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Multiple positive periodic solutions to a predator-prey system with harvesting terms and Holling II type functional response

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Abstract—In this paper, a periodic predator-prey system with harvesting terms and Holling II type functional response is considered. Sufficient criteria for the existence of at least sixteen periodic solutions are established by using the well known continuation theorem due to Mawhin. An example is given to illustrate the main result.

Keywords—Periodic solution; Predator-prey system; Harvesting terms; Continuation theorem.

I. Introduction

RECENTLY, the existence of positive periodic solutions for biological models has been widely studied by many researchers. Models with harvesting terms are often considered. Generally, the predator-prey model with harvesting terms is described as follows:

$$\begin{cases} \dot{x} = xf(x,y) - h, \\ \dot{y} = yg(x,y) - k, \end{cases}$$

where x and y stand for the population of the prey and the predator, respectively; h and k are harvesting terms standing for the harvests (see [1]). Particularly, the ratio-dependent type predator-prey model with harvesting terms is described by the following system of ordinary differential equations

$$\begin{cases} \dot{x} = x\left(a - bx - \dfrac{cy}{my+x}\right) - h, \\ \dot{y} = y\left(-d + \dfrac{fx}{my+x}\right) - k, \end{cases} \tag{1}$$

where a, c, d, f, m are the prey intrinsic growth rate, capture rate, death rate of predator, conversion rate, and half saturation-parameter, respectively. Moreover, on account of the biological background of model (1), we always assume that all of the parameters are positive constants. For the detailed biological meanings, we refer to [2-5] and the references cited therein.

Considering the inclusion of the effect of changing environment, Zhang and Hou [6] investigate the following ratio-dependent predator-prey system with multiple harvesting terms:

$$\begin{cases} \dot{x}(t) = x(t)\left(a(t) - b(t)x(t) - \dfrac{c(t)y(t)}{m(t)y(t)+x(t)}\right) - h_1(t), \\ \dot{y}(t) = y(t)\left(-d(t) + \dfrac{f(t)x(t)}{m(t)y(t)+x(t)}\right) - h_2(t), \end{cases} \tag{2}$$

where the parameter in system (2) are positive continuous ω -periodic functions. By using the coincidence degree theory, the authors established the existence of at least four positive periodic solutions.

The main purpose of this paper is by using continuation theorem to establish new criteria to guarantee the existence of at least sixteen periodic solutions of the periodic predator-prey system with harvesting terms and Holling II type functional response:

$$\begin{cases} \dot{x_1}(t) &= x_1(t)\Big(r_1(t) - a_{11}(t)x_1(t) \\ &\quad - \frac{a_{12}(t)x_2(t)}{m(t)x_2(t)+x_1(t)}\Big) - h_1(t), \\ \dot{x_2}(t) &= x_2(t)\Big(r_2(t) + \frac{\theta_1(t)a_{12}(t)x_1(t)}{m(t)x_2(t)+x_1(t)} \\ &\quad - a_{22}(t)x_2(t) - \frac{a_{23}(t)x_3(t)}{n(t)x_3(t)+x_2(t)}\Big) - h_2(t), \\ \dot{x_3}(t) &= x_3(t)\Big(r_3(t) + \frac{\theta_2(t)a_{23}(t)x_2(t)}{n(t)x_3(t)+x_2(t)} \\ &\quad - a_{33}(t)x_3(t) - \frac{a_{34}(t)x_4(t)}{\alpha(t)x_4(t)+x_3(t)}\Big) - h_3(t), \\ \dot{x_4}(t) &= x_4(t)\Big(r_4(t) + \frac{\theta_3(t)a_{34}(t)x_3(t)}{\alpha(t)x_4(t)+x_3(t)} \\ &\quad - a_{44}(t)x_4(t)\Big) - h_4(t), \end{cases} \tag{3}$$

where $x_i(t)$ stands for the density of the ith species; $r_i(t)$ represents the ith species intrinsic growth rate; $a_{ii}(t)$ denotes the intra-specific competition rate of the ith species; $\theta_1(t)$, $\theta_2(t), \theta_3(t)$ are the nutrition conversion rates for the first species to the second species, the second species to the third species, the third species to the fourth species, respectively; $h_i(t)$ is the harvesting term for the ith species. Moreover, $r_i(t), a_{ii}(t), a_{12}(t), a_{23}(t), a_{34}(t), m(t), n(t), \alpha(t), \theta_1(t), \theta_2(t)$, $\theta_3(t)$ are continuous, bounded and strictly positive ω-periodic functions defined on $[0, \infty)$ $(i = 1, 2, 3, 4)$.

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II. Existence of multiple positive periodic solutions

We recall some basic tools in the framework of Mawhin's coincidence degree [7] that will be used to investigate the existence of periodic solutions.

Let X and Z be two Banach spaces, $L : \mathrm{Dom}\, L \subset X \to Z$ be a linear mapping and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker\, L = co\dim \mathrm{Im}\, L < +\infty$ and $\mathrm{Im}\, L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}\, P = \ker\, L$ and $\mathrm{Im}\, L = \ker\, Q = \mathrm{Im}(I - Q)$. It follows that $L|_{\mathrm{Dom}\, L \cap \ker\, P} : (I - P)X \to \mathrm{Im}\, L$ is invertible. We denote the inverse of the map $L|_{\mathrm{Dom}\, L \cap \ker\, P}$ by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega} \times [0, 1]$ if $(QN)(\overline{\Omega} \times [0, 1])$ is bounded and $K_P(I - Q)N : \overline{\Omega} \times [0, 1] \to X$ is compact. Since $\mathrm{Im}\, Q$ is a isomorphic to $\ker\, L$, there exists an isomorphism $J : \mathrm{Im}\, Q \to \ker L$.

Lemma 1. *(Continuation Theorem ([7])) Let L be a Fredholm mapping of index zero and let N be L -compact on $\overline{\Omega}$. Suppose*

(1) *for each $\lambda \in (0, 1)$, every solution x of $Lx \neq \lambda N(x, \lambda)$ is such that $x \in \partial\Omega \cap \mathrm{Dom}\, L$;*
(2) *$QN(x, 0) \neq 0$, for each $x \in \partial\Omega \cap \ker\, L$;*
(3) *$\deg(JQN(x, 0), \Omega \cap \ker\, L, 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{Dom}\, L \cap \partial\overline{\Omega}$.

For convenience, we denote

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t, \quad f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \sup_{t\in[0,\omega]} f(t),$$

where $f(t)$ is a continuous ω -periodic function.

We need the following assumptions:

(H_1) $r_1^L > 2\sqrt{a_{11}^M h_1^M} + \dfrac{a_{12}^M}{m^L}$;
(H_2) $r_2^L > 2\sqrt{a_{22}^M h_2^M} + \dfrac{a_{23}^M}{n^L}$;
(H_3) $r_3^L > 2\sqrt{a_{33}^M h_3^M} + \dfrac{a_{34}^M}{\alpha^L}$;
(H_4) $r_4^L > 2\sqrt{a_{44}^M h_4^M}$.

For the sake of convenience, we also introduce some notations as follows:

$$l_1^\pm = \frac{r_1^M \pm \sqrt{(r_1^M)^2 - 4a_{11}^L h_1^L}}{2a_{11}^L},$$

$$l_2^\pm = \frac{(r_2^M + a_{12}^M) \pm \sqrt{(r_2^M + a_{12}^M)^2 - 4a_{22}^L h_2^L}}{2a_{22}^L},$$

$$l_3^\pm = \frac{(r_3^M + a_{23}^M) \pm \sqrt{(r_3^M + a_{23}^M)^2 - 4a_{33}^L h_3^L}}{2a_{33}^L},$$

$$l_4^\pm = \frac{(r_4^M + a_{44}^M) \pm \sqrt{(r_4^M + a_{44}^M)^2 - 4a_{44}^L h_4^L}}{2a_{44}^L},$$

$$v_1^\pm = \frac{(r_1^L - \frac{a_{12}^M}{m^L}) \pm \sqrt{(r_1^L - \frac{a_{12}^M}{m^L})^2 - 4a_{11}^M h_1^M}}{2a_{11}^M},$$

$$v_2^\pm = \frac{(r_2^L - \frac{a_{23}^M}{n^L}) \pm \sqrt{(r_2^L - \frac{a_{23}^M}{n^L})^2 - 4a_{22}^M h_2^M}}{2a_{22}^M},$$

$$v_3^\pm = \frac{(r_3^L - \frac{a_{34}^M}{\alpha^L}) \pm \sqrt{(r_3^L - \frac{a_{34}^M}{\alpha^L})^2 - 4a_{33}^M h_3^M}}{2a_{33}^M},$$

$$v_4^\pm = \frac{r_4^L \pm \sqrt{(r_4^L)^2 - 4a_{44}^M h_4^M}}{2a_{44}^M}.$$

Lemma 2. ([8]) *Let $x > 0$, $y > 0$, $z > 0$ and $x > 2\sqrt{yz}$, for the functions $f(x, y, z) = \frac{x+\sqrt{x^2-4yz}}{2z}$ and $g(x, y, z) = \frac{x-\sqrt{x^2-4yz}}{2z}$, the following assertions hold.*

(1) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically increasing and monotonically decreasing on the variable $x \in (0, \infty)$, respectively;*
(2) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $y \in (0, \infty)$, respectively;*
(3) *$f(x, y, z)$ and $g(x, y, z)$ are monotonically decreasing and monotonically increasing on the variable $z \in (0, \infty)$, respectively.*

Theorem 1. *Assume that (H_1)-(H_4) hold, then system (3) has at least sixteen ω-periodic solutions.*

Proof: Since we are concerned with positive solution of (3), we make the change of variables

$$x_i(t) = e^{u_i(t)} \quad (i = 1, 2, 3, 4),$$

the system (3) is rewritten as

$$\begin{cases} \dot{u}_1(t) = r_1(t) - a_{11}(t)e^{u_1(t)} \\ \qquad - \frac{a_{12}(t)e^{u_2(t)}}{m(t)e^{u_2(t)}+e^{u_1(t)}} - h_1(t)e^{-u_1(t)}, \\ \dot{u}_2(t) = r_2(t) + \frac{\theta_1(t)a_{12}(t)e^{u_1(t)}}{m(t)e^{u_2(t)}+e^{u_1(t)}} - a_{22}(t)e^{u_2(t)} \\ \qquad - \frac{a_{23}(t)e^{u_3(t)}}{n(t)e^{u_3(t)}+e^{u_2(t)}} - h_2(t)e^{-u_2(t)}, \\ \dot{u}_3(t) = r_3(t) + \frac{\theta_2(t)a_{23}(t)e^{u_2(t)}}{n(t)e^{u_3(t)}+e^{u_2(t)}} - a_{33}(t)e^{u_3(t)} \\ \qquad - \frac{a_{34}(t)e^{u_4(t)}}{\alpha(t)e^{u_4(t)}+e^{u_3(t)}} - h_3(t)e^{-u_3(t)}, \\ \dot{u}_4(t) = r_4(t) + \frac{\theta_3(t)a_{34}(t)e^{u_3(t)}}{\alpha(t)e^{u_4(t)}+e^{u_3(t)}} \\ \qquad -a_{44}(t)e^{u_4(t)} - h_4(t)e^{-u_4(t)}. \end{cases}$$

Let $X = Z = \{u = (u_1(t), u_2(t), u_3(t), u_4(t))^T \in C(R, R^4) : u(t + \omega) = u(t), t \in R\}$ with the norm defined by $\|u\| = \sum_{i=1}^4 \max_{t\in[0,\omega]} |u_i(t)|, u \in Z$ or X, then X and Z are Banach spaces. Let

$$N(u, \lambda) = \begin{pmatrix} r_1(t) - a_{11}(t)e^{u_1(t)} - \frac{\lambda a_{12}(t)e^{u_2(t)}}{m(t)e^{u_2(t)}+e^{u_1(t)}} - h_1(t)e^{-u_1(t)} \\ r_2(t) + \frac{\lambda\theta_1(t)a_{12}(t)e^{u_1(t)}}{m(t)e^{u_2(t)}+e^{u_1(t)}} - a_{22}(t)e^{u_2(t)} - \frac{\lambda a_{23}(t)e^{u_3(t)}}{n(t)e^{u_3(t)}+e^{u_2(t)}} - h_2(t)e^{-u_2(t)} \\ r_3(t) + \frac{\lambda\theta_2(t)a_{23}(t)e^{u_2(t)}}{n(t)e^{u_3(t)}+e^{u_2(t)}} - a_{33}(t)e^{u_3(t)} - \frac{\lambda a_{34}(t)e^{u_4(t)}}{\alpha(t)e^{u_4(t)}+e^{u_3(t)}} - h_3(t)e^{-u_3(t)} \\ r_4(t) + \frac{\lambda\theta_3(t)a_{34}(t)e^{u_3(t)}}{\alpha(t)e^{u_4(t)}+e^{u_3(t)}} -a_{44}(t)e^{u_4(t)} - h_4(t)e^{-u_4(t)} \end{pmatrix}$$

$Lu = u = \dfrac{\mathrm{d}u(t)}{\mathrm{d}t}$, $Pu = \dfrac{1}{\omega}\displaystyle\int_0^\omega u(t)\,\mathrm{d}t$, $Qz = \dfrac{1}{\omega}\displaystyle\int_0^\omega z(t)\,\mathrm{d}t$, $u \in X, z \in Z, \lambda \in (0,1)$. Thus it follows that $\ker L = R^4$, $\mathrm{Im}\, L = \{z \in Z : \int_0^\omega z(t)\,\mathrm{d}t = 0\}$ is closed in Z, $\dim\ker\ L = 4 = co\dim\mathrm{Im}\ L$, and P, Q are continuous projectors such that

$$\mathrm{Im}\, P = \ker\ L, \qquad \mathrm{Im}\, L = \ker Q = \mathrm{Im}(I - Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \mathrm{Im}L \to \mathrm{Dom}\, L \cap \ker\ P$ is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,\mathrm{d}s\mathrm{d}t.$$

Then

$$QN(u,\lambda) = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega f_1(s)\,\mathrm{d}s\mathrm{d}t \\ \frac{1}{\omega}\int_0^\omega f_2(s)\,\mathrm{d}s\mathrm{d}t \\ \frac{1}{\omega}\int_0^\omega f_3(s)\,\mathrm{d}s\mathrm{d}t \\ \frac{1}{\omega}\int_0^\omega f_4(s)\,\mathrm{d}s\mathrm{d}t \end{pmatrix}$$

and

$$\begin{aligned} & K_P(I-Q)N(u,\lambda) \\ = & \begin{pmatrix} \int_0^t f_1(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t f_1(s)\,\mathrm{d}s\mathrm{d}t \\ \int_0^t f_2(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t f_2(s)\,\mathrm{d}s\mathrm{d}t \\ \int_0^t f_3(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t f_3(s)\,\mathrm{d}s\mathrm{d}t \\ \int_0^t f_4(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t f_4(s)\,\mathrm{d}s\mathrm{d}t \end{pmatrix. \\ & \begin{matrix} +(\frac{1}{2} - \frac{t}{\omega})\int_0^\omega f_1(s)\,\mathrm{d}s \\ +(\frac{1}{2} - \frac{t}{\omega})\int_0^\omega f_2(s)\,\mathrm{d}s \\ +(\frac{1}{2} - \frac{t}{\omega})\int_0^\omega f_3(s)\mathrm{d}s \\ +(\frac{1}{2} - \frac{t}{\omega})\int_0^\omega f_4(s)\mathrm{d}s \end{matrix}\right), \end{aligned}$$

where

$$\begin{aligned} f_1(s) = &\ r_1(s) - a_{11}(s)e^{u_1(s)} - \frac{\lambda a_{12}(s)e^{u_2(s)}}{m(s)e^{u_2(s)} + e^{u_1(s)}} \\ & - h_1(s)e^{-u_1(s)}, \end{aligned}$$

$$\begin{aligned} f_2(s) = &\ r_2(s) + \frac{\lambda\theta_1(s)a_{12}(s)e^{u_1(s)}}{m(s)e^{u_2(s)} + e^{u_1(s)}} - a_{22}(s)e^{u_2(s)} \\ & - \frac{\lambda a_{23}(s)e^{u_3(s)}}{n(s)e^{u_3(s)} + e^{u_2(s)}} - h_2(s)e^{-u_2(s)}, \end{aligned}$$

$$\begin{aligned} f_3(s) = &\ r_3(s) + \frac{\lambda\theta_2(s)a_{23}(s)e^{u_2(s)}}{n(s)e^{u_3(s)} + e^{u_2(s)}} - a_{33}(s)e^{u_3(s)} \\ & - \frac{\lambda a_{34}(s)e^{u_4(s)}}{\alpha(s)e^{u_4(s)} + e^{u_3(s)}} - h_3(s)e^{-u_3(s)}, \end{aligned}$$

$$\begin{aligned} f_4(s) = &\ r_4(s) + \frac{\lambda\theta_3(s)a_{34}(s)e^{u_3(s)}}{\alpha(s)e^{u_4(s)} + e^{u_3(s)}} - a_{44}(s)e^{u_4(s)} \\ & - h_4(s)e^{-u_4(s)}. \end{aligned}$$

Obviously, QN and $K_P(I-Q)N$ are continuous, and $K_P(I-Q)N(\bar\Omega)$ is compact for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\bar\Omega)$ is clearly bounded. Thus, N is L-compact on $\bar\Omega$ with any open bounded set $\Omega \subset X$.

In order to use Lemma 1, we have to find at least sixteen appropriate open bounded subsets in X. Corresponding to the operator equation $Lu = \lambda N(u,\lambda)$, $\lambda \in (0,1)$, we have

$$\begin{cases} \dot u_1(t) = \lambda f_1(t), \\ \dot u_2(t) = \lambda f_2(t), \\ \dot u_3(t) = \lambda f_3(t), \\ \dot u_4(t) = \lambda f_4(t). \end{cases} \tag{4}$$

Assume that $u \in X$ is an ω-periodic solution of the system (4) for some $\lambda \in (0,1)$, then, there exist $\xi_i, \eta_i \in [0,\omega]$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]}, \quad u_i(\eta_i) = \min_{t\in[0,\omega]}, \quad i = 1,2,3,4.$$

It is clear that $\dot u_i(\xi_i) = 0$, $\dot u_i(\eta_i) = 0 (i = 1,2,3,4)$. From this and (4), we have

$$\begin{cases} f_1(\xi_1) = 0, & (a) \\ f_2(\xi_2) = 0, & (b) \\ f_3(\xi_3) = 0, & (c) \\ f_4(\xi_4) = 0, & (d) \end{cases} \tag{5}$$

and

$$\begin{cases} f_1(\eta_1) = 0, & (a) \\ f_2(\eta_2) = 0, & (b) \\ f_3(\eta_3) = 0, & (c) \\ f_4(\eta_4) = 0. & (d) \end{cases} \tag{6}$$

According to (5) (a), we have

$$\begin{aligned} & a_{11}^L e^{u_1(\xi_1)} + h_1^L e^{-u_1(\xi_1)} \\ \le &\ a_{11}e^{u_1(\xi_1)} + h_1 e^{-u_1(\xi_1)} \\ = &\ r_1(\xi_1) - \frac{\lambda a_{12}(\xi_1)e^{u_2(\xi_1)}}{m(\xi_1)e^{u_2(\xi_1)} + e^{u_1(\xi_1)}} \\ < &\ r_1^M, \end{aligned}$$

that is, $a_{11}^L e^{2u_1(\xi_1)} - r_1^M e^{u_1(\xi_1)} + h_1^L < 0$, which implies that

$$\ln l_1^- < u_1(\xi_1) < \ln l_1^+, \tag{7}$$

similarly, by (6) (a), we get

$$\ln l_1^- < u_1(\eta_1) < \ln l_1^+. \tag{8}$$

By the same method, according to (5) (b), we obtain

$$\begin{aligned} & a_{22}^L e^{u_2(\xi_2)} + h_2^L e^{-u_2(\xi_2)} \\ \le &\ a_{22}(\xi_2)e^{u_2(\xi_2)} + h_2(\xi_2)e^{-u_2(\xi_2)} \\ < &\ r_2(\xi_2) + a_{12}(\xi_2) \\ < &\ r_2^M + a_{12}^M, \end{aligned}$$

that is, $a_{22}^L e^{2u_2(\xi_2)} - (r_2^M + a_{12}^M)e^{u_2(\xi_2)} + h_2^L < 0$, which implies that

$$\ln l_2^- < u_2(\xi_2) < \ln l_2^+, \tag{9}$$

similarly, by (6) (b), we get

$$\ln l_2^- < u_2(\eta_2) < \ln l_2^+. \tag{10}$$

From (5) (c), we obtain

$$\begin{aligned}&a_{33}^L e^{u_3(\xi_3)} + h_3^L e^{-u_3(\xi_3)}\\ &\le a_{33}(\xi_3)e^{u_3(\xi_3)} + h_3(\xi_3)e^{-u_3(\xi_3)}\\ &< r_3(\xi_3) + a_{23}(\xi_3)\\ &< r_3^M + a_{23}^M,\end{aligned}$$

that is, $a_{33}^L e^{2u_3(\xi_3)} - (r_3^M + a_{23}^M)e^{u_3(\xi_3)} + h_3^L < 0$, which implies that

$$\ln l_3^- < u_3(\xi_3) < \ln l_3^+, \tag{11}$$

similarly, by (6) (c), we get

$$\ln l_3^- < u_3(\eta_3) < \ln l_3^+. \tag{12}$$

From (5) (d), we obtain

$$\begin{aligned}&a_{44}^L e^{u_4(\xi_4)} + h_4^L e^{-u_4(\xi_4)}\\ &\le a_{44}(\xi_4)e^{u_4(\xi_4)} + h_4(\xi_4)e^{-u_4(\xi_4)}\\ &< r_4(\xi_4) + a_{34}(\xi_4)\\ &< r_4^M + a_{34}^M,\end{aligned}$$

that is, $a_{44}^L e^{2u_4(\xi_4)} - (r_4^M + a_{34}^M)e^{u_4(\xi_4)} + h_4^L < 0$, which implies that

$$\ln l_4^- < u_4(\xi_4) < \ln l_4^+, \tag{13}$$

similarly, by (6) (d), we get

$$\ln l_4^- < u_4(\eta_4) < \ln l_4^+. \tag{14}$$

On the other hand, from (5) (a), we have

$$a_{11}^M e^{2u_1(\xi_1)} - (r_1^L - \frac{a_{12}^M}{m^L})e^{u_1(\xi_1)} + h_1^M > 0,$$

which implies that

$$u_1(\xi_1) > \ln v_1^+ \quad \text{or} \quad u_1(\xi_1) < \ln v_1^-, \tag{15}$$

similarly, by (6) (a), we get

$$u_1(\eta_1) > \ln v_1^+ \quad \text{or} \quad u_1(\eta_1) < \ln v_1^-. \tag{16}$$

We claim that $\ln l_1^+ > \ln v_1^+$ and $\ln l_1^- < \ln v_1^-$. In fact, by Lemma 2, we have

$$\ln l_1^+ = f(r_1^M, h_1^L, a_{11}^L) > f(r_1^L - \frac{a_{12}^M}{m^L}, h_1^M, a_{11}^M) = \ln v_1^+,$$

$$\ln l_1^- = g(r_1^M, h_1^L, a_{11}^L) < g(r_1^L - \frac{a_{12}^M}{m^L}, h_1^M, a_{11}^M) = \ln v_1^-.$$

From (7), (8), (15) and (16), we can get

$$\ln l_1^- < u_1(\eta_1) < u_1(\xi_1) < \ln v_1^-$$

or

$$\ln v_1^+ < u_1(\eta_1) < u_1(\xi_1) < \ln l_1^+,$$

so, for all $t \in R$

$$\ln l_1^- < u_1(t) < \ln v_1^- \quad \text{or} \quad \ln v_1^+ < u_1(t) < \ln l_1^+. \tag{17}$$

According to (5) (b), we obtain

$$a_{22}^M e^{2u_2(\xi_2)} - (r_2^L - \frac{a_{23}^M}{n^L})e^{u_2(\xi_2)} + h_2^M > 0,$$

which implies

$$u_2(\xi_2) > \ln v_2^+ \quad \text{or} \quad u_2(\xi_2) < \ln v_2^-, \tag{18}$$

similarly, by (6) (b), we get

$$u_2(\eta_2) > \ln v_2^+ \quad \text{or} \quad u_2(\eta_2) < \ln v_2^-. \tag{19}$$

We claim that $\ln l_2^+ > \ln v_2^+$ and $\ln l_2^- < \ln v_2^-$. In fact, in view of Lemma 2, we have

$$\ln l_2^+ = f(r_2^M + a_{12}^M, h_2^L, a_{22}^L) > f(r_2^L - \frac{a_{23}^M}{n^L}, h_2^M, a_{22}^M) = \ln v_2^+,$$

$$\ln l_2^- = g(r_2^M + a_{12}^M, h_2^L, a_{22}^L) < g(r_2^L - \frac{a_{23}^M}{n^L}, h_2^M, a_{22}^M) = \ln v_2^-.$$

From (9), (10), (18) and (19), we can get

$$\ln l_2^- < u_2(\eta_2) < u_2(\xi_2) < \ln v_2^-$$

or

$$\ln v_2^+ < u_2(\eta_2) < u_2(\xi_2) < \ln l_2^+,$$

so, for all $t \in R$,

$$\ln l_2^- < u_2(t) < \ln v_2^- \quad \text{or} \quad \ln v_2^+ < u_2(t) < \ln l_2^+.$$

By (5) (c), we obtain

$$a_{33}^M e^{2u_3(\xi_3)} - (r_3^L - \frac{a_{34}^M}{\alpha^L})e^{u_3(\xi_3)} + h_3^M > 0,$$

which implies

$$u_3(\xi_3) > \ln v_3^+ \quad \text{or} \quad u_3(\xi_3) < \ln v_3^-, \tag{20}$$

similarly, by (6) (c), we get

$$u_3(\eta_3) > \ln v_3^+ \quad \text{or} \quad u_3(\eta_3) < \ln v_3^-. \tag{21}$$

We claim that $\ln l_3^+ > \ln v_3^+$ and $\ln l_3^- < \ln v_3^-$. In fact, in view of Lemma 2, we have

$$\ln l_3^+ = f(r_3^M + a_{23}^M, h_3^L, a_{33}^L) > f(r_3^L - \frac{a_{34}^M}{\alpha^L}, h_3^M, a_{33}^M) = \ln v_3^+,$$

$$\ln l_3^- = g(r_3^M + a_{23}^M, h_3^L, a_{33}^L) < g(r_3^L - \frac{a_{34}^M}{\alpha^L}, h_3^M, a_{33}^M) = \ln v_3^-.$$

From (11), (12), (2.19) and (2.20), we can get

$$\ln l_3^- < u_3(\eta_3) < u_3(\xi_3) < \ln v_3^-$$

or

$$\ln v_3^+ < u_3(\eta_3) < u_3(\xi_3) < \ln l_3^+,$$

so, for all $t \in R$,

$$\ln l_3^- < u_3(t) < \ln v_3^- \quad \text{or} \quad \ln v_3^+ < u_3(t) < \ln l_3^+.$$

By (5) (d), we obtain

$$a_{44}^M e^{2u_4(\xi_4)} - r_4^L e^{u_4(\xi_4)} + h_4^M > 0,$$

which implies

$$u_4(\xi_4) > \ln v_4^+ \quad \text{or} \quad u_4(\xi_4) < \ln v_4^-,$$

similarly, by (6) (d), we get

$$u_4(\eta_4) > \ln v_4^+ \quad \text{or} \quad u_4(\eta_4) < \ln v_4^-. \tag{22}$$

We claim that $\ln l_4^+ > \ln v_4^+$ and $\ln l_4^- < \ln v_4^-$. In fact, by Lemma 2, we have

$$\ln l_4^+ = f(r_4^M + a_{34}^M, h_4^L, a_{44}^L) > f(r_4^L, h_4^M, a_{44}^M) = \ln v_4^+,$$
$$\ln l_3^- = g(r_4^M + a_{34}^M, h_4^L, a_{44}^L) < g(r_4^L, h_4^M, a_{44}^M) = \ln v_4^-.$$

From (13), (14), (2.22) and (22), we can get

$$\ln l_4^- < u_4(\eta_4) < u_4(\xi_4) < \ln v_4^-$$

or

$$\ln v_4^+ < u_4(\eta_4) < u_4(\xi_4) < \ln l_4^+,$$

so, for all $t \in R$,

$$\ln l_4^- < u_4(t) < \ln v_4^- \quad \text{or} \quad \ln v_4^+ < u_4(t) < \ln l_4^+. \quad (23)$$

Clearly, $\ln l_1^\pm,\ \ln l_2^\pm,\ \ln l_3^\pm,\ \ln l_4^\pm,\ \ln v_1^\pm,\ \ln v_2^\pm,\ \ln v_3^\pm,\ \ln v_4^\pm$ are independent of λ. Now, let

$$\begin{aligned}\Omega_1 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_2 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^+, \ln v_4^+)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_3 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_4 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^+, \ln v_4^+)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_5 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^+, \ln v_2^+), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_6 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^+, \ln v_2^+), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^+, \ln v_4^+)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_7 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^+, \ln v_2^+), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_8 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^-, \ln v_1^-), u_2 \in (\ln l_2^+, \ln v_2^+), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^+, \ln v_4^+)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_9 = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_{10} = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^+, \ln v_4^+)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_{11} = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_{12} = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^+, \ln v_4^+)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_{13} = \{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^+, \ln v_2^+), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_{14} = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^+, \ln v_2^+), \\ &u_3 \in (\ln l_3^-, \ln v_3^-), u_4 \in (\ln l_4^+, \ln v_4^+)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_{15} = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^-, \ln v_2^-), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^-, \ln v_4^-)\big\},\end{aligned}$$

$$\begin{aligned}\Omega_{16} = \big\{&u = (u_1, u_2, u_3, u_4)^T \in X : \\ &u_1 \in (\ln l_1^+, \ln v_1^+), u_2 \in (\ln l_2^+, \ln v_2^+), \\ &u_3 \in (\ln l_3^+, \ln v_3^+), u_4 \in (\ln l_4^+, \ln v_4^+)\big\}.\end{aligned}$$

Then $\Omega_i (i = 1, 2, \ldots, 16)$ are bounded open subsets of X, $\Omega_i \cap \Omega_j = \phi$, $i \neq j$, $(i, j = 1, 2, \ldots, 16)$, thus $\Omega_i (i = 1, 2, \ldots, 16)$ satisfy the requirement (1) in Lemma 1.

Now, we show that condition (2) of Lemma 1 holds, that is, $QN(u, 0) \neq (0, 0, 0, 0)^T$ for $u \in \partial\Omega_i \cap \ker L = \partial\Omega_i \cap R^4 (i = 1, 2, \ldots, 16)$. If it is not true, then there exists constant vector $u = (u_1, u_2, u_3, u_4)^T \in \partial\Omega_i$ satisfies

$$\begin{cases}\int_0^\omega r_1(t)\mathrm{d}t - \int_0^\omega a_{11}(t)e^{u_1}\mathrm{d}t - \int_0^\omega h_1(t)e^{-u_1}\mathrm{d}t = 0, \\ \int_0^\omega r_2(t)\mathrm{d}t - \int_0^\omega a_{22}(t)e^{u_2}\mathrm{d}t - \int_0^\omega h_2(t)e^{-u_1}\mathrm{d}t = 0, \\ \int_0^\omega r_3(t)\mathrm{d}t - \int_0^\omega a_{33}(t)e^{u_3}\mathrm{d}t - \int_0^\omega h_3(t)e^{-u_1}\mathrm{d}t = 0, \\ \int_0^\omega r_4(t)\mathrm{d}t - \int_0^\omega a_{44}(t)e^{u_4}\mathrm{d}t - \int_0^\omega h_4(t)e^{-u_1}\mathrm{d}t = 0.\end{cases}$$

In view of differential mean value theorem, there exist four points $t_j \in [0,\omega](j=1,2,3,4)$ such that

$$\begin{cases} r_1(t_1) - a_{11}(t_1)e^{u_1} - h_1(t_1)e^{-u_1} = 0, \\ r_2(t_2) - a_{22}(t_2)e^{u_2} - h_2(t_2)e^{-u_2} = 0, \\ r_3(t_3) - a_{33}(t_3)e^{u_3} - h_3(t_3)e^{-u_3} = 0, \\ r_4(t_4) - a_{44}(t_4)e^{u_4} - h_4(t_4)e^{-u_4} = 0, \end{cases}$$

therefore, we have

$$\ln l_1^- < u_1(t) < \ln v_1^- \quad \text{or} \quad \ln v_1^+ < u_1(t) < \ln l_1^+,$$

$$\ln l_2^- < u_2(t) < \ln v_2^- \quad \text{or} \quad \ln v_2^+ < u_2(t) < \ln l_2^+,$$

$$\ln l_3^- < u_3(t) < \ln v_3^- \quad \text{or} \quad \ln v_3^+ < u_3(t) < \ln l_3^+,$$

$$\ln l_4^- < u_4(t) < \ln v_4^- \quad \text{or} \quad \ln v_4^+ < u_4(t) < \ln l_4^+,$$

which imply $u \in \Omega_i \cap R^4$, it contradicts the fact that $u \in \partial\Omega_i \cap R^4 (i=1,2,\ldots,16)$. The condition (2) of Lemma 1 holds. Finally, we show that condition (3) of Lemma 1 is valid. Noting that the system of algebraic equations

$$\begin{cases} r_1(t_1) - a_{11}(t_1)e^{x} - h_1(t_1)e^{-x} = 0, \\ r_2(t_2) - a_{22}(t_2)e^{y} - h_2(t_2)e^{-y} = 0, \\ r_3(t_3) - a_{33}(t_3)e^{z} - h_3(t_3)e^{-z} = 0, \\ r_4(t_4) - a_{44}(t_4)e^{p} - h_4(t_4)e^{-p} = 0 \end{cases}$$

has sixteen distinct solutions:

$$\begin{aligned}
(x_1^*, y_1^*, z_1^*, p_1^*) &= (\ln x_-, \ln y_-, \ln z_-, \ln p_-), \\
(x_2^*, y_2^*, z_2^*, p_2^*) &= (\ln x_-, \ln y_-, \ln z_-, \ln p_+), \\
(x_3^*, y_3^*, z_3^*, p_3^*) &= (\ln x_-, \ln y_-, \ln z_+, \ln p_-), \\
(x_4^*, y_4^*, z_4^*, p_4^*) &= (\ln x_-, \ln y_-, \ln z_+, \ln p_+), \\
(x_5^*, y_5^*, z_5^*, p_5^*) &= (\ln x_-, \ln y_+, \ln z_-, \ln p_-), \\
(x_6^*, y_6^*, z_6^*, p_6^*) &= (\ln x_-, \ln y_+, \ln z_-, \ln p_+), \\
(x_7^*, y_7^*, z_7^*, p_7^*) &= (\ln x_-, \ln y_+, \ln z_+, \ln p_-), \\
(x_8^*, y_8^*, z_8^*, p_8^*) &= (\ln x_-, \ln y_+, \ln z_+, \ln p_+), \\
(x_9^*, y_9^*, z_9^*, p_9^*) &= (\ln x_+, \ln y_-, \ln z_-, \ln p_-), \\
(x_{10}^*, y_{10}^*, z_{10}^*, p_{10}^*) &= (\ln x_+, \ln y_-, \ln z_-, \ln p_+), \\
(x_{11}^*, y_{11}^*, z_{11}^*, p_{11}^*) &= (\ln x_+, \ln y_-, \ln z_+, \ln p_-), \\
(x_{12}^*, y_{12}^*, z_{12}^*, p_{12}^*) &= (\ln x_+, \ln y_-, \ln z_+, \ln p_+), \\
(x_{13}^*, y_{13}^*, z_{13}^*, p_{13}^*) &= (\ln x_+, \ln y_+, \ln z_-, \ln p_-), \\
(x_{14}^*, y_{14}^*, z_{14}^*, p_{14}^*) &= (\ln x_+, \ln y_+, \ln z_-, \ln p_+), \\
(x_{15}^*, y_{15}^*, z_{15}^*, p_{15}^*) &= (\ln x_+, \ln y_+, \ln z_+, \ln p_-), \\
(x_{16}^*, y_{16}^*, z_{16}^*, p_{16}^*) &= (\ln x_+, \ln y_+, \ln z_+, \ln p_+),
\end{aligned}$$

where

$$x_\pm = \frac{r_1(t_1) \pm \sqrt{(r_1(t_1))^2 - 4a_{11}(t_1)h_1(t_1)}}{2a_{11}(t_1)},$$

$$y_\pm = \frac{r_2(t_2) \pm \sqrt{(r_2(t_2))^2 - 4a_{22}(t_2)h_2(t_2)}}{2a_{22}(t_2)},$$

$$z_\pm = \frac{r_3(t_3) \pm \sqrt{(r_3(t_3))^2 - 4a_{33}(t_3)h_3(t_3)}}{2a_{33}(t_3)},$$

$$p_\pm = \frac{r_4(t_4) \pm \sqrt{(r_4(t_4))^2 - 4a_{44}(t_4)h_4(t_4)}}{2a_{44}(t_4)},$$

by Lemma 2, it easy to verify that

$$\ln l_1^- < \ln x_- < \ln v_1^- < \ln v_1^+ < \ln x_+ < \ln l_1^+,$$

$$\ln l_2^- < \ln y_- < \ln v_2^- < \ln v_2^+ < \ln y_+ < \ln l_2^+,$$

$$\ln l_3^- < \ln z_- < \ln v_3^- < \ln v_3^+ < \ln z_+ < \ln l_3^+,$$

$$\ln l_4^- < \ln p_- < \ln v_4^- < \ln v_4^+ < \ln p_+ < \ln l_4^+,$$

therefore

$$(x_1^*, y_1^*, z_1^*, p_1^*) \in \Omega_1,\ (x_2^*, y_2^*, z_2^*,\ p_2^*) \in \Omega_2,$$

$$(x_3^*,\ y_3^*, z_3^*, p_3^*) \in \Omega_3,\ (x_4^*, y_4^*, z_4^*, p_4^*) \in \Omega_4,$$

$$(x_5^*, y_5^*, z_5^*, p_5^*) \in \Omega_5,\ (x_6^*, y_6^*, z_6^*, p_6^*) \in \Omega_6,$$

$$(x_7^*, y_7^*, z_7^*, p_7^*) \in \Omega_7,\ (x_8^*, y_8^*, z_8^*, p_8^*) \in \Omega_8,$$

$$(x_9^*, y_9^*, z_9^*, p_9^*) \in \Omega_9,\ (x_{10}^*, y_{10}^*, z_{10}^*, p_{10}^*) \in \Omega_{10},$$

$$(x_{11}^*, y_{11}^*, z_{11}^*, p_{11}^*) \in \Omega_{11}, (x_{12}^*, y_{12}^*, z_{12}^*, p_{12}^*) \in \Omega_{12},$$

$$(x_{13}^*, y_{13}^*, z_{13}^*, p_{13}^*) \in \Omega_{13},\ (x_{14}^*, y_{14}^*, z_{14}^*, p_{14}^*) \in \Omega_{14},$$

$$(x_{15}^*, y_{15}^*, z_{15}^*, p_{15}^*) \in \Omega_{15},\ (x_{16}^*, y_{16}^*, z_{16}^*, p_{16}^*) \in \Omega_{16}.$$

Since $\ker L = \operatorname{Im} P$, we take $J = I$, a direct computation yields that

$$\begin{aligned}
&\deg(\mathrm{JQN}(x,0), \Omega_i \cap \ker L, (0,,0,0,0)^T) \\
&= \operatorname{sign} \begin{bmatrix} -a_{11}(t_1)x^* + \frac{h_1(t_1)}{x^*} & 0 & 0 & 0 \\ 0 & -a_{22}(t_2)y^* + \frac{h_2(t_2)}{y^*} & 0 & 0 \\ 0 & 0 & -a_{33}(t_3)z^* + \frac{h_3(t_3)}{z^*} & 0 \\ 0 & 0 & 0 & -a_{44}(t_4)p^* + \frac{h_4(t_4)}{p^*} \end{bmatrix} \\
&= \operatorname{sign}\Bigg[\bigg(-a_{11}(t_1)x^* + \frac{h_1(t_1)}{x^*}\bigg)\bigg(-a_{22}(t_2)y^* + \frac{h_2(t_2)}{y^*}\bigg) \\
&\quad \times\bigg(-a_{33}(t_3)z^* + \frac{h_3(t_3)}{z^*}\bigg)\bigg(-a_{44}(t_4)p^* + \frac{h_4(t_4)}{p^*}\bigg)\Bigg]
\end{aligned}$$

together with the fact

$$\begin{cases} r_1(t_1) - a_{11}(t_1)x^* + \frac{h_1(t_1)}{x^*} = 0, \\ r_2(t_2) - a_{22}(t_2)y^* + \frac{h_2(t_2)}{y^*} = 0, \\ r_3(t_3) - a_{33}(t_3)z^* + \frac{h_3(t_3)}{z^*} = 0, \\ r_4(t_4) - a_{44}(t_4)p^* + \frac{h_4(t_4)}{p^*} = 0, \end{cases}$$

then

$$\begin{aligned}
& \deg(\mathrm{JQN}(x,0), \Omega_i \cap \ker L, (0,,0,0,0)^T) \\
= \ & \operatorname{sign}[(r_1(t_1) - 2a_{11}(t_1)x^*)(r_2(t_2) - 2a_{22}(t_2)y^*) \\
& \times(r_3(t_3) - 2a_{33}(t_3)z^*)(r_4(t_4) - 2a_{44}(t_4)p^*)], \\
& i = 1,2,\ldots,16.
\end{aligned}$$

Thus

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_1\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_2\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_3\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_4\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_5\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_6\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_7\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_8\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_-)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_9\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_{10}\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_{11}\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_{12}\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_-)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_{13}\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_{14}\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_-)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_{15}\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_-)]\\&=-1,\end{aligned}$$

$$\begin{aligned}&\deg(\mathrm{JQN}(x,0),\Omega_{16}\cap\ker L,(0,,0,0,0)^T)\\=&\ \mathrm{sign}[(r_1(t_1)-2a_{11}(t_1)x_+)(r_2(t_2)-2a_{22}(t_2)y_+)\\&\times(r_3(t_3)-2a_{33}(t_3)z_+)(r_4(t_4)-2a_{44}(t_4)p_+)]\\&=1.\end{aligned}$$

So far, we have proved that $\Omega_i(i=1,2,\ldots,16)$ satisfies (1)-(3) of Lemma 1. Hence, system (3) has at least sixteen positive ω-periodic solutions. The proof of Theorem 2.1 is complete. ■

III. An Example

Consider the following four-species predator-prey with harvesting terms:

$$
\begin{cases}
\dot{x_1} = x_1\left(3+\sin t-\frac{4+\cos t}{10}x_1-\frac{\frac{1}{15+\cos t}x_2}{(2+\cos t)x_2+x_1}\right) \\
\quad -\frac{9+\cos t}{20}, \\
\dot{x_2} = x_2\left(3+\sin t+\frac{\frac{2+\sin t}{10}\times\frac{1}{15+\cos t}x_1}{(2+\cos t)x_2+x_1}-\frac{6+\cos t}{10}x_2\right. \\
\quad \left.-\frac{\frac{1}{11+\sin t}x_3}{(4+\cos t)x_3+x_2}\right)-\frac{2+\cos t}{9}, \\
\dot{x_3} = x_3\left(3+\cos t+\frac{\frac{2+\sin t}{9}\times\frac{1}{11+\sin t}x_2}{(4+\cos t)x_3+x_2}-\frac{4+\sin t}{10}x_3\right. \\
\quad \left.-\frac{\frac{1}{10+\sin t}x_4}{(3+\sin t)x_4+x_3}\right)-\frac{4+\cos t}{10}, \\
\dot{x_4} = x_4\left(3+\cos t+\frac{\frac{2+\sin t}{12}\times\frac{1}{10+\sin t}x_3}{(3+\cos t)x_4+x_3}-\frac{4+\cos t}{10}x_4\right) \\
\quad -\frac{2+\cos t}{10}.
\end{cases}
$$

In this case, $r_1(t)=3+\sin t$, $a_{11}(t)=\frac{4+\cos t}{10}$, $a_{12}(t)=\frac{1}{15+\cos t}$, $m(t)=2+\cos t$, $h_1(t)=\frac{9+\cos t}{20}$, $r_2(t)=3+\sin t$, $\theta_1(t)=\frac{2+\sin t}{10}$, $a_{22}(t)=\frac{6+\cos t}{10}$, $a_{23}(t)=\frac{1}{11+\sin t}$, $n(t)=4+\cos t$, $h_2(t)=\frac{2+\cos t}{9}$, $r_3(t)=3+\cos t$, $\theta_2(t)=\frac{2+\sin t}{9}$, $a_{33}(t)=\frac{4+\sin t}{10}$, $a_{34}(t)=\frac{1}{10+\sin t}$, $\alpha(t)=3+\sin t$, $h_3(t)=\frac{4+\cos t}{10}$, $r_4(t)=3+\cos t$, $\theta_3(t)=\frac{2+\sin t}{12}$, $a_{44}(t)=\frac{4+\cos t}{10}$, $h_4(t)=\frac{1+\cos t}{10}$. Since $r_1^L=2$, $a_{11}^M=\frac{1}{2}$, $a_{12}^M=\frac{1}{14}$, $m^L=1$, $h_1^M=\frac{1}{2}$, $r_2^L=2$, $a_{22}^M=\frac{7}{10}$, $a_{23}^M=\frac{1}{10}$, $n^L=3$, $h_2^M=\frac{1}{3}$, $r_3^L=2$, $a_{33}^M=\frac{1}{2}$, $a_{34}^M=\frac{1}{9}$, $\alpha^L=2$, $h_3^M=\frac{1}{2}$, $r_4^L=2$, $a_{44}^M=\frac{1}{2}$, $h_4^M=\frac{1}{5}$, then

$$r_1^L=2>2\sqrt{\frac{1}{2}\times\frac{1}{2}}+\frac{1}{14}=2\sqrt{a_{11}^Mh_1^M}+\frac{a_{12}^M}{m^L},$$

$$r_2^L=2>2\sqrt{\frac{7}{10}\times\frac{1}{3}}+\frac{1}{30}=2\sqrt{a_{22}^Mh_2^M}+\frac{a_{23}^M}{n^L},$$

$$r_3^L=2>2\sqrt{\frac{1}{2}\times\frac{1}{2}}+\frac{1}{18}=2\sqrt{a_{33}^Mh_3^M}+\frac{a_{34}^M}{\alpha^L},$$

$$r_4^L=2>2\sqrt{\frac{1}{2}\times\frac{1}{5}}=2\sqrt{a_{44}^Mh_4^M}.$$

According to Theorem 1, it is easy to see the above system has at least sixteen positive 2π-periodic solutions.

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