International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:1, 2010

New application of EHTA for the generalized (2+1)-dimensional nonlinear evolution equations

Mohammad Taghi Darvishi, Maliheh Najafi and Mohammad Najafi

Abstract—In this paper, the generalized (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (shortly CBS) equations are investigated. We employ the Hirota's bilinear method to obtain the bilinear form of CBS equations. Then by the idea of extended homoclinic test approach (shortly EHTA), some exact soliton solutions including breather type solutions are presented.

Keywords—EHTA, (2+1)-dimensional CBS equations, (2+1)-dimensional breaking soliton equation, Hirota's bilinear form.

I. Introduction

THE generalized (2+1)-dimensional nonlinear evolution equations is

$$u_{xt} + au_x u_{xy} + bu_{xx}u_y + u_{xxxy} = 0, \qquad (1)$$

where a and b are constant parameters. For different values of these parameters we have special kinds of equation (1), e.g., for $a = 4$ and $b = 2$, we have the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0, \qquad (2)$$

the (2+1)-dimensional breaking soliton equation for $a = -4$, $b = -2$,

$$u_{xt} - 4u_x u_{xy} - 2u_{xx}u_y + u_{xxxy} = 0, \qquad (3)$$

and the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation for $a = b = 4$,

$$u_{xt} + 4u_x u_{xy} + 4u_{xx}u_y + u_{xxxy} = 0. \qquad (4)$$

In recent years, many kinds of powerful methods have been proposed to find solutions of nonlinear partial differential equations, numerically and/or analytically, e.g., the variational iteration method [1], [2], [3], the homotopy perturbation method [4], [5], [6], [7], [8], parameter expansion method [9], [10], [11], spectral collocation method [12], [13], [14], [15], [16], homotopy analysis method [17], [18], [19], [20], [21], [22], and the Exp-function method [23], [24], [25], [26], [27], [28].

In this paper, we solve equation (1) by the EHTA and obtain some exact and new solutions for (2), (3) and (4).
There are some studies on CBS equations. [29] obtained some new traveling wave solutions for the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation. Wazwaz [30] considered the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation. He employed the Cole-Hopf transformation and the Hirota's bilinear method to derive multiple-soliton solutions and multiple singular soliton solutions for the equation. Also he derived the necessary conditions for complete integrability of the equation. Also he [31] employed the Hirota's bilinear method to the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation.

M.T. Darvishi, Maliheh Najafi and **Mohammad Najafi**: Department of Mathematics, Faculty of Science, Razi University, Kermanshah 67149, Iran. **e-mail**: darvishimt@yahoo.com, malihe_math87@yahoo.com and m_najafi82@yahoo.com.

II. Description of extended homoclinic test approach

For a governing equation like (1), we consider a general form of a higher dimensional nonlinear evolution equation

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \cdots) = 0, \qquad (5)$$

where $u = u(x, y, t)$ and F is a polynomial of u and its derivatives. The basic idea of this method applies the Painlevè analysis to make a transformation as

$$u = T(f) \qquad (6)$$

for some new and unknown function f. Then we use this transformation in a high dimensional nonlinear equation of the general form. By substituting (6) in (5), the first one converts into the Hirota's bilinear form, which it will solve by taking a special form for function f and assuming that the obtained Hirota's bilinear form has solutions in EHTA, then we can specify the unknown function f; cf. [32].

III. New application of EHTA

In this paper, we investigate explicit formula of solutions of the following generalized (2+1)-dimensional nonlinear evolution equations

$$u_{xt} + au_x u_{xy} + bu_{xx}u_y + u_{xxxy} = 0, \qquad (7)$$

To solve eq. (8), we consider the following cases:

A. Case 1: *a=b*

Suppose $a = b$, hence we have

$$u_{xt} + au_x u_{xy} + au_{xx}u_y + u_{xxxy} = 0. \qquad (8)$$

To solve eq. (8) we introduce a new dependent variable w by

$$w = \frac{6}{a}(\ln f)_x \qquad (9)$$

where $f(x, y, t)$ is an unknown real function which will be determined. Substituting eq. (9) into eq. (8), we obtain the following Hirota's bilinear form

$$(D_x^3 D_y + D_x D_t) f \cdot f = 0, \tag{10}$$

where the D-operator is defined by

$$D_x^m D_y^k D_t^n f(x, y, t) \cdot g(x, y, t) =$$

$$(\tfrac{\partial}{\partial x_1} - \tfrac{\partial}{\partial x_2})^m (\tfrac{\partial}{\partial y_1} - \tfrac{\partial}{\partial y_2})^k (\tfrac{\partial}{\partial t_1} - \tfrac{\partial}{\partial t_2})^n$$

$$[f(x_1, y_1, t_1) g(x_2, y_2, t_2)],$$

and the right hand side is computed in

$$x_1 = x_2 = x,\ y_1 = y_2 = y,\ t_1 = t_2 = t.$$

Now we suppose the solution of eq. (10) as

$$f(x, y, t) = e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 e^{\xi_1} \tag{11}$$

where

$$\xi_i = a_i x + b_i y + c_i t, \quad i = 1, 2 \tag{12}$$

and a_i, b_i,c_i and δ_i are some constants to be determined later. Substituting eq. (11) into eq. (10), and equating all coefficients of $\exp(\xi_1)$, $\exp(-\xi_1)$, $\sin(\xi_2)$ and $\cos(\xi_2)$ to zero, we get the following set of algebraic equations for a_i,b_i,c_i,δ_i, $(i = 1, 2)$

$$b_2 a_2{}^3 + c_1 a_1 + a_1{}^3 b_1 - a_2 c_2 - 3\, a_1{}^2 b_2 a_2 - 3\, a_2{}^2 a_1 b_1 = 0,$$

$$a_2{}^3 b_1 - 3\, a_1{}^2 b_1 a_2 + 3\, b_2 a_2{}^2 a_1 - c_2 a_1 - c_1 a_2 - a_1{}^3 b_2 = 0,$$

$$4\,\delta_1{}^2 b_2 a_2{}^3 - \delta_1{}^2 c_2 a_2 + 4\, c_1 a_1 \delta_2 + 16\, a_1{}^3 b_1 \delta_2 = 0. \tag{13}$$

Solving the system of equations (13) with the aid of Maple, yields the following cases:

case i:

$$b_1 = 0, c_1 = 2\, a_1 b_2 a_2, c_2 = b_2\left(-a_1{}^2 + a_2{}^2\right), \quad \delta_2 = -\frac{\delta_1{}^2\left(a_1{}^2 + 3\, a_2{}^2\right)}{8\, a_1{}^2} \tag{14}$$

for some arbitrary complex constants a_1, a_2, b_2 and δ_1. Substitute eq. (14) into eq. (9) with eq. (11), we obtain the solution as

$$f(x, y, t) = e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 e^{\xi_1}$$

and

$$u(x, y, t) = 6\,\frac{-a_1 e^{-\xi_1} + \delta_1 \sin(\xi_2)\, a_2 + \delta_2 a_1 e^{\xi_1}}{a\left(e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 e^{\xi_1}\right)} \tag{15}$$

for

$$\xi_1 = a_1 x + 2\, a_1 b_2 a_2 t,$$
$$\xi_2 = a_2 x + b_2 y + \left(-a_1{}^2 b_2 + b_2 a_2{}^2\right) t$$

and

$$\delta_2 = -\frac{\delta_1{}^2\left(a_1{}^2 + 3\, a_2{}^2\right)}{8\, a_1{}^2} \tag{16}$$

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x, y, t) = 6\,\frac{2\, a_1 \sqrt{\delta_2}\sinh(\xi_1 - \theta) + \delta_1 \sin(\xi_2)\, a_2}{a\left(2\sqrt{\delta_2}\cosh(\xi_1 - \theta) + \delta_1 \cos(\xi_2)\right)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x, y, t) = 6\,\frac{2\, a_1 \sqrt{-\delta_2}\cosh(\xi_1 - \theta) + \delta_1 \sin(\xi_2)\, a_2}{a\left(2\sqrt{-\delta_2}\sinh(\xi_1 - \theta) + \delta_1 \cos(\xi_2)\right)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2).$$

case ii:

$$a_2 = 0, b_1 = 0, c_1 = 0, c_2 = -a_1{}^2 b_2 \tag{17}$$

for some arbitrary complex constants $a_1, b_2, \delta_1, \delta_2$. Substitute eq. (17) into eq. (9) with eq. (11), we obtain the solution as follows

$$f(x, y, t) = e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 e^{\xi_1}$$

and

$$u(x, y, t) = 6\,\frac{-a_1 e^{-\xi_1} + \delta_2 a_1 e^{\xi_1}}{a\left(e^{-\xi_1} + \delta_1 \cos(\xi) + \delta_2 e^{\xi_1}\right)} \tag{18}$$

for

$$\xi_1 = a_1 x \quad , \quad \xi_2 = b_2 y - a_1{}^2 b_2 t$$

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x, y, t) = 6\,\frac{-2 a_1 \sqrt{\delta_2}\sinh(\xi_1 - \theta)}{a\left(2\sqrt{\delta_2}\cosh(\xi_1 - \theta) + \delta_1 \cos(\xi)\right)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x, y, t) = 6\,\frac{-2 a_1 \sqrt{-\delta_2}\cosh(\xi_1 - \theta)}{a\left(2\sqrt{-\delta_2}\sinh(\xi_1 - \theta) + \delta_1 \cos(\xi)\right)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2).$$

case iii:

$$a_1 = i a_2, c_1 = 4\, i a_2{}^2 b_2 - i c_2 + 4\, a_2{}^2 b_1, \delta_2 = \tfrac{\delta_1{}^2}{4} \tag{19}$$

for some arbitrary complex constants $a_2, b_1, b_2, c_2, \delta_1$. Substitute eq. (19) into eq. (9) with eq. (11), we obtain the solution as follows

$$f(x, y, t) = e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 e^{\xi_1}$$

and

$$u(x, y, t) = 6\,\frac{-i a_2 e^{-\xi_1} - \delta_1 \sin(\xi_2)\, a_2 + i \delta_2 a_2 e^{\xi_1}}{\left(e^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 e^{\xi_1}\right) a}, \tag{20}$$

for

$$\xi_1 = ia_2x + b_1y + \left(4\, ia_2{}^2b_2 - ic_2 + 4\, a_2{}^2b_1\right)t,$$

$$\xi_2 = a_2x + b_2y + c_2t,$$

and

$$\delta_2 = \frac{\delta_1{}^2}{4}. \tag{21}$$

We make the dependent variable transformation in equation (20) as follows

$$a_2 = i\,A_2,$$
$$b_2 = i\,B_2, \tag{22}$$
$$c_2 = i\,C_2,$$

where A_2, B_2 and C_2 are real. We obtain new form for equation (20) as

$$u\,(x,y,t) = 6\,\frac{A_2\mathrm{e}^{-\xi_1^*} - \delta_1 \sinh\left(\xi_2^*\right)A_2 - \delta_2 A_2 \mathrm{e}^{\xi_1^*}}{a\left(\mathrm{e}^{-\xi_1^*} + \delta_1\cosh\left(\xi_2^*\right) + \delta_2\mathrm{e}^{\xi_1^*}\right)} \tag{23}$$

for

$$\xi_1^* = -A_2x + b_1y + \left(4\,A_2{}^2B_2 + C_2 - 4\,A_2{}^2b_1\right)t$$

$$\xi_2^* = -A_2x + B_2y + C_2t.$$

If $\delta_2 > 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = \frac{6}{a}\,\frac{A_2\left(2\sqrt{\delta_2}\sinh(\xi_1^* - \beta) - \delta_1\sinh\left(\xi_2^*\right)\right)}{2\sqrt{\delta_2}\cosh(\xi_1^* - \beta) + \delta_1\cosh\left(\xi_2^*\right)}$$

for

$$\beta = \frac{1}{2}\,\ln(\delta_2) \quad , \quad \delta_2 = \frac{\delta_1{}^2}{4}.$$

If $\delta_2 < 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = \frac{6}{a}\,\frac{A_2\left(2\sqrt{-\delta_2}\cosh(\xi_1^* - \beta) - \delta_1\sinh\left(\xi_2^*\right)\right)}{2\sqrt{-\delta_2}\sinh(\xi_1^* - \beta) + \delta_1\cosh\left(\xi_2^*\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_2) \quad , \quad \delta_2 = \frac{\delta_1{}^2}{4}.$$

B. *Case 2:* $\boldsymbol{a \neq b}$

For $a \neq b$, we have

$$u_{xt} + \tfrac{a+b}{2}u_xu_{xy} + \tfrac{a+b}{2}u_{xx}u_y + u_{xxxy} + \tfrac{a-b}{2}u_xu_{xy} + \tfrac{b-a}{2}u_{xx}u_y = 0. \tag{24}$$

To solve eq. (24), we introduce a new dependent variable w by

$$w = \frac{12}{a+b}(\ln f)_x \tag{25}$$

where $f(x, y, t)$ is an unknown real function which will be determined. Substituting eq. (25) into eq. (24), we obtain the following Hirota's bilinear form

$$(D_x^3D_y + D_xD_t)f\cdot f + \tfrac{(a-b)}{2}\,f^2\,\partial_x^{-1}(D_x(\ln f)_{xx}\cdot(\ln f)_{xy}) = 0. \tag{26}$$

We suppose that

$$\partial_x^{-1}(D_x(\ln f)_{xx}\cdot(\ln f)_{xy}) = 0, \tag{27}$$

then eq. (26) reduces to

$$(D_xD_t + D_yD_x^3)f\cdot f = 0. \tag{28}$$

Now we suppose the solution of eq. (28) as

$$f\,(x,y,t) = \mathrm{e}^{-\xi_1} + \delta_1\,\cos\left(\xi_2\right) + \delta_2\,\mathrm{e}^{\xi_1} \tag{29}$$

where

$$\xi_i = a_ix + b_iy + c_it, \quad i = 1, 2 \tag{30}$$

and a_i, b_i,c_i and δ_i are some constants to be determined later. Substituting eq. (29) into eq. (28) and eq. (27), and equating all coefficients of $\exp(\xi_1)$, $\exp(-\xi_1)$, $\sin\left(\xi_2\right)$ and $\cos\left(\xi_2\right)$ to zero, we get the following set of algebraic equations for a_i,b_i,c_i,δ_i, $(i = 1, 2)$

$$\begin{aligned}
&b_2a_2{}^3 + c_1a_1 + a_1{}^3b_1 - a_2c_2 - 3\,a_1{}^2b_2a_2 - 3\,a_2{}^2a_1b_1 = 0,\\
&a_2{}^3b_1 - 3\,a_1{}^2b_1a_2 + 3\,b_2a_2{}^2a_1 - c_2a_1 - c_1a_2 - a_1{}^3b_2 = 0,\\
&4\,\delta_1{}^2b_2a_2{}^3 - \delta_1{}^2c_2a_2 + 4\,c_1a_1\delta_2 + 16\,a_1{}^3b_1\delta_2 = 0,\\
&-4\,a_2{}^2b_2a_1{}^2 + 4\,b_1a_1{}^3a_2 + 4\,b_1a_1a_2{}^3 - 4\,a_1{}^4b_2 = 0,\\
&-a_2{}^3b_2a_1 + b_1a_1{}^2a_2{}^2 + a_2{}^4b_1 - a_2b_2a_1{}^3 = 0,
\end{aligned} \tag{31}$$

Solving the system of equations (31) with the aid of Maple, we obtain the following cases:

case i:

$$\begin{aligned}
&b_1 = \tfrac{b_2a_1}{a_2}, c_1 = -\tfrac{b_2a_1\left(-3\,a_2{}^2 + a_1{}^2\right)}{a_2},\\
&c_2 = b_2a_2{}^2 - 3\,a_1{}^2b_2, \delta_2 = -\tfrac{\delta_1{}^2a_2{}^2}{4\,a_1{}^2}
\end{aligned} \tag{32}$$

for some arbitrary complex constants a_1, a_2, b_1 and δ_1. Substitute eq. (32) into eq. (25) with eq. (29), we obtain the solution as

$$f\,(x,y,t) = \mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\mathrm{e}^{\xi_1}$$

and

$$u\,(x,y,t) = 12\,\frac{-a_1\mathrm{e}^{-\xi_1} + \delta_1\sin\left(\xi_2\right)a_2 + \delta_2a_1\mathrm{e}^{\xi_1}}{(a+b)\left(\mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\mathrm{e}^{\xi_1}\right)} \tag{33}$$

for

$$\xi_1 = a_1x + \tfrac{b_2a_1y}{a_2} - \tfrac{b_2a_1\left(-3\,a_2{}^2 + a_1{}^2\right)t}{a_2},$$

$$\xi_2 = a_2x + b_2y + \left(b_2a_2{}^2 - 3\,a_1{}^2b_2\right)t$$

and

$$\delta_2 = -\frac{\delta_1{}^2a_2{}^2}{4\,a_1{}^2}. \tag{34}$$

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = 12\,\frac{-2\,a_1\sqrt{\delta_2}\sinh(\xi_1 - \theta) + \delta_1\sin\left(\xi_2\right)a_2}{(a+b)(2\sqrt{\delta_2}\cosh(\xi_1 - \theta) + \delta_1\cos(\xi_2))}$$

for

$$\theta = \frac{1}{2}\,\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = 12\,\frac{-2\,a_1\sqrt{-\delta_2}\cosh(\xi_1-\theta)+\delta_1\sin\left(\xi_2\right)a_2}{(a+b)(2\sqrt{-\delta_2}\sinh(\xi_1-\theta)+\delta_1\cos(\xi_2))}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_2).$$

case ii:

$$a_1 = ia_2, c_1 = 4\,{a_2}^2 b_1, c_2 = 4b_2{a_2}^2, \qquad (35)$$

for some arbitrary complex constants $a_2, b_1, b_2, \delta_1, \delta_2$. Substitute eq. (35) into eq. (25) with eq. (29), we obtain the solution as follows

$$f\,(x,y,t) = \mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\mathrm{e}^{\xi_1}$$

and

$$u\,(x,y,t) = 12\,\frac{-ia_2\mathrm{e}^{-\xi_1} - \delta_1\sin\left(\xi_2\right)a_2 + i\delta_2 a_2\mathrm{e}^{\xi_1}}{(a+b)\left(\mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\mathrm{e}^{\xi_1}\right)} \qquad (36)$$

for

$$\xi_1 = ia_2x + b_1y + 4\,{a_2}^2 b_1 t\,, \quad \xi_2 = a_2x + b_2y + 4\,b_2{a_2}^2 t.$$

We make the dependent variable transformation in equation (36) as

$$a_2 = -i\,A_2, \qquad (37)$$

where A_2 is real. We obtain new form for equation (36) as follows

$$u\,(x,y,t) = 12\,\frac{A_2\left(-\mathrm{e}^{-\xi_1^*} + i\delta_1\sin\left(\xi_2^*\right) + \delta_2\mathrm{e}^{\xi_1^*}\right)}{(a+b)\left(\mathrm{e}^{-\xi_1^*} + \delta_1\cos\left(\xi_2^*\right) + \delta_2\mathrm{e}^{\xi_1^*}\right)} \qquad (38)$$

for

$$\xi_1^* = A_2x + b_1y - 4\,{A_2}^2 b_1 t$$

$$\xi_2^* = -iA_2x + b_2y - 4\,b_2{A_2}^2 t.$$

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = 12\,\frac{A_2\left(-2\sqrt{\delta_2}\sinh(\xi_1^*-\theta) + i\delta_1\sin\left(\xi_2^*\right)\right)}{(a+b)\left(2\sqrt{\delta_2}\cosh(\xi_1^*-\theta) + \delta_1\cos\left(\xi_2^*\right)\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = 12\,\frac{A_2\left(-2\sqrt{-\delta_2}\cosh(\xi_1^*-\theta) + i\delta_1\sin\left(\xi_2^*\right)\right)}{(a+b)\left(2\sqrt{-\delta_2}\sinh(\xi_1^*-\theta) + \delta_1\cos\left(\xi_2^*\right)\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_2).$$

case iii:

$$a_1 = ia_2, b_1 = ib_2, c_1 = i\left(8\,b_2{a_2}^2 - c_2\right), \delta_2 = \tfrac{{\delta_1}^2}{4} \qquad (39)$$

for some arbitrary complex constants $a_2, b_1, b_2, c_2, \delta_1$. Substitute eq. (39) into eq. (25) with eq. (29), we obtain the solution as

$$f\,(x,y,t) = \mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\mathrm{e}^{\xi_1}$$

and

$$u\,(x,y,t) = -12\,\frac{a_2\left(i\mathrm{e}^{-\xi_1} + \delta_1\sin\left(\xi_2\right) - i\delta_2\mathrm{e}^{\xi_1}\right)}{(a+b)\left(\mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\mathrm{e}^{\xi_1}\right)}, \qquad (40)$$

for

$$\xi_1 = i\left(a_2x + b_2y + 8\,b_2{a_2}^2 t - c_2t\right),$$

$$\xi_2 = a_2x + b_2y + c_2t,$$

and

$$\delta_2 = \frac{{\delta_1}^2}{4}. \qquad (41)$$

We make the dependent variable transformation in equation (40) as follows

$$a_2 = i\,A_2,$$

$$b_2 = i\,B_2, \qquad (42)$$

$$c_2 = i\,C_2,$$

where A_2, B_2 and C_2 are real. We obtain new form for equation (40) as

$$u\,(x,y,t) = 12\,\frac{A_2\left(\mathrm{e}^{-\xi_1^*} + \delta_1\sinh\left(\xi_2^*\right) - \delta_2\mathrm{e}^{\xi_1^*}\right)}{(a+b)\left(\mathrm{e}^{-\xi_1^*} + \delta_1\cosh\left(\xi_2^*\right) + \delta_2\mathrm{e}^{\xi_1^*}\right)} \qquad (43)$$

for

$$\xi_1^* = -A_2x - B_2y + 8\,B_2{A_2}^2 t + C_2t$$

$$\xi_2^* = A_2x + B_2y + C_2t.$$

If $\delta_2 > 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = \frac{12}{a+b}\,\frac{A_2\left(2\sqrt{\delta_2}\sinh(\xi_1^*-\beta) - \delta_1\sinh\left(\xi_2^*\right)\right)}{2\sqrt{\delta_2}\cosh(\xi_1^*-\beta) + \delta_1\cosh\left(\xi_2^*\right)}$$

for

$$\beta = \frac{1}{2}\,\ln(\delta_2) \quad , \quad \delta_2 = \frac{{\delta_1}^2}{4}.$$

If $\delta_2 < 0$ then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = \frac{12}{a+b}\,\frac{A_2\left(2\sqrt{-\delta_2}\cosh(\xi_1^*-\beta) - \delta_1\sinh\left(\xi_2^*\right)\right)}{2\sqrt{-\delta_2}\sinh(\xi_1^*-\beta) + \delta_1\cosh\left(\xi_2^*\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_2) \quad , \quad \delta_2 = \frac{{\delta_1}^2}{4}.$$

IV. Exact solution of the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff (CBS) equation

In this section, we investigate explicit formula of solutions of the following (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation

$$u_{xt} + 4u_x u_{xy} + 2u_{xx}u_y + u_{xxxy} = 0, \tag{44}$$

using section IV, we have the following exact solutions:

Exact solution I:

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = 2\,\frac{-2\,a_1\sqrt{\delta_2}\sinh(\xi_1-\theta) + \delta_1\sin(\xi_2)\,a_2}{(2\sqrt{\delta_2}\cosh(\xi_1-\theta) + \delta_1\cos(\xi_2))}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = 2\,\frac{-2\,a_1\sqrt{-\delta_2}\cosh(\xi_1-\theta) + \delta_1\sin(\xi_2)\,a_2}{(2\sqrt{-\delta_2}\sinh(\xi_1-\theta) + \delta_1\cos(\xi_2))}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2)$$

where

$$\xi_1 = a_1x + \tfrac{b_2a_1y}{a_2} - \tfrac{b_2a_1\left(-3\,a_2^2 + a_1^2\right)t}{a_2},$$

$$\xi_2 = a_2x + b_2y + \left(b_2a_2^2 - 3\,a_1^2b_2\right)t$$

and

$$\delta_2 = -\frac{{\delta_1}^2{a_2}^2}{4\,{a_1}^2}. \tag{45}$$

Exact solution II:

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = 2\,\frac{A_2\left(-2\sqrt{\delta_2}\sinh(\xi_1^*-\theta) + i\delta_1\sin(\xi_2^*)\right)}{\left(2\sqrt{\delta_2}\cosh(\xi_1^*-\theta) + \delta_1\cos(\xi_2^*)\right)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = 2\,\frac{A_2\left(-2\sqrt{-\delta_2}\cosh(\xi_1^*-\theta) + i\delta_1\sin(\xi_2^*)\right)}{\left(2\sqrt{-\delta_2}\sinh(\xi_1^*-\theta) + \delta_1\cos(\xi_2^*)\right)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2)$$

where

$$\xi_1 = a_1x + \tfrac{b_2a_1y}{a_2} - \tfrac{b_2a_1\left(-3\,a_2^2 + a_1^2\right)t}{a_2},$$

$$\xi_2 = a_2x + b_2y + \left(b_2a_2^2 - 3\,a_1^2b_2\right)t$$

and

$$\delta_2 = -\frac{{\delta_1}^2{a_2}^2}{4\,{a_1}^2}. \tag{46}$$

Exact solution III:

If $\delta_2 > 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) = 2\frac{A_2\left(2\sqrt{\delta_2}\sinh(\xi_1^*-\beta) - \delta_1\sinh(\xi_2^*)\right)}{2\sqrt{\delta_2}\cosh(\xi_1^*-\beta) + \delta_1\cosh(\xi_2^*)}$$

for

$$\beta = \frac{1}{2}\ln(\delta_2) \quad , \quad \delta_2 = \frac{{\delta_1}^2}{4}.$$

If $\delta_2 < 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) = 2\frac{A_2\left(2\sqrt{-\delta_2}\cosh(\xi_1^*-\beta) - \delta_1\sinh(\xi_2^*)\right)}{2\sqrt{-\delta_2}\sinh(\xi_1^*-\beta) + \delta_1\cosh(\xi_2^*)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2) \quad , \quad \delta_2 = \frac{{\delta_1}^2}{4}$$

and

$$\xi_1^* = -A_2x - B_2y + 8\,B_2{A_2}^2t + C_2t$$

$$\xi_2^* = A_2x + B_2y + C_2t.$$

V. Exact solution of the (2+1)-dimensional breaking soliton equation

In this section, we investigate explicit formula of solutions of the following (2+1)-dimensional Breaking soliton equation

$$u_{xt} - 4u_x u_{xy} - 2u_{xx}u_y + u_{xxxy} = 0, \tag{47}$$

using section IV, we have the following exact solutions:

Exact solution I:

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = -2\,\frac{-2\,a_1\sqrt{\delta_2}\sinh(\xi_1-\theta) + \delta_1\sin(\xi_2)\,a_2}{(2\sqrt{\delta_2}\cosh(\xi_1-\theta) + \delta_1\cos(\xi_2))}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = -2\,\frac{-2\,a_1\sqrt{-\delta_2}\cosh(\xi_1-\theta) + \delta_1\sin(\xi_2)\,a_2}{(2\sqrt{-\delta_2}\sinh(\xi_1-\theta) + \delta_1\cos(\xi_2))}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2)$$

where

$$\xi_1 = a_1x + \tfrac{b_2a_1y}{a_2} - \tfrac{b_2a_1\left(-3\,a_2^2 + a_1^2\right)t}{a_2},$$

$$\xi_2 = a_2x + b_2y + \left(b_2a_2^2 - 3\,a_1^2b_2\right)t$$

and

$$\delta_2 = -\frac{{\delta_1}^2{a_2}^2}{4\,{a_1}^2}. \tag{48}$$

Exact solution II:

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = -2\,\frac{A_2\left(-2\sqrt{\delta_2}\sinh(\xi_1^* - \theta) + i\delta_1\sin(\xi_2^*)\right)}{\left(2\sqrt{\delta_2}\cosh(\xi_1^* - \theta) + \delta_1\cos(\xi_2^*)\right)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2)$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = -2\,\frac{A_2\left(-2\sqrt{-\delta_2}\cosh(\xi_1^* - \theta) + i\delta_1\sin(\xi_2^*)\right)}{\left(2\sqrt{-\delta_2}\sinh(\xi_1^* - \theta) + \delta_1\cos(\xi_2^*)\right)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2)$$

where

$$\xi_1 = a_1x + \tfrac{b_2a_1y}{a_2} - \tfrac{b_2a_1\left(-3\,a_2{}^2 + a_1{}^2\right)t}{a_2},$$

$$\xi_2 = a_2x + b_2y + \left(b_2a_2{}^2 - 3\,a_1{}^2b_2\right)t$$

and

$$\delta_2 = -\frac{\delta_1{}^2a_2{}^2}{4\,a_1{}^2} \tag{49}$$

Exact solution III:

If $\delta_2 > 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) = -2\frac{A_2\left(2\sqrt{\delta_2}\sinh(\xi_1^* - \beta) - \delta_1\sinh(\xi_2^*)\right)}{2\sqrt{\delta_2}\cosh(\xi_1^* - \beta) + \delta_1\cosh(\xi_2^*)}$$

for

$$\beta = \frac{1}{2}\ln(\delta_2) \quad , \quad \delta_2 = \frac{\delta_1{}^2}{4}.$$

If $\delta_2 < 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) = -2\frac{A_2\left(2\sqrt{-\delta_2}\cosh(\xi_1^* - \beta) - \delta_1\sinh(\xi_2^*)\right)}{2\sqrt{-\delta_2}\sinh(\xi_1^* - \beta) + \delta_1\cosh(\xi_2^*)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2) \quad , \quad \delta_2 = \frac{\delta_1{}^2}{4}$$

and

$$\xi_1^* = -A_2x - B_2y + 8\,B_2A_2{}^2t + C_2t$$

$$\xi_2^* = A_2x + B_2y + C_2t.$$

VI. Exact solution of the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation

In this section, we investigate explicit formula of solutions of the following (2+1)-dimensional Bogoyavlenskii's breaking soliton equation

$$u_{xt} + 4u_xu_{xy} + 4u_{xx}u_y + u_{xxxy} = 0, \tag{50}$$

using section IV, we have the following exact solutions:

Exact solution I:

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = \frac{3}{2}\,\frac{2\,a_1\sqrt{\delta_2}\sinh(\xi_1 - \theta) + \delta_1\sin(\xi_2)\,a_2}{\left(2\sqrt{\delta_2}\cosh(\xi_1 - \theta) + \delta_1\cos(\xi_2)\right)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = \frac{3}{2}\,\frac{2\,a_1\sqrt{-\delta_2}\cosh(\xi_1 - \theta) + \delta_1\sin(\xi_2)\,a_2}{\left(2\sqrt{-\delta_2}\sinh(\xi_1 - \theta) + \delta_1\cos(\xi_2)\right)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2)$$

where

$$\xi_1 = a_1x + 2\,a_1b_2a_2t, \qquad \xi_2 = a_2x + b_2y + \left(-a_1{}^2b_2 + b_2a_2{}^2\right)t$$

and

$$\delta_2 = -\frac{\delta_1{}^2\left(a_1{}^2 + 3\,a_2{}^2\right)}{8\,a_1{}^2}. \tag{51}$$

Exact solution II:

If $\delta_2 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = \frac{3}{2}\,\frac{-2a_1\sqrt{\delta_2}\sinh(\xi_1 - \theta)}{\left(2\sqrt{\delta_2}\cosh(\xi_1 - \theta) + \delta_1\cos(\xi)\right)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_2).$$

If $\delta_2 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t) = \frac{3}{2}\,\frac{-2a_1\sqrt{-\delta_2}\cosh(\xi_1 - \theta)}{\left(2\sqrt{-\delta_2}\sinh(\xi_1 - \theta) + \delta_1\cos(\xi)\right)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2)$$

and

$$\xi_1 = a_1x, \qquad \xi_2 = b_2y - a_1{}^2b_2t.$$

Exact solution III:

If $\delta_2 > 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) = \frac{3}{2}\frac{A_2\left(2\sqrt{\delta_2}\sinh(\xi_1^* - \beta) - \delta_1 \sinh(\xi_2^*)\right)}{2\sqrt{\delta_2}\cosh(\xi_1^* - \beta) + \delta_1 \cosh(\xi_2^*)}$$

for

$$\beta = \frac{1}{2}\ln(\delta_2), \qquad \delta_2 = \frac{{\delta_1}^2}{4}.$$

If $\delta_2 < 0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t) = \frac{3}{2}\frac{A_2\left(2\sqrt{-\delta_2}\cosh(\xi_1^* - \beta) - \delta_1 \sinh(\xi_2^*)\right)}{2\sqrt{-\delta_2}\sinh(\xi_1^* - \beta) + \delta_1 \cosh(\xi_2^*)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_2), \qquad \delta_2 = \frac{{\delta_1}^2}{4},$$

and

$$\xi_1^* = -A_2 x + b_1 y + \left(4\,{A_2}^2 B_2 + C_2 - 4\,{A_2}^2 b_1\right)t$$

$$\xi_2^* = -A_2 x + B_2 y + C_2 t.$$

VII. Conclusions

In this paper, using the EHTA we obtained some explicit formulas of solutions for the generalized (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation in some special cases of its parameters. EHTA with the aid of a symbolic computation like Maple or Mathematica is an easy and straightforward method which can be apply to other nonlinear partial differential equations. It must be noted that, all obtained solutions have checked in the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equations. All solutions satisfy in the equations.

References

[1] J.H. He, *Variational iteration method-a kind of non-linear analytical technique: some examples*, Int. J. Non-linear Mech. 34(4) (1999) 699–708.
[2] M.T. Darvishi, F. Khani, A.A. Soliman, *The numerical simulation for stiff systems of ordinary differential equations*, Comput. Math. Appl. 54(7-8) (2007) 1055–1063.
[3] M.T. Darvishi, F. Khani, *Numerical and explicit solutions of the fifth-order Korteweg-de Vries equations*, Chaos, Solitons and Fractals 39 (2009) 2484–2490.
[4] J.H. He, *New interpretation of homotopy perturbation method,* Int. J. Mod. Phys. B 20(18) (2006) 2561–2568.
[5] J.H. He, *Application of homotopy perturbation method to nonlinear wave equations*, Chaos, Solitons and Fractals 26(3) (2005) 695–700.
[6] J.H. He, *Homotopy perturbation method for bifurcation of nonlinear problems*, Int. J. Nonlinear Sci. Numer. Simul. 6(2) (2005) 207–208.
[7] M.T. Darvishi, F. Khani, *Application of He's homotopy perturbation method to stiff systems of ordinary differential equations*, Zeitschrift fur Naturforschung A, 63a (1-2) (2008) 19–23.
[8] M.T. Darvishi, F. Khani, S. Hamedi-Nezhad, S.-W. Ryu, *New modification of the HPM for numerical solutions of the sine-Gordon and coupled sine-Gordon equations*, Int. J. Comput. Math. 87(4) (2010) 908–919.
[9] J.H. He, *Bookkeeping parameter in perturbation methods,* Int. J. Nonlin. Sci. Numer. Simul. 2 (2001) 257–264.
[10] M.T. Darvishi, A. Karami, B.-C. Shin, *Application of He's parameter-expansion method for oscillators with smooth odd nonlinearities,* Phys. Lett. A 372(33) (2008) 5381–5384.
[11] B.-C. Shin, M.T. Darvishi, A. Karami, *Application of He's parameter-expansion method to a nonlinear self-excited oscillator system,* Int. J. Nonlin. Sci. Num. Simul. 10(1) (2009) 137–143.
[12] M.T. Darvishi, *Preconditioning and domain decomposition schemes to solve PDEs*, Int'l J. of Pure and Applied Math. 1(4) (2004) 419–439.
[13] M.T. Darvishi, S. Kheybari and F. Khani, *A numerical solution of the Korteweg-de Vries equation by pseudospectral method using Darvishi's preconditionings,* Appl. Math. Comput. 182(1) (2006) 98–105.
[14] M.T. Darvishi, M. Javidi, *A numerical solution of Burgers' equation by pseudospectral method and Darvishi's preconditioning*, Appl. Math. Comput. 173(1) (2006) 421–429.
[15] M.T. Darvishi, F. Khani and S. Kheybari, *Spectral collocation solution of a generalized Hirota-Satsuma KdV equation,* Int. J. Comput. Math. 84(4) (2007) 541–551.
[16] M.T. Darvishi, F. Khani, S. Kheybari, *Spectral collocation method and Darvishi's preconditionings to solve the generalized Burgers-Huxley equation*, Commun., Nonlinear Sci. Numer. Simul. 13(10) (2008) 2091–2103.
[17] S.J. Liao, *An explicit, totally analytic approximate solution for Blasius viscous flow problems,* Int. J. Non-Linear Mech. 34 (1999) 759–778.
[18] S.J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method,* Chapman & Hall/CRC Press, Boca Raton, 2003.
[19] S.J. Liao, *On the homotopy analysis method for nonlinear problems,* Appl. Math. Comput. 147 (2004) 499–513.
[20] S.J. Liao, *A new branch of solutions of boundary-layer flows over an impermeable stretched plate,* Int. J. Heat Mass Transfer 48 (2005) 2529–2539.
[21] S.J. Liao, *A general approach to get series solution of non-similarity boundary-layer flows*, Commun. Nonlinear Sci. Numer. Simul. 14(5) (2009) 2144–2159.
[22] M.T. Darvishi, F. Khani, *A series solution of the foam drainage equation*, Comput. Math. Appl. 58 (2009) 360–368.
[23] J.H. He, M.A. Abdou, *New periodic solutions for nonlinear evolution equations using Exp-function method*, Chaos, Solitons and Fractals 34 (2007) 1421–1429.
[24] J.H. He, X.H. Wu, *Exp-function method for nonlinear wave equations*, Chaos, Solitons and Fractals, 30(3) (2006) 700–708.
[25] J.H. He, X.H. Wu, *Construction of solitary solution and compacton-like solution by variational iteration method,* Chaos, Solitons and Fractals, 29 (2006) 108–113.
[26] F. Khani, S. Hamedi-Nezhad, M.T. Darvishi, S.-W. Ryu, *New solitary wave and periodic solutions of the foam drainage equation using the Exp-function method*, Nonlin. Anal.: Real World Appl. 10 (2009) 1904–1911.
[27] B.-C. Shin, M.T. Darvishi, A. Barati, *Some exact and new solutions of the Nizhnik-Novikov-Vesselov equation using the Exp-function method*, Comput. Math. Appl. 58(11/12) (2009) 2147–2151.
[28] X.H. Wu, J.H. He, *Exp-function method and its application to nonlinear equations,* Chaos, Solitons and Fractals 38(3) (2008) 903–910.
[29] A.M. Wazwaz, *New solutions of distinct physical structures to high-dimensional nonlinear evolution equations,* Appl. Math. Comput., 196 (2008) 363–370.
[30] A.M. Wazwaz, *The (2+1) and (3+1)-Dimensional CBS Equations: Multiple Soliton Solutions and Multiple Singular Soliton Solutions,* Zeitschrift fur Naturforschung A 65a, (2010) 173–181.
[31] A.M. Wazwaz, *Multiple-soliton solutions for the Calogero-Bogoyavlenskii-Schiff, Jimbo-Miwa and YTSF equations,* Appl. Math. Comput., 203 (2008) 592–597.
[32] Z.-H. Zhao, Z. Dai, S. Han, *The EHTA for nonlinear evolution equations*, Appl. Math. Comput., (in press), doi:10.1016/j.amc.2010.09.069.