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Necessary and Sufficient Condition for the Quaternion Vector Measure

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Abstract—In this paper, the definitions of the quaternion measure and the quaternion vector measure are introduced. The relation between the quaternion measure and the complex vector measure as well as the relation between the quaternion linear functional and the complex linear functional are discussed respectively. By using these relations, the necessary and sufficient condition to determine the quaternion vector measure is given.

Keywords—Quaternion, Quaternion measure, Quaternion vector measure, Quaternion Banach space, Quaternion linear functional.

I. Introduction

LET X be a Banach space over complex field and (Ω, Σ) be a measurable space. A function m: $\sum \to X$ is said to be a vector measure if m satisfies

$$m(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$$

for all sequences of pairwise disjoint sets $\{E_n\}_{n=1}^{\infty} \subseteq \sum$, where the series is convergent in the norm topology of X.

The study of vector measure is a very active field of research, and it is already very old, too. For the case of vector measure on $\sigma-$ algebra Σ to the real or complex Banach space, in 1936, J. A. Clarkson [1] used vector measure-theoretic ideas to prove that, many Banach spaces do not admit equivalent uniformly convex norms. In 1938, I. Gel'fand [2] also used vector measure-theoretic methods to prove that $L_1[0, 1]$ is not isomorphic to a dual of a Banach space. From the forties to the mid-sixties, many mathematicians, for example, R. G. Bartle [3], N. Dinculeanu and I. Kluvánek [4], N. Dunford and J. T. Schwartz [5], J. Lindenstrauss and A. Pelczyński [6], etc., gave many classical results on vector measure.

In 1977, J. Diestel and J. J. Uhl Jr [7] gave a comprehensive survey of vector measures. After the Seventies, for example, in 1980, I. Kluvánek [8] discussed the applications of vector measures. In 1984, J. Diestel and J. J. Uhl Jr [9] again gave the progress in vector measures. In 1997, A. Fernández and Faranjo [10] studied the Rybakov's theorem for vector measures in Fréchet spaces. In 2007, G. P. Curbera and W. J. Ricker [11] gave survey on vector measure, integration and application. For more information on vector measures, the reader is referred to [7], [9], [11] and its references.

Recent years, there are some interests in the quaternion Banach space and quaternion Hilbert space. For example, in 1987, C. S. Sharma and T. J. Coulson [12] discussed the spectral theory for unitary operators on a quaternion Hilbert space; in 1992, S. H. Kulkarni [13] gave the representation of a class of real B*-algebras as algebras of quaternion-valued functions; in 2007, Chi-Keung Ng [14] gave some results on quaternion normed spaces, S. V. Ladkovsky [15] studied the algebra of operators in Banach spaces over the quaternion skew field. For more details about quaternion analysis and its applications, we refer to [16] and references there in.

Consider the differences between the complex Banach space and the quaternion Banach space, and the applications of the quaternion measure and quaternion vector measure to quantum mechanics [17], we naturally discuss the following question.

Question. Does the quaternion vector measure have some properties which are analogous to that of complex vector measure ?

In this paper, we introduce the definition of the quaternion vector measure, and discuss the above question, give some properties on the quaternion measure and quaternion vector measure. By using these obtained properties, we also prove that Lemma 3, Theorem 1 and 2, which are necessary and sufficient conditions for the quaternion measure and the quaternion vector measure. Moreover, Theorem 1 and 2 are similar to complex vector measure case.

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II. Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ be the real number field and the complex number field, respectively. The quaternion skew field, denoted by $\mathbb{H}$, is the set of all elements with the form $q_0 + q_1 i + q_2 j + q_3 k$, where q_0, q_1, q_2 and $q_4 \in \mathbb{R}$, moreover,

$$i^2 = j^2 = k^2 = ijk = -1;$$
$$ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

It is clear that $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, and the multiplication operation is noncommutative in $\mathbb{H}$, it is easy to imply that $jc = \overline{c}j$ for any complex number c. Furthermore, for every $q \in \mathbb{H}$, q can be uniquely expressed as

$$q = q_1 + q_2 j, \tag{1}$$

where $q_1, q_2 \in \mathbb{C}$ for $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$. The conjugate and norm of q are respectively defined as

$$\bar{q} = q_0 - q_{1i} - q_{2j} - q_{3k},$$

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

Let $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{H})$) be the collection of all $n \times n$ matrices with complex entries (resp. quaternion entries). For $A \in M_n(\mathbb{H})$, then there exist A_1 and A_2 in $M_n(\mathbb{C})$ such that $A = A_1 + A_2 j$ and such representation is unique. We call the $2n \times 2n$ complex matrix

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \tag{2}$$

as the complex adjoint matrix of the quaternion matrix A and denote it by χ_A.

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Analogy to the classical measure theory, we extend the definition of complex measure to the quaternion setting and have Definition 1.

Definition 1. Let (Ω, Σ) be a measurable space. A function $\mu : \Sigma \to \mathbb{H}$ is called a quaternion measure if

$$\mu(\cup_{n=1}^{\infty} E_n) = \Sigma_{n=1}^{\infty} \mu(E_n)$$

whenever $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ is a sequence of pairwise disjoint sets.

In this paper, we call the Banach space over complex field as the complex Banach space and the vector measure from Σ to the complex Banach space as the complex vector measure.

Similar to the definition of the complex vector measure, we introduce the definition of a quaternion vector measure.

Definition 2. Let (Ω, Σ) be a measurable space and $X_{\mathbb{H}}$ be a quaternion Banach space. A function $m : \Sigma \to X_{\mathbb{H}}$ is called a quaternion vector measure if m satisfies

$$m(\cup_{n=1}^{\infty} E_n) = \Sigma_{i=1}^{\infty} m(E_n)$$

for all sequences of pairwise disjoint sets $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$, where the series is convergent in the norm topology of $X_{\mathbb{H}}$.

III. Necessary and Sufficient Conditions for the Quaternion Vector Measure

Owing to the quaternion multiplication is noncommutative, according to the left scalar multiplication and the right scalar multiplication, we call a vector space over the quaternion field $\mathbb{H}$ as a left or right quaternion vector space. For convenience, the left quaternion vector space and the left quaternion Banach space are also said to be the quaternion vector space and the quaternion Banach space, respectively.

Throughout this paper, we assume that $X_{\mathbb{H}}$ is a quaternion vector space, $\{e_i\}_{i=1}^{\infty} \subset X_{\mathbb{H}}$ is a basis of $X_{\mathbb{H}}$,

$$X_{\mathbb{C}} = \{x \mid x = \textstyle\sum_{i=1}^{\infty} \alpha_i e_i,\ \alpha_i \in \mathbb{C}\},$$
$$Y_{\mathbb{C}} = \{x \mid x = \textstyle\sum_{i=1}^{\infty} \alpha_i j e_i,\ \alpha_i \in \mathbb{C}\},$$

then $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are vector spaces over $\mathbb{C}$ with respect to the addition operation and the scalar multiplication operation of $X_{\mathbb{H}}$, respectively.

In order to give some necessary and sufficient conditions for the quaternion measure and quaternion vector measure, we need the following auxiliary lemmas.

Lemma 1. Let $a_n, b_n \in \mathbb{C}$ and $q_n = a_n + b_n j$, then the series $\sum_{n=1}^{\infty} q_n$ is convergent if and only if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, respectively.

Proof. Let $\sum_{n=1}^{\infty} q_n = q$, where $q = a + bj$ and $a, b \in \mathbb{C}$. By

$$|\sum_{i=1}^{n} q_i - q| = ((|\sum_{i=1}^{n} a_i - a|)^2 + (|\sum_{i=1}^{n} b_i - b|)^2)^{\frac{1}{2}},$$

let $n \to \infty$, then the conclusion is valid. □

Lemma 2. Under the hypotheses of $X_{\mathbb{H}}$, then

$$X_{\mathbb{H}} = X_{\mathbb{C}} + Y_{\mathbb{C}}.$$

Moreover, if $X_{\mathbb{H}}$ is a quaternion Banach space, then $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are complex Banach spaces under the norm of $X_{\mathbb{H}}$.

Proof. Let $x \in X_{\mathbb{H}}$, then $x = \sum_{n=1}^{\infty} \alpha_i e_i, \alpha_i \in \mathbb{H}$. By (1), then α_i can be represented as $\alpha_i = \alpha_{i1} + \alpha_{i2} j$, where $\alpha_{i1}, \alpha_{i2} \in \mathbb{C}$. By simple computation, then

$$x = \textstyle\sum_{n=1}^{\infty} \alpha_{i1} e_i + \sum_{n=1}^{\infty} \alpha_{i2} j e_i.$$

By using the definitions of $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$, we have

$$X_{\mathbb{H}} = X_{\mathbb{C}} + Y_{\mathbb{C}}.$$

Let $\{y_n\}_{n=1}^{\infty} \subseteq X_{\mathbb{C}}$, then $y_n = \sum_{i=1}^{\infty} \beta_{in} e_i$, where $\beta_{in} \in \mathbb{C}$. If $\{y_n\}_{n=1}^{\infty}$ is a cauchy sequence of the complex vector space $X_{\mathbb{C}}$ under the norm of $X_{\mathbb{H}}$, note that $X_{\mathbb{H}}$ is a Banach space, then y_n is convergent to $y \in X_{\mathbb{H}}$.

Let $y = \sum_{i=1}^{\infty} \alpha_i e_i$, $\alpha_i \in \mathbb{H}$. By (1), there exist α_{i1} and $\alpha_{i2} \in \mathbb{C}$ such that $\alpha_i = \alpha_{i1} + \alpha_{i2} j$, thus

$$y_n - y = \sum_{i=1}^{\infty} (\beta_{in} - \alpha_{i1}) e_i - \sum_{i=1}^{\infty} \alpha_{i2} j e_i.$$

Since y_n is convergent to y, we have

$$\lim_{n \to \infty} \sum_{i=1}^{\infty} (\beta_{in} - \alpha_{i1}) e_i = \sum_{i=1}^{\infty} \alpha_{i2} j e_i.$$

Note that $\{e_i\}_{i=1}^{\infty}$ is the basis of $X_{\mathbb{H}}$ and $X_{\mathbb{C}} \subset X_{\mathbb{H}}$, hence $\{e_i\}_{i=1}^{\infty}$ is also basis of the complex vector space $X_{\mathbb{C}}$. Since $\beta_{in} - \alpha_{i1} \in \mathbb{C}$ and $\{e_i\}_{i=1}^{\infty}$ is a basis of $X_{\mathbb{C}}$, we imply that $\alpha_{i2} = 0$ for $i = 1, 2, \cdots$. Thus $y = \sum_{i=1}^{\infty} \alpha_{i1} e_i$ and $y \in X_{\mathbb{C}}$. So $X_{\mathbb{C}}$ is a Banach space.

Analogue of the above proof, we can also show that $Y_{\mathbb{C}}$ is a Banach space. Here we omit its proof. □

Lemma 3. Let (Ω, Σ) be a measurable space and $X_{\mathbb{H}}$ a quaternion Banach space. Then $m : \Sigma \to X_{\mathbb{H}}$ is a quaternion vector measure if and only if there exist complex vector measures $m_1 : \Sigma \to X_{\mathbb{C}}$ and $m_2 : \Sigma \to Y_{\mathbb{C}}$ such that

$$m = m_1 + m_2.$$

Proof. For each $E \in \Sigma$, since $m : \Sigma \to X_{\mathbb{H}}$, by Lemma 2, $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ are complex Banach spaces under the norm of $X_{\mathbb{H}}$, moreover, $m(E)$ can be uniquely expressed as

$$m(E) = m_1(E) + m_2(E),$$

where

$$m_1(E) = \textstyle\sum_{i=1}^{\infty} m_{i1}(E) e_i \in X_{\mathbb{C}},\ \ m_{i1}(E) \in \mathbb{C},$$
$$m_2(E) = \textstyle\sum_{i=1}^{\infty} m_{i2}(E) j e_i \in Y_{\mathbb{C}},\ \ m_{i2}(E) \in \mathbb{C}.$$

Note that $m : \Sigma \to X_{\mathbb{H}}$ is a function, thus $m_1 : \Sigma \to X_{\mathbb{C}}$ and $m_2 : \Sigma \to Y_{\mathbb{C}}$ are well defined and

$$m = m_1 + m_2. \tag{3}$$

For all sequences of pairwise disjoint sets $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$, by (3), then $m(\cup_{n=1}^{\infty} E_n) = m_1(\cup_{n=1}^{\infty} E_n) + m_2(\cup_{n=1}^{\infty} E_n)$.

According to the uniqueness of the representation of the equality $m(E) = m_1(E) + m_2(E)$ and

$$\Sigma_{i=1}^{\infty} m(E_n) = \Sigma_{i=1}^{\infty} m_1(E_n) + \Sigma_{i=1}^{\infty} m_2(E_n),$$

the proof follows. □

In following, We list a result in [18] as our Lemma 4.

Lemma 4 ([18]). Let $A, B \in M_n(\mathbb{H})$, then

(1). $\chi_{A+B} = \chi_A + \chi_B$,

(2). $\|A\| = \|\chi_A\|$.

Theorem 1 reflects a relation between the quaternion measure and the complex vector measure.

Theorem 1. Let (Ω, Σ) be a measurable space, $\mu_{\mathbb{H}} : \Sigma \to \mathbb{H}$ be a function. Then $\mu_{\mathbb{H}}$ is a quaternion measure if and only if $m : \Sigma \to M_2(\mathbb{C})$ defined by $m(E) = \chi_{\mu_{\mathbb{H}}(E)}$ is a complex vector measure.

Proof. By (1), then $\mu_{\mathbb{H}}(E)$ can be uniquely expressed as

$$\mu_{\mathbb{H}}(E) = \mu_{\mathbb{H}}^{(1)}(E) + \mu_{\mathbb{H}}^{(2)}(E) j \tag{4}$$

where $\mu_{\mathbb{H}}^{(1)}(E)$, $\mu_{\mathbb{H}}^{(2)}(E) \in \mathbb{C}$.

By (2), then

$$m(E) = \chi_{\mu_{\mathbb{H}}}(E) = \begin{bmatrix} \mu_{\mathbb{H}}^{(1)}(E) & \mu_{\mathbb{H}}^{(2)}(E) \\ -\mu_{\mathbb{H}}^{(2)}(E) & \mu_{\mathbb{H}}^{(1)}(E) \end{bmatrix}. \quad (5)$$

Sufficiency: For all sequences of pairwise disjoint sets $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$, since $m : \Sigma \to M_2(\mathbb{C})$ is a complex vector measure, we have

$$m(\cup_{n=1}^{\infty} E_n) = \Sigma_{n=1}^{\infty} m(E_n)$$

in the norm topology of $M_2(\mathbb{C})$. By (5),

$$m(\cup_{n=1}^{\infty} E_n) = \begin{bmatrix} \mu_{\mathbb{H}}^{(1)}(\cup_{n=1}^{\infty} E_n) & \mu_{\mathbb{H}}^{(2)}(\cup_{n=1}^{\infty} E_n) \\ -\mu_{\mathbb{H}}^{(2)}(\cup_{n=1}^{\infty} E_n) & \mu_{\mathbb{H}}^{(1)}(\cup_{n=1}^{\infty} E_n) \end{bmatrix},$$

$$\begin{aligned} \Sigma_{n=1}^{\infty} m(E_n) &= \lim_{n\to\infty} \Sigma_{i=1}^{n} m(E_i) \\ &= \lim_{n\to\infty} \begin{bmatrix} \Sigma_{i=1}^{n} \mu_{\mathbb{H}}^{(1)}(E_i) & \Sigma_{i=1}^{n} \mu_{\mathbb{H}}^{(2)}(E) \\ -\Sigma_{i=1}^{n} \mu_{\mathbb{H}}^{(2)}(E) & \Sigma_{i=1}^{n} \mu_{\mathbb{H}}^{(1)}(E) \end{bmatrix} \\ &= \lim_{n\to\infty} \begin{bmatrix} \mu_{\mathbb{H}}^{(1)}(\cup_{i=1}^{n} E_n) & \mu_{\mathbb{H}}^{(2)}(\cup_{i=1}^{n} E_n) \\ -\mu_{\mathbb{H}}^{(2)}(\cup_{i=1}^{n} E_n) & \mu_{\mathbb{H}}^{(1)}(\cup_{i=1}^{n} E_n) \end{bmatrix} \\ &= \begin{bmatrix} \mu_{\mathbb{H}}^{(1)}(\cup_{n=1}^{\infty} E_n) & \mu_{\mathbb{H}}^{(2)}(\cup_{n=1}^{\infty} E_n) \\ -\mu_{\mathbb{H}}^{(2)}(\cup_{n=1}^{\infty} E_n) & \mu_{\mathbb{H}}^{(1)}(\cup_{n=1}^{\infty} E_n) \end{bmatrix}. \end{aligned}$$

Consequently,

$$\mu_{\mathbb{H}}^{(1)}(\cup_{n=1}^{\infty} E_n) = \Sigma_{n=1}^{\infty} \mu_{\mathbb{H}}^{(1)}(E_n),$$

$$\mu_{\mathbb{H}}^{(2)}(\cup_{n=1}^{\infty} E_n) = \Sigma_{n=1}^{\infty} \mu_{\mathbb{H}}^{(2)}(E_n).$$

Thus, $\mu_{\mathbb{H}}^{(1)}$ and $\mu_{\mathbb{H}}^{(2)}$ are complex measures form Σ to $\mathbb{C}$. By (4) and Definition 1, then $\mu_{\mathbb{H}}$ is a quaternion measure. The sufficiency is proved.

Necessity: If $\mu_{\mathbb{H}} : \Sigma \to \mathbb{H}$ is a quaternion measure, note that the representation of the equality (4) is unique, by Lemma 1, we can imply that $\mu_{\mathbb{H}}^{(1)}$ and $\mu_{\mathbb{H}}^{(2)}$ are complex measures.

Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets in Σ, note that $\mu_{\mathbb{H}}^{(1)}$ and $\mu_{\mathbb{H}}^{(2)}$ are complex measures, by Lemma 4, we can imply that

$$\|\chi_{\mu_{\mathbb{H}}(\cup_{i=1}^{\infty} E_i)} - \chi_{\Sigma_{i=1}^{\infty} \mu_{\mathbb{H}}(E_i)}\| \to 0.$$

By (5), then $m : \Sigma \to M_2(\mathbb{C})$ defined by $m(E) = \chi_{\mu_{\mathbb{H}}(E)}$ is a complex vector measure. □

In the rest of this section, we will give a necessary and sufficient condition for quaternion vector measure. Due to the noncommutative of quaternion, there are two types of linear functional on quaternion Banach space, left linear functional and right linear functional. Here we are interested in the left linear functional, so the introduction to the right linear functional is omitted.

A left quaternion linear functional on a quaternion Banach space X is a map $f : X \to \mathbb{H}$ satisfying

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in X$ and $\alpha, \beta \in \mathbb{H}$. For convenience, we also call the left quaternion linear functional as the quaternion linear functional.

Lemma 5. Let $X_{\mathbb{H}}$ be a quaternion Banach space and $f : X_{\mathbb{C}} \to \mathbb{C}$ a bounded complex linear functional. If

$$F(x) = f(x_1) - jf(jx_2)$$

where $x \in X_{\mathbb{H}}$ with form $x = x_1 + x_2$, $x_1 \in X_{\mathbb{C}}$ and $x_2 \in Y_{\mathbb{C}}$, then $F : X_{\mathbb{H}} \to \mathbb{H}$ is a bounded quaternion linear functional.

Proof. Let $x, y \in X_{\mathbb{H}}$, by Lemma 2, then

$$x = x_1 + x_2, \ y = y_1 + y_2,$$

where $x_1, y_1 \in X_{\mathbb{C}}$ and $x_2, y_2 \in Y_{\mathbb{C}}$.

Note that $jx_2 \in X_{\mathbb{C}}$, thus

$$F(x) = f(x_1) - jf(jx_2)$$

is well defined for each $x \in X_{\mathbb{H}}$.

Let $\alpha, \beta \in \mathbb{C}$, since $\alpha x = \alpha x_1 + \alpha x_2$, $jx = jx_1 + jx_2$, $f : X_{\mathbb{C}} \to \mathbb{C}$ is a linear functional, by simple computation, we can imply that

$$\begin{aligned} F(x+y) &= F(x) + F(y), \\ F(\alpha x) &= f(\alpha x_1) - jf(j\alpha x_2) \\ &= \alpha f(x_1) - jf(\overline{\alpha}\, jx_2) \\ &= \alpha f(x_1) - j\,\overline{\alpha} f(\,jx_2) \\ &= \alpha f(x_1) - \alpha\, jf(\,jx_2) \\ &= \alpha F(x), \\ F(jx) &= f(jx_2) - jf(j^2\, x_1) \\ &= f(jx_2) + jf(x_1) \\ &= j(-jf(jx_2) + f(x_1)) \\ &= jF(x). \end{aligned}$$

By the above arguments, we have that

$$\begin{aligned} F((\alpha + \beta\, j)x) &= F(\alpha\, x) + F(\beta\, jx) \\ &= (\alpha + \beta\, j)F(x). \end{aligned}$$

Thus, F is a linear functional on $X_{\mathbb{H}}$. Note that

$$|f(x)| \le |F(x)| \le |f(x)| + |f(jx)| \le 2\|f\|\|x\|.$$

Hence, $F(x)$ is a bounded quaternion linear functional. □

Lemma 6. Let $X_{\mathbb{H}}$ be the quaternion Banach space and F a bounded quaternion linear functional on $X_{\mathbb{H}}$, then there exist bounded complex linear functionals f_1 and $f_2 : X_{\mathbb{H}} \to \mathbb{C}$ such that

$$F(x) = f_1(x) + f_2(x)j$$

for each $x \in X_{\mathbb{H}}$.

Proof. Since F is a bounded quaternion linear functional, for each $x \in X_{\mathbb{H}}$, by (1), then $F(x)$ can be uniquely expressed as

$$F(x) = f_1(x) + f_2(x)j, \quad (6)$$

where $f_1(x), f_2(x) \in \mathbb{C}$.

Let x, $y \in X_{\mathbb{H}}$, note that $F(x+y) = F(x) + F(y)$, by (6), we can imply that

$$\begin{aligned} F(x+y) &= f_1(x+y) + f_2(x+y)j, \\ F(x) + F(y) &= f_1(x) + f_2(x)j + f_1(y) + f_2(y)j, \\ &= (f_1(x) + f_1(y)) + (f_2(x) + f_2(y))j. \end{aligned}$$

Hence

$$f_1(x+y) = f_1(x) + f_1(y),$$
$$f_2(x+y) = f_2(x) + f_2(y).$$

Let $\alpha \in \mathbb{C}$, since $F(\alpha x) = \alpha F(x)$, by (6), we have

$$F(\alpha x) = f_1(\alpha x) + f_2(\alpha x)j,$$
$$\alpha F(x) = \alpha f_1(x) + \alpha f_2(x)j.$$

Hence, $f_1(\alpha x) = \alpha f_1(x)$, $f_1(\alpha x) = \alpha f_1(x)$. Consequently, f_1 and f_2 are complex linear functionals from $X_{\mathbb{H}}$ to $\mathbb{C}$.

Note that

$$|f_1(x)| < |F(x)|, |f_2(x)| < |F(x)|.$$

Then, f_1 and f_2 are bounded complex linear functionals. □

Lemma 7. Let (Ω, Σ) be a measurable space and $X_{\mathbb{H}}$ a quaternion Banach space. If m, m_1 and m_2 are the same as Lemma 3, and $m : \Sigma \to X_{\mathbb{H}}$ satisfies that $F(m) : \Sigma \to \mathbb{H}$ defined by $E \to F(m(E))$ is quaternion measure for each bounded quaternion linear functional F. Then $m_1 : \Sigma \to X_{\mathbb{C}}$ and $m_2 : \Sigma \to Y_{\mathbb{C}}$ are complex vector measures, respectively.

Proof. Let $f : X_{\mathbb{C}} \to \mathbb{C}$ be an arbitrary bounded complex linear functional, by Lemma 5, then

$$F(x) = f(x_1) - jf(jx_2),$$

is a bounded quaternion linear functional on $X_{\mathbb{H}}$, where $x \in X_{\mathbb{H}}$ with the form $x = x_1 + x_2$, $x_1 \in X_{\mathbb{C}}$ and $x_2 \in Y_{\mathbb{C}}$.

By Lemma 3, for every $E \in \Sigma$, then

$$m(E) = m_1(E) + m_2(E),$$

$m_1(E) \in X_{\mathbb{C}}$ and $m_2(E) \in Y_{\mathbb{C}}$. Hence

$$F(m(E)) = f(m_1(E)) - jf(jm_2(E)).$$

Let $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ be a pairwise disjoint sequence of sets, since $F(m(E))$ is a quaternion measure, we have

$$F(m(\cup_{n=1}^{\infty} E_n))) = \sum_{n=1}^{\infty} F(m(E_n)).$$

Note that

$$F(m(\cup_{n=1}^{\infty} E_n))) = f(m_1(\cup_{n=1}^{\infty} E_n)) - jf(jm_2(\cup_{n=1}^{\infty} E_n)),$$
$$\sum_{n=1}^{\infty} F(m(E_n)) = \sum_{n=1}^{\infty} (f(m_1(E_n)) - jf(jm_2(E_n))),$$

by Lemma 1, then

$$f(m_1(\cup_{n=1}^{\infty} E_n)) = \sum_{n=1}^{\infty} f(m_1(E_n)). \qquad (7)$$

By Lemma 2, $X_{\mathbb{C}}$ is a complex Banach space, note that f is an arbitrary bounded complex linear functional, apply [19, Proposition I,1] to (7), we can imply that $m_1 : \Sigma \to X_{\mathbb{C}}$ is a complex vector measure.

Similar to the above proof, we can also show that $m_2 : \Sigma \to Y_{\mathbb{C}}$ is a complex vector measure. Here its proof is omitted. $\square$

The following theorem is the main result in this paper.

Theorem 2. Let (Ω, Σ) be a measurable space and $X_{\mathbb{H}}$ a quaternion Banach space. If $m : \Sigma \to X_{\mathbb{H}}$ is a function, then m is a quaternion vector measure if and only if $F(m) : \Sigma \to \mathbb{H}$ defined by $E \to F(m(E))$ is a quaternion measure for each bounded quaternion linear functional F.

Proof. Let F be a bounded quaternion linear functional, by Lemma 6, then $F(x) = f_1(x) + f_2(x)j$, where f_1 and $f_2 : X_{\mathbb{H}} \to \mathbb{C}$ are bounded complex linear functionals, respectively.

Necessity: Let $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ be a sequence of pairwise disjoint sets, if $m : \Sigma \to X_{\mathbb{H}}$ is a quaternion vector measure, then

$$F(m(\cup_{n=1}^{\infty} E_n)) = f_1(m(\cup_{n=1}^{\infty} E_i)) + f_2(m(\cup_{n=1}^{\infty} E_i))j.$$

Since $X_{\mathbb{H}}$ can be also regard as a Banach space over $\mathbb{C}$, we regard m as a complex vector measure. Note that f_1 and $f_2 : X_{\mathbb{H}} \to \mathbb{C}$ are bounded complex linear functionals, apply [19, Proposition I.1] to f_1 and f_2, we have

$$f_1(m(\cup_{n=1}^{\infty} E_i)) = f_1(\sum_{n=1}^{\infty} (m(E_n))) = \sum_{n=1}^{\infty} f_1(m(E_n)),$$

$$f_2(m(\cup_{n=1}^{\infty} E_i)) = f_2(\sum_{n=1}^{\infty} (m(E_n))) = \sum_{n=1}^{\infty} f_2(m(E_n)).$$

By the above equalities, then

$$f_1(m(\cup_{n=1}^{\infty} E_i)) + f_2(m(\cup_{n=1}^{\infty} E_i))j$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f_1(m(E_i)) + \lim_{n\to\infty} \sum_{i=1}^{n} f_2(m(E_i))j$$
$$= \sum_{n=1}^{\infty} (f_1(m(E_n)) + f_2(m(E_n))j) = \sum_{n=1}^{\infty} F(m(E_n)).$$

Hence, for each sequence $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ of pairwise disjoint sets, we have

$$F(m(\cup_{n=1}^{\infty} E_n)) = \sum_{n=1}^{\infty} F(m(E_n)).$$

Consequently, the function $F(m) : \Sigma \to \mathbb{H}$ defined by $E \to F(m(E))$ is a quaternion measure. The proof for the necessity of Theorem 2 is complete.

Sufficiency: By using Lemma 7 and Lemma 3, then m is a quaternion vector measure. The proof is completed. $\square$

By Theorem 1 and 2, the following corollary is valid.

Corollary 1. With the same notations as Theorem 2. Then $m : \Sigma \to X_{\mathbb{H}}$ is a quanternionic vector measure if and only if $\chi_{F(m(E))} : \Sigma \to M_2(\mathbb{C})$ is a complex vector measure.

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References

[1] J. A. Clarkson, *Abstrakte Funktionen und lineare Operatoren*, Trans. Amer. Math. Soc., Vol. 40, 1936, pp. 396-414.

[2] I. M. Gel'fand, *Uniformly convex spaces*, Mat. Sb. (N. S.) 4(46), 1938, pp. 235-286.

[3] R. G. Bartle, *A general bilinear vector integral*, Studia Math., Vol. 15, 1956, pp. 337-352.

[4] N. Dinculeanu and I. Kluvánek, *On vector measures*, Proc. London Math. Soc.,17(3), 1967, pp. 505-512.

[5] N. Dunford and J. T. Schwartz, *Linear operators. Part I*, Interscience, New York and London, 1958.

[6] J. Lindenstrauss and A. Pelczyński, *Absolutely summing operators in Lp-spaces and their applications*, Studia Math., Vol. 29, 1968, pp. 275-326.

[7] J. Diestel and J. J. Uhl, Jr, *Vector measures*, Math Surveys Monographs, Providence, 1977.

[8] I. Kluvánek, *Applications of vector measures*, Proceedings of the Conference on Integration, Topology, and Geometry in Linear Spaces (Univ. North Carolina, Chapel Hill, N.C.), 1980, pp.101-134.

[9] J. Diestel and J. J. Uhl, Jr, *Progress in vector measuresł 1977C83*, Indagationes Mathematicae, Vol. 8(1), 1997, pp.33-42.

[10] A. Fernández and Faranjo, *Rybakov's theorem for vector measures in Fréchet spaces*, Measure Theory and its Applications, Lecture Notes in Mathematics Vol. 1033, 1984, pp.144-192.

[11] G. P. Curbera and W. J. Ricker, *Vector Measures, Integration and Applications*, Positivity, Birkhäuser Basel, 2007, pp.127-160.

[12] C. S. Sharma and T. J. Coulson, *Spectral theory for unitary operators on a quaternionic Hilbert space*, Journal of Mathematical Physics, Vol. 28(9), 1987, pp.1941-1946.

[13] S. H. Kulkarni, *Representation of a real B^*-algebra on a quaternionic Hilbert space*, Proceedings of the American Mathematical Society, Vol.121(2), 1994, pp.505-509.

[14] Chi-Keung Ng, *On quaternionic functional analysis*, Mathematical Proceedings of the Cambridge Philosophical Society, Vol.143(2), 2007, pp.391-406.

[15] S. V. Ladkovsky, *Algebras of operators in Banach spaces over the quaternion skew field and the octonion algebra*, Journal of Mathematical Sciences, Vol. 144(4), 2007, pp.4301-4366.

[16] F. Brackx, J. S. R. Chisholm and V Souček, *Clifford analysis and its applications*, Kluwer Academic Publishers, 2001.

[17] S. L. Adler, *Quaternionic quantum mechanics and quantum filed*, Oxford university press, 1995.

[18] H. C. Lee, *Eigenvalues of canonical forms of matrices with quaternion coefficients*, Proc. R. I. A., Vol. 52, 1949, pp.253-260.

[19] W. J. Ricker, *Operator Algebras Generated by Commuting Projections: A Vector Measure Approach*, Springer Berlin Heidelberg, 1999.