\nu_j^0 \nu_k^0, \forall j,k = 1,\dots,m$ and to consider that $\nu_j^0 \ne 1$.

This procedure allows us to construct an algorithm of the iterative process for solving of the nonlinear polynomial equations systems.

Consider a finite system of equations (3)

$$\begin{cases} P_{S_1}^1(x_1,\dots,x_m) = 0 \\ \dots \\ P_{S_m}^m(x_1,\dots,x_m) = 0 \end{cases}. \quad (6)$$

Applying the method of obtaining the expression (5), we get (in general) a system of m inhomogeneous linear algebraic equations

$$\begin{cases} P_{S_1}^1(r_1^0,\dots,r_m^0) - G_{S-1}^1 E^0 = 0 \\ \dots \\ P_{S_m}^m(r_1^0,\dots,r_m^0) - G_{S-1}^m E^0 = 0 \end{cases}. \quad (7)$$

In matrix form the system (7) has the kind

$$\mathrm{P}_S^0 - \Gamma_{S-1}^0 \mathrm{E}^0 = 0, \quad (8)$$

where

$S = \max(S_j)$; $\mathrm{P}_S^0 = (P_{S_1}^1,\dots,P_{S_m}^m)^T\Big|_{R^0}$; $\Gamma_{S-1}^0 = (G_{S-1}^1,\dots,G_{S-1}^m)^T\Big|_{R^0}$; $\mathrm{E}^0 = (\nu_1^0,\dots,\nu_m^0)^T$.

Assuming that $\det \Gamma_{S-1}^0 \ne 0$, we multiply (8) on $(\Gamma_{S-1}^0)^{-1}$. As a result, we obtain

$$\mathrm{E}^0 = (\Gamma_{S-1}^0)^{-1} \mathrm{P}_S^0. \quad (9)$$

If $\|\mathrm{E}^0\| = \sqrt{(\nu_1^0)^2 + \dots + (\nu_m^0)^2} \le \varepsilon$, then $R^0 = (r_1^0,\dots,r_m^0)^T$ - ε - p solution of (6). Otherwise, accept $R^1 = \left(\frac{r_1^0}{1-\nu_1^0},\dots,\frac{r_m^0}{1-\nu_m^0}\right)^T$ and move on to the next iteration. The iterative process continues as long as at the iteration number K the following condition will not be satisfied

$$\|\mathrm{E}^K\| = \sqrt{(\nu_1^K)^2 + \dots + (\nu_m^K)^2} \le \varepsilon \quad (10)$$

or until the number of iterations does not exceed the permissible value $K_{\max}$.

Let $I_{m\times m}$- identity matrix of the size $m \times m$, and $I_{m\times 1} = (1,1,\dots,1)$ - column matrix of the $m \times 1$ of single elements. Then, using (4) and (9), identifying m- dimensional vectors

$R^k = (r_1^k,\dots,r_m^k)^T$, $\mathrm{E}^k = (\Gamma_{S-1}^k)^{-1} \mathrm{P}_S^k = (\nu_1^k,\dots,\nu_m^k)^T$:{ $k \in \mathbb{N}_0$: $k \le K_{\max}$ } with column matrices of the size $m \times 1$, we can write

$$R^{k+1} = I_{m\times m}(I_{m\times 1} - \mathrm{E}^k)^{-1} R^k, \quad (11)$$

or

$$R^{k+1} = I_{m\times m}\left[I_{m\times 1} - \left(\Gamma_{S-1}^{k}\right)^{-1} \mathrm{P}_S^k\right]^{-1} R^k . \qquad (11a)$$

It is easy to see that when $m = 1$ the iterative formula (11 a) is transformed to (1) and, therefore, belongs to the Laguerre's type family of formulas.

Considering that $R^k = I_{m\times m}\left[I_{m\times 1} - \left(\Gamma_{S-1}^{k-1}\right)^{-1} \mathrm{P}_S^{k-1}\right] R^{k-1}$ and so on, we obtain

$$R^{k+1} = I_{m\times m}\left[I_{m\times 1} - \left(\Gamma_{S-1}^{k}\right)^{-1} \mathrm{P}_S^k\right]^{-1} \cdot \ldots \cdot I_{m\times m}\left[I_{m\times 1} - \left(\Gamma_{S-1}^{0}\right)^{-1} \mathrm{P}_S^0\right]^{-1} R^0 ,$$

from which it follows that the iterative process (11) is convergent, if for some K : $R^{K+1} = R^K$, i.e.

$$\exists K \in \mathbb{N}: \lim_{k\to K}\left(\Gamma_{S-1}^{k}\right)^{-1} \mathrm{P}_S^k = 0 \text{ or } \left\|\mathrm{E}^k\right\| \to 0 .$$

If $m = 1$ this condition is as follows:

$$r_{k+1} = \frac{1}{1-\nu_k} r_k = \frac{1}{(1-\nu_k)\cdot\ldots\cdot(1-\nu_0)} r_0$$

and, if we consider that $\nu_k = \dfrac{p_n(r_k)}{g_{n-1}(r_k)}$, then

$$\exists K \in \mathbb{N}: \lim_{k\to K} \frac{p_n(r_k)}{g_{n-1}(r_k)} = 0 \text{ or } |\nu_k| \to 0 .$$

Define in the metric space $\mathbb{R}^m$ a multimetric

$$Dist\left(R^1, R^2\right) = \left(dist\left(R_1^1, R_1^2\right), \ldots, dist\left(R_m^1, R_m^2\right)\right)^T \in \mathbb{R}_+^m .$$

Definition The mapping $F: \ X \to X$ of the multivariate space X with multimetric $Dist: X \to \mathbb{R}_+^m$ is called L-compressing if there exists a nonnegative Lipchitz matrix L of the size $m \times m$ with spectral radius $\rho(L) < 1$, such that the following inequality holds

$$Dist\left(F\left(R^1\right), F\left(R^2\right)\right) \le L \cdot Dist\left(R^1, R^2\right).$$

Fixed point Schroder's theorem [9] Let the mapping $F: \mathbb{R}^m \supseteq X \to \mathbb{R}^m$ is compressing on a closed subset X of $\mathbb{R}^m$ space with multimetric $Dist$. Then, for any R^0 sequence of iterations $R^{k+1} = F\left(R^k\right), \quad k = 0,1,2,...$ converges to the unique fixed point R^* of the mapping F in X and we have the estimation

$$Dist\left(R^k, R^*\right) \le (I - L)^{-1} L \cdot Dist\left(R^k, R^{k-1}\right).$$

Thus, not for all R^0 (in the case of one dimension - r_0) the iterative process (11) is convergent. To ensure convergence it is necessary that the mapping (11) was compressing and the Schroeder's theorem about the fixed point was held.

B. Test examples

In order to evaluate the iterative processes implemented in accordance with (11), was made their analysis when searching for ε-solutions of various systems of polynomial equations. As a test cases several systems that are characterized by specific features that were previously used in [15] for similar purposes were chosen. In this section, these examples are used to assess the functional properties of the iterative formula (11). Fig. 5.1 presents illustrations showing the dynamics of convergent iterative processes for the chosen (in general) arbitrary initial approximations.

1. Consider the system of equations

$$\begin{cases} 4x_1^3 - 3x_1 - x_2 = 0 \\ x_1^2 - x_2 = 0 \end{cases}, \qquad (12)$$

which has three real solutions: $[0,0], [-0.75, 0.5625], [1.0, 1.0]$.

Let's represent the iterative process (11) in detail, considering that the Jacobian of the system $J = 12x_1^2 - 2x_1 - 3$ goes to zero at $x_1 = x_{J1} = -0.4236$ and $x_1 = x_{J2} = 0.5902$.

For arbitrary x_1 and x_2 the system (8) is as follows:

$$\begin{pmatrix} 4x_1^3 - 3x_1 - x_2 \\ x_1^2 - x_2 \end{pmatrix} + \begin{pmatrix} 6x_1 + 3x_2 & -4x_1^3 + 3x_1 \\ 2x_2 & -x_1^2 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (13)$$

The solution of (13) is a vector

$$\mathrm{E} = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \dfrac{x_2\left(4x_1^2 - x_1 - 3\right)}{2x_1^2\left(4x_2 - 3\right) - 3x_1\left(2x_1 + x_2\right)} \\ \dfrac{2x_1^3\left(4x_2 - 3\right) + x_2\left(x_2 - 3x_1^2\right)}{x_1\left[2x_1^2\left(4x_2 - 3\right) - 3x_1\left(2x_1 + x_2\right)\right]} \end{pmatrix} .$$

Thus, the mapping $R = I_{2\times 2}\left(I_{2\times 1} - E\right)^{-1} R$ can be represented as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dfrac{x_1\left[2x_1^2\left(4x_2 - 3\right) - 3x_2\left(x_1 + 2\right)\right]}{2x_1^2\left(2x_2 - 3\right) - x_2\left(2x_1 + 3\right)} \\ -\dfrac{x_1\left[2x_1^2\left(4x_2 - 3\right) - 3x_2\left(x_1 + 2\right)\right]}{6x_1 + x_2} \end{pmatrix} \qquad (14)$$

It is easy to see that system (14) is an identity in two cases: 1) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.75 \\ 0.5625 \end{pmatrix}$; 2) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}$, that is when x_1 and x_2 both are non-zero solutions of (12). In the case $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the system (14) is not defined.

Therefore, the zero solution cannot be obtained by an iterative process (11).

As an initial approximation for finding ε - solutions of (12) the numbers $x_1^{\ 0} = x_{J1} = -0.4236$ and $x_2^{\ 0} = 1.5$ were chosen. That is, considered one of the cases, when $J = 0$.

An iterative process was converging and on the seventh iteration the ε-solution of the system $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -0.75 \\ 0.5625 \end{pmatrix}$ was obtained with the required accuracy (Fig.1).

Similar dynamics of the iterative process was observed when choosing as an initial approximation $x_1^{\ 0} = x_{J2} = 0.5902$, but in this case the second non-zero ε-solution of the system $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.0 \\ 1.0 \end{pmatrix}$ was obtained.

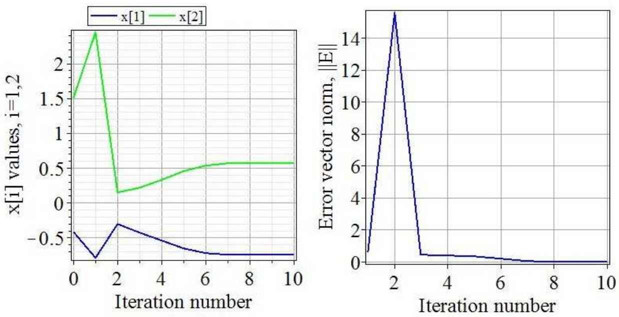

Fig. 1 The dynamics of the iterative process of solving the system of equations (12)

Thus, it is important to note that the iterative processes (11) converge in cases where the Jacobi matrix of the system of equations is ill-conditioned or even singular. This is explained by the structure of formula (11), which special case when $m = 1$ is the formula (1). At the same time, obviously, iterative processes are divergent when $\left\| I_{m\times 1} - \left(\Gamma_{S-1}^{k}\right)^{-1} \mathrm{P}_S^k \right\| \approx 0$. This fact can be used in the design of software modules for solving of the polynomial equations systems.

`

2. Let's now consider the iterative process of solving an almost linear Brown's system of equations

$$\begin{cases} 2x_1 + x_2 + x_3 + x_4 + x_5 = 6 \\ x_1 + 2x_2 + x_3 + x_4 + x_5 = 6 \\ x_1 + x_2 + 2x_3 + x_4 + x_5 = 6 \\ x_1 + x_2 + x_3 + 2x_4 + x_5 = 6 \\ x_1 x_2 x_3 x_4 x_5 = 1 \end{cases} . \tag{15}$$

As an initial approximation was chosen numbers sufficiently large in absolute value for the given system: $x_1^{\ 0} = 8.0$; $x_2^{\ 0} = 6.0$; $x_3^{\ 0} = 4.0$; $x_5^{\ 0} = -2.0$. The dynamics of the iterative process is shown in Fig.2.

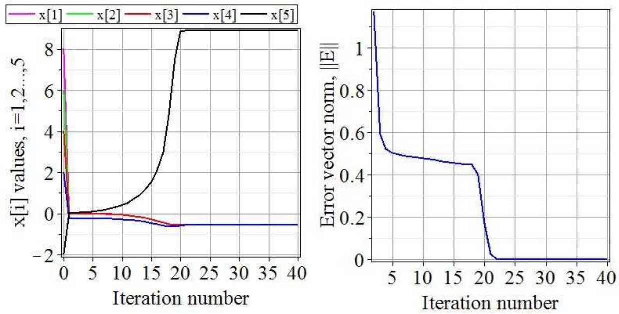

Fig. 2 The dynamics of the iterative process of solving the system of equations (15)

It should be noted that in this example to obtain ε-solutions with the required accuracy a large number of iterations $K = 20$ was taken. However, when choosing the initial approximations in sufficiently small neighborhoods of the roots required solutions were obtained for a small number of iterations.

3. The problem of the intersection of the circles

$$\begin{cases} (x_1 - a_{10})^2 + (x_2 - a_{20})^2 = r^2 \\ (x_1 - b_{10})^2 + (x_2 - 0.5)^2 = (0.5 - b_{10})^2 \end{cases}, \tag{16}$$

where $a_{10} = 100$; $b_{10} = -100$; $a_{20} = 0.5 + (a_{10} - 0.5) tg\,\vartheta$; $\vartheta = 1'$; $r^2 = (0.5 - a_{10})^2 (1 + tg^2 \vartheta)$.

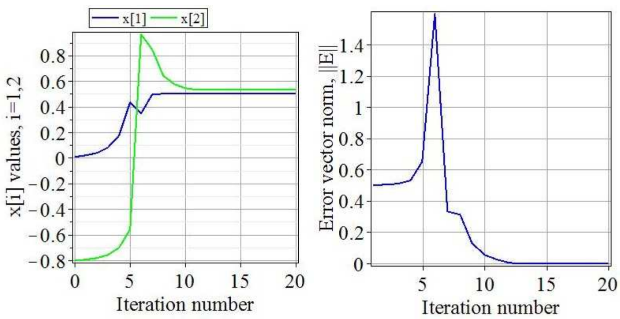

Fig. 3 The dynamics of the iterative process of solving a system of equations (16)

As noted in [15], the process of the roots finding of the system (16) by the method of bisection for the given input data is very difficult. When using the formula (11) with the initial approximations $x_1^{\ 0} = 0.01$, $x_2^{\ 0} = -0.8$, the iterative process was convergent, and the required solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.4999576 \\ 0.52918069 \end{pmatrix}$ was obtained after $K = 12$ iterations. The dynamics of the iterative process is shown in Fig.3.

4. Combustion chemistry problem

This is a real practical problem of a hydrocarbons combustion in the case of excess of fuel.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:9, 2012

Its mathematical model can be represented by a system of four polynomial equations

$$\begin{cases} \alpha_1 x_2 x_4 + \alpha_2 x_2 + \alpha_3 x_1 x_4 + \alpha_4 x_1 + \alpha_5 x_4 = 0 \\ \beta_1 x_2 x_4 + \beta_2 x_1 x_3 + \beta_3 x_1 x_4 + \beta_4 x_3 x_4 + \beta_5 x_3 + \beta_6 x_4 + \beta_7 = 0 \\ x_1^2 - x_2 = 0 \\ x_4^2 - x_3 = 0 \end{cases} \quad (17)$$

where $\alpha_1 = -1.697 \cdot 10^7$; $\alpha_2 = 2.177 \cdot 10^7$; $\alpha_3 = 0.55$; $\alpha_4 = 0.45$; $\alpha_5 = -1.0$; $\beta_1 = 1.585 \cdot 10^{14}$; $\beta_2 = 4.126 \cdot 10^7$; $\beta_3 = -8.285 \cdot 10^6$; $\beta_4 = 2.284 \cdot 10^7$; $\beta_5 = -1.918 \cdot 10^7$; $\beta_6 = 48.4$; $\beta_7 = -27.73$.

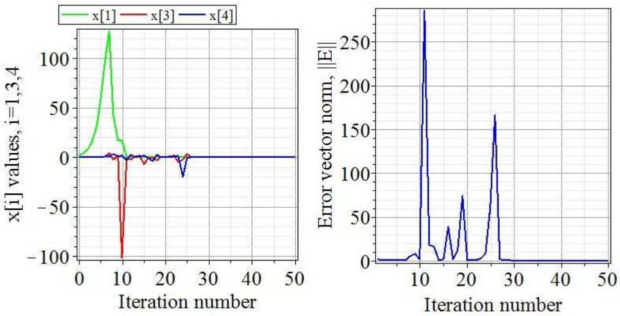

Fig. 4 The dynamics of the iterative process of solving a system of equations (17)

As you can see, the main problems that may arise during the iterative solution of (17) are determined by a significant difference between the values of the coefficients and their absolute values. When using the formula (11), these problems were manifested in the initial iterations (see Fig. 4). Nevertheless, as a result of $K = 28$ iterations the only solution was obtained for the small values of the variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -0.0001588472 \\ 2.523242375 \cdot 10^{-8} \\ 0.1478747693 \\ 0.3845448859 \end{pmatrix}.$$

At the same time as the initial approximations were adopted the following values:

$$x_1^{\ 0} = 2.0: \ x_2^{\ 0} = x_3^{\ 0} = x_4^{\ 0} = 0.01.$$

5. A robot kinematics problem

A mathematical model of the kinematics of the robot manipulator with rotational kinematic pairs represented by a system of eight polynomial equations

$$\begin{cases} \alpha_1 x_1 x_3 + \alpha_2 x_2 x_3 + \alpha_3 x_1 + \alpha_4 x_2 + \alpha_5 x_4 + \alpha_6 x_7 + \alpha_7 = 0 \\ \alpha_8 x_1 x_3 + \alpha_9 x_2 x_3 + \alpha_{10} x_1 + \alpha_{11} x_2 + \alpha_{12} x_4 + \alpha_{13} = 0 \\ \alpha_{14} x_6 x_8 + \alpha_{15} x_1 + \alpha_{16} x_2 = 0 \\ \alpha_{17} x_1 + \alpha_{18} x_2 + \alpha_{19} = 0 \\ x_1^2 + x_2^2 - 1.0 = 0 \\ x_3^2 + x_4^2 - 1.0 = 0 \\ x_5^2 + x_6^2 - 1.0 = 0 \\ x_7^2 + x_8^2 - 1.0 = 0 \end{cases} \quad (18)$$

where $\alpha_1 = 4.731 \cdot 10^{-3}$; $\alpha_2 = -0.3578$; $\alpha_3 = -0.1238$; $\alpha_4 = -1.637 \cdot 10^{-3}$; $\alpha_5 = -0.9338$; $\alpha_5 = -0.9338$; $\alpha_6 = 1.0$; $\alpha_7 = -0.3571$; $\alpha_8 = 0.2238$; $\alpha_9 = 0.7623$; $\alpha_{10} = 0.2638$; $\alpha_{11} = -0.7745 \cdot 10^{-1}$; $\alpha_{12} = -0.6734$; $\alpha_{13} = -0.6022$; $\alpha_{14} = 1.0$; $\alpha_{15} = 0.3578$; $\alpha_{16} = 4.731 \cdot 10^{-3}$; $\alpha_{17} = -0.7623$; $\alpha_{18} = 0.2238$; $\alpha_{19} = 0.3461$.

The system (18) represents the transformed system of four trigonometric equations by introducing new variables $x_1 = \cos\varphi_1$; $x_2 = \sin\varphi_1$,.., $x_7 = \cos\varphi_4$; $x_8 = \sin\varphi_4$ and its supplementation with four trigonometric identities of the form $x_i^2 + x_{i+1}^2 - 1 = 0$, $i = 1,3,5,7$. During its solution the initial approximation were chosen arbitrarily in general, but with the allowable ranges of cos and sin functions values and were taken the following: $x_1^{\ 0} = 0.15$; $x_2^{\ 0} = -0.7$; $x_3^{\ 0} = 0.3$; $x_4^{\ 0} = -0.6$; $x_5^{\ 0} = -0.7$; $x_6^{\ 0} = 0.1$; $x_7^{\ 0} = -0.8$; $x_8^{\ 0} = -0.8$.

For these values turned out that an iterative process converged rapidly (see Fig. 5) and already at $K = 6$ iteration desired ε-solution with the required accuracy has been obtained:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} 0.1644316658 \\ -0.9863884769 \\ 0.2396160174 \\ -0.9708677374 \\ -0.9976353915 \\ 0.06872853938 \\ -0.6155084072 \\ -0.7881303206 \end{pmatrix}.$$

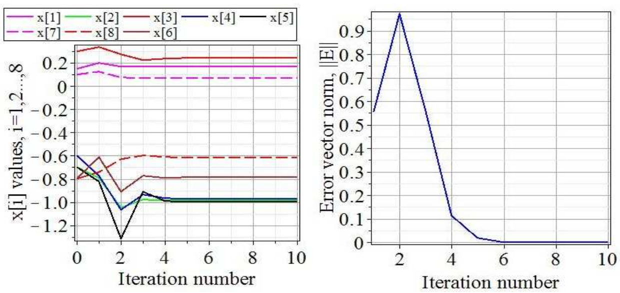

Fig. 5 The dynamics of the iterative process of solving the system of equations (18)

III. Discussion

Analysis of the results of numerical experiments on solving polynomial systems of equations, selected as the test, shows the effectiveness of the iterative search process of ε-solutions vectors in accordance with the formula (11). These iterative processes can be attributed to the Laguerre's type family. But at the same time, they are similar to a family of quasi-Newton's methods. For example, it is easy to see that in the one-dimensional case when $(n-1)\dfrac{P_n(r_k)}{r_k} \to 0$, the formula (1) is a well-known iterative formula of Newton-Raphson. This situation arises in almost every iteration process in the final iteration, when $P_n(r_k) \to 0$ and is particularly evident in cases where the desired real root of the equation is significantly greater than zero. Even more this becomes noticeable in cases, when $\left|(n-1)\dfrac{P_n(r_k)}{r_k}\right| << \left|P_n^{'}(r_k)\right|$. It can be shown that a similar effect is observed in the solution of systems of equations. Consequently, the method proposed above, in a sense has the advantages inherent in Newton's method. But at the same time, the implementation of the method is practically independent of the properties of the Jacobi matrix of the system of equations. As shown by the results of the numerical experiments, the iterative process implemented in accordance with the formula (11), are rapidly converging in cases where the Jacobi matrices are ill-conditioned or even singular. Analysis of the dynamics of iterative processes of the solutions of different systems allows noting that the randomly selected initial approximations can be nonmonotonic, or divergent. However, the iterative processes (11)are strictly monotonic with a successful choice of initial approximations.As we know, this is typical of other methods for solving systems of nonlinear equations.

One of the important advantages of the method is the lack of differentiation operations needed in the calculation of the Jacobi matrix elements, but, as in Newton's method, the iterative process (11) requires a procedure for solving systems of linear equations. In cases of large-scale systems it is a significant disadvantage, so in practical calculations it is desirable to bring the original systems to the systems of smaller dimension. For polynomial systems of equations, there are many algorithms that reduce the dimension, for example, the method of resultant of two polynomials or the method of expanding the system to a Grobner basis [12]. This approach is appropriate, for example, when solving the problems of analysis and synthesis of linkages. At their solution the formula (11) is very effective.

As noted above, the numerical solutions of some systems of equations can be obtained only in the case of right choice of initial approximations. If such a choice is difficult, there is a need for the procedure of localization of the roots in the iterative process, or at least, the definition of their boundaries.

In the one-dimensional case, such procedures are implemented fairly simple, but much more complicated in the localization of the roots in a multidimensional space. Significant advantages can be achieved by using the interval arithmetic operations. Moreover, considering the similarity of the formula (11) and the iterative Newton's formula, we can assume that it is possible to generalize it for the case of spaces of interval variables $\mathbb{IR}$, in which Newton's methods are essential for solving linear and nonlinear equations and systems. Further work in this area will be devoted to solving this particular problem.

References

[1] G.N. Vanderplaats, *Numerical Optimization Techniques for Engineering Design: With Applications*, New York: McGraw Hill, 1984.
[2] R.T. Haftka, and Z. Gurdal, *Elements of Structural Optimization*, Dordrecht: Kluwer Academic Publishers, 1992.
[3] R. Aviles, G. Ajuria, V. Gomez-Garay, and S. Navalpotro, "Comparison Among Nonlinear Optimization Methods for the Static Equilibrium Analysis of Multibody Systems With Rigid And Elastic Elements," *Mechanism and Machine Theory*, vol. 35, no. 8, pp. 1151–1168, August 2000.
[4] Y. Luo, X. Fan, D. Li, and X. Wu, "Hyper-Chaotic Mapping Newton Iterative Method to Mechanism Synthesis," *Journal of Mechanical Engineering*, vol. 54, no. 5, pp. 372-378, 2008.
[5] A. Hernandez, , and V. Petuya, "Position Analysis of Planar Mechanisms With R-Pairs Using a Geometrical–Iterative Method," *Mechanism and Machine Theory*, vol. 39, no. 2, pp. 133–152, February 2004,
[6] U.S. Chavan, and S.V. Joshi, "Synthesis And Analysis of Coupler Curves With Combined Planar Cam Follower Mechanisms," *International Journal of Engineering, Science and Technology*, vol. 2, no. 6, pp. 231–243, 2010.
[7] A.E. Albanesi, V.D. Fachinotti, and M.A. Pucheta, "Review on Design Methods for Compliant Mechanisms," *Mecanica Computacional*, vol. XXIX, pp. 59–72, November 2010.
[8] A. Polyakov, and M. Polyakova, "Recurrent Formula for Finding of a Real Roots of a Nonlinear Algebraic Equations In Application to the Problems of a Mechanisms Mechanics," *New materials and technologies in metallurgy and mechanical engineering*, no. 2, pp. 93–96, 2002.
[9] J.M. Ortega, and W.C. Rheinboldt, *Iterative Solution of nonlinear equation in several variables.* New York - London: Academic Press, 1970.
[10] E. Hansen, and M. Patric, "A Family of Root Finding Methods," *Numerical Mathematics*, vol. 27, no. 3, pp. 257–269, 1976/1977.
[11] O. Tikhonov, "On the Rapid Computation of the Largest Zeros of a Polynomial," *Notes of the Leningrad Mining Institute*, vol. 48, no. 3, pp.98–103, 1968.
[12] A. Polyakov, M. Kolesova, M. Kalinin, and P. Shtanko, "Algorithm of Numerical Solving of the Nonlinear Algebraic Equations in the Conditions of Absence of the Information's About Neighbourhoods of Roots," in *Proc. volume from the 6-th IFAC workshop DECOM-TT 2009*, Scopje, 2009, pp. 301–306.
[13] A. Polyakov, "A Numerical Solving Algorithm for Polynomial Equation Systems," in *Proc. of Selected AAS 2009 Conf. Papers*, Scopje-Istanbul, 2009, pp. 109–112.
[14] A. Polyakov, M. Kolesova, M. Kalinin, and V. Kramar, "Manipulator Synthesis on the Given Properties of Working Space. Application of Polynomials with Linear Real Parameter," in *Proc. of 2010 IEEE International Conference on Systems Man and Cybernetics (SMC)*, Istanbul, 2010, pp. 4325–4332.
[15] R.P. Kearfott, "Some Tests of Generalized Bisection," *ACM Transactions on Mathematical Software*, vol. 13, no. 3, pp.197–220, September 1987.