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Modern Method for Solving Pure Integer Programming Models

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Abstract**—** In this paper, all variables are supposed to be integer and positive. In this modern method, objective function is assumed to be maximized or minimized but constraints are always explained like less or equal to. In this method, choosing a dual combination of ideal nonequivalent and omitting one of variables. With continuing this act, finally, having one nonequivalent with (n-m+1) unknown quantities in which final nonequivalent, m is counter for constraints, n is counter for variables of decision.

Keywords**—**Integer, Programming, Operation Research, Variables of decision.

I. Introduction

ONE of the most important branches of Operation Research which wide usage of that is not covered to any other science, is pure integer programming in which this model all variables of decision are integer and positive[1,2]. Until knowing many methods for solving pure integer programming models which had explained by many scientists of this branch, but none of them has proper outcomes, especially, some of those methods with long calculations may not have optimal result [3,4]. This method, guaranties the optimal result, because it is based on solving equivalent apparatus by omitting method. In near future, software of this method will be introduced.

II. Manuscript

Supposing pure integer models with two faces:

A: $MAX\ Z = \sum_{j=1}^{n} C_j x_j$

s.t: $Ax \leq b$

$x \in W$; $W=\{0,1,2,3,\ldots\}$

Defining Case "A":

A(1): $C_k > 0$, $a_{ik} > 0$; $i = 1,\ldots,m$

$k \in j = 1,\ldots,n$

A(2): $C_k < 0$, $a_{ik} < 0$; $i = 1,\ldots,m$; $k \in j = 1,\ldots,n$

B: $MIN\ Z = \sum_{j=1}^{n} C_j x_j$

s.t: $Ax \leq b$

$x \in W$; $W=\{0,1,2,3,\ldots\}$

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Defining Case "B" :

B(1): $C_k > 0$, $a_{ik} < 0$; $i = 1,\ldots,m$; $k \in j = 1,\ldots,n$

B(2): $C_k < 0$, $a_{ik} > 0$; $i = 1,\ldots,m$; $k \in j = 1,\ldots,n$

Without putting any contrary in totally logic, supposing case "A(1)" indefeasible:

1) $MAX\ Z = \sum_{j=1}^{n} C_j x_j$; $C_k > 0$, $a_{ik} > 0$

s.t: $Ax \leq b$; $i = 1\ldots m$;

$k \in j = 1,\ldots,n$ $x \in W$

In this satiation, supposing objective function as: $-\sum_{j=1}^{n} C_j x_j \leq -Z$ with "m" nonequivalent constraints. Forming an apparatus with (m+1) nonequivalent with (n+1) unknown quantities.

$$2)\quad \begin{cases} \mathbf{(1)}\ -C_1x_1 - \cdots - C_nx_n \leq -Z \\ \mathbf{(2)}\ \ a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \\ \mathbf{(3)}\ \ a_{21}x_1 + \cdots + a_{1n}x_n \leq b_2 \\ \quad\vdots \\ \mathbf{(m+1)}\ \ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{cases}$$

$(x_1,\ldots,x_n) \in W$

Supposing all dual combination from nonequivalent of (1) with all other nonequivalenst:

$$3)\quad (1,2)\begin{cases} -C_1x_1 - \cdots - C_nx_n \leq -Z \\ a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1 \end{cases}$$

$$(1,3)\begin{cases} -C_1x_1 - \cdots - C_nx_n \leq -Z \\ a_{21}x_1 + \cdots + a_{1n}x_n \leq b_2 \end{cases}$$

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$$(1,(m+1))\begin{cases} -C_1x_1 - \cdots - C_nx_n \leq -Z \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \end{cases}$$

In upon apparatus, C_1, a_{11}, a_{21},…, a_{m1} are supposed to be positives, therefore, Omitting x_1 from all upon apparatus, hence, in apparatus (1,2) having:

4) $\begin{cases} -a_{11}C_1x_1 - \cdots - a_{11}C_nx_n \ \leq -a_{11}Z \\ a_{11}C_1x_1 + \cdots + a_{1n}C_1x_n \ \leq C_1b_1 \end{cases}$

From totaling upon dual nonequivalent, having:

5) $-a_{11}C_2x_2 + a_{12}C_1x_2 - \cdots - a_{11}C_nx_n + a_{1n}C_1x_n \leq -a_{11}Z + C_1b_1$

Omitting x_1 continues, having:

6) $-a_{m1}C_2x_2 + a_{m2}C_1x_2 - \cdots - a_{m1}C_nx_n + a_{mn}C_1x_n \leq -a_{m1}Z + C_1b_m$

Having an apparatus with "m" nonequivalent with "n" unknown quantities, (5) and (6) are samples of them. We can continuing the last step and Omitting x_2 and then having (m-1) nonequivalent with (n-1) unknown quantities.

Finally, with continuing upon step and Omitting $x_3, x_4, \ldots, x_l$ $(l \in j)$, reaching to nonequivalent with (n-m+1) unknown quantities, interducing:

7) $x_{l+1}, x_{l+2}, \ldots, x_n$, Z

(End of section 2)

III. Examples

A. *Example for "Case A(1)"*

$$MAX\ Z = 2x_1 + x_2 - 3x_3 + 5x_4$$

s.t:

8) $\begin{cases} 3x_1 - x_2 + x_3 + 2x_4 \ \leq 8 \\ x_1 + 7x_2 + 3x_3 + 7x_4 \ \leq 46 \\ 2x_1 + 3x_2 - x_3 + x_4 \ \leq 10 \end{cases}$

$(x_1, x_2, x_3, x_4) \in W$; $W=\{0,1,2,\ldots\}$

Resolve:

9) $\begin{Bmatrix} -2x_1 - x_2 + 3x_3 - 5x_4 \ \leq -Z \\ 3x_1 - x_2 + x_3 + 2x_4 \ \leq 8 \end{Bmatrix}$

Omitting x_1 in apparatus (9),then having:

10) $3Z \leq 5x_2 - 11x_3 + 11x_4 + 16$

11) $\begin{cases} -2x_1 - x_2 + 3x_3 - 5x_4 \ \leq -Z \\ x_1 + 7x_2 + 3x_3 + 7x_4 \ \leq 46 \end{cases}$

Omitting x_1 in apparatus (11), then having:

12) $Z \leq -13x_2 - 9x_3 - 9x_4 + 92$

13) $\begin{cases} -2x_1 - x_2 + 3x_3 - 5x_4 \ \leq -Z \\ 2x_1 + 3x_2 - x_3 + x_4 \ \leq 10 \end{cases}$

Omitting x_1 in apparatus (13), then having:

14) $Z \leq -2x_2 - 2x_3 + 4x_4 + 10$

Here, forming an apparatus using nonequivalents of (10), (12), (14) :

15) $\begin{cases} 3Z \ \leq 5x_2 - 11x_3 + 11x_4 + 16 \\ Z \ \leq -13x_2 - 9x_3 - 9x_4 + 92 \\ Z \ \leq -2x_2 - 2x_3 + 4x_4 + 10 \end{cases}$

Choosing a dual combination form above apparatus:

16) $\begin{cases} 3Z \ \leq 5x_2 - 11x_3 + 11x_4 + 16 \\ Z \ \leq -13x_2 - 9x_3 - 9x_4 + 92 \end{cases}$

Omitting x_2 in apparatus (16), then having:

17) $44Z \leq -188x_3 + 98x_4 + 668$

Choosing a dual combination from above apparatus:

18) $\begin{cases} 3Z \ \leq 5x_2 - 11x_3 + 11x_4 + 16 \\ Z \ \leq -2x_2 - 2x_3 + 4x_4 + 10 \end{cases}$

Omitting x_2 in apparatus (18), then having:

19) $11Z \leq -32x_3 + 42x_4 + 82$

Variables of x_3 and x_4 can not be omitted in (17), (19), because, the sign of x_3 in both of them is negative and the sign of x_4, in both of them is positive.

Supposing (17), (19) :

20) $\begin{cases} 44Z \ \leq -188x_3 + 98x_4 + 668 \\ 44Z \ \leq -128x_3 + 168x_4 + 328 \end{cases}$

Having 3 conditions:

Condition 1:

21) $-188x_3 + 98x_4 + 668 = -128x_3 + 168x_4 + 328$

Simplifying (21), then having:

22) $6x_3 + 7x_4 = 34$

According to (22) and integering variables of decision:

23) $x_4 = 2q => 3x_3 + 7q = 17$

According to (23) , having:

24) $q + 1 = 3d => q = 3d - 1 => 3x_3 + 21d = 24$

Therefore, having:

25) $x_3 = 8 - 7d \ \ , \ \ x_4 = 6d - 2$

Hence, for d=1 having:

26) $x_3 = 1 \ \ , \ \ x_4 = 4$

Putting values of x_3 and, x_4 in nonequivalents of (20) having:

27) $44Z \leq -188(1) + 98(4) + 668 => Z \leq 19$

Putting values of x_3, x_4 and Z in nonequivalent (8) having:

28) $$\begin{cases} 2x_1 + x_2 - 3(1) + 5(4) \leq 19 \quad => \\ \quad 2x_1 + x_2 \leq 2 \\ 3x_1 - x_2 + 1 + 2(4) \leq 8 \quad => \\ \quad 3x_1 - x_2 \leq -1 \\ x_1 + 7x_2 + 3(1) + 7(4) \leq 46 \quad => \\ \quad x_1 + 7x_2 \leq 15 \\ 2x_1 + 3x_2 - 1 + 4 \leq 10 \quad => \\ \quad 2x_1 + 3x_2 \leq 7 \end{cases}$$

Simply , According to (28) having:

29) $x_1 = 0 \ , \ x_2 = 2 \ , \ Z = 19$

Condition 2:

30) $-188x_3 + 98x_4 + 668 > -128x_3 + 168x_4 + 328$

Simplifying (30), then having:

31) $6x_3 + 7x_4 < 34$

Therefore having:
32) $44Z \leq -128x_3 + 168x_4 + 328$

Hence, regarding to (31) and (32) having:

33) $x_3 = 0 \ , \ x_4 = 4 \ , \ Z \leq 22$

Thereupon:

34) $$\begin{cases} 2x_1 + x_2 \leq 2 \\ 3x_1 - x_2 \leq 0 \\ x_1 + 7x_2 \leq 18 \\ 2x_1 + 3x_2 \leq 6 \end{cases} => 5x_1 \leq 2 => x_1 = 0, x_2 = 2 ,$$

$$Z = 22$$

Condition 3:

35) $-188x_3 + 98x_4 + 668 < -128x_3 + 168x_4 + 328$

Sampling:

36) $6x_3 + 7x_4 > 34$

Hence:

37) $44Z \leq -188x_3 + 98x_4 + 668$
Regarding to (36) and constraint of $x_1 + 7x_2 + 3x_3 + 7x_4 \leq 46$ having:

38) $x_3 = 0 \ , \ x_4 = 5 \ \ or \ \ x_4 = 6$

Putting values in (37) :

39) $(x_3 = 0 \, , x_4 = 5) => Z \leq 26$

Putting values in objective function:

40) $2x_1 + x_2 \leq 1 => x_1 = 0 => x_2 \leq 1$

Putting values in first constraint:

41) $-x_2 + 10 \leq 8 => x_2 \geq 2$

Nonequivalents of (40), (41) reverse each other.
Putting values in (37):

42) $(x_3 = 0 \, , x_4 = 6) => Z \leq 28$

Putting values in objective function:

43) $2x_1 + x_2 \leq -2$

Unequal of (43) is an impossible tie (x_j is an integer variable).
Therefore optimal result is attained from (34):

44) $x_1 = 0 \ , \ x_2 = 2 \ , \ x_3 = 0 \ , \ x_4 = 4 \ , \ Z = 22$
End of example A.

B. Example for "Case A(2)"

$MAX \ Z = -3x_1 + x_2 - 7x_3 + 3x_4$
s.t:

45) $$\begin{cases} -x_1 + x_2 - x_3 + x_4 \leq -2 \\ x_1 - x_2 - x_3 \leq 1 \end{cases}$$

$(x_1, x_2, x_3, x_4) \in W \ ; \ W=\{0,1,2,\ldots\}$

Resolve:

46) $$\begin{cases} 3x_1 - x_2 + 7x_3 - 3x_4 \leq -Z \\ -x_1 + x_2 - x_3 + x_4 \leq -2 \end{cases}$$

Omitting x_1 in apparatus (46),then having:

47) $2x_2 + 4x_3 \leq -Z - 6$

48) $$\begin{cases} -x_1 + x_2 - x_3 + x_4 \leq -2 \\ x_1 - x_2 - x_3 \leq 1 \end{cases}$$

Omitting x_1 in apparatus of (48),then having:

49) $-2x_3 + x_4 \leq -1 => x_3 \geq 1$

Forming (47) and (49) together, Omitting x_3:

50) $$\begin{cases} 2x_2 + 4x_3 \leq -Z - 6 \\ -2x_3 + x_4 \leq 1 \end{cases} =>$$
$$2(x_2 + x_4) \leq -Z - 8$$

51) $Z \leq -2x_2 - 2x_4 - 8$

Regarding to (49):

52) $x_3 = 1 => x_4 \leq 1 => x_4 = 1 \ or \ x_4 = 0$

Putting values in objective function:

53) $x_4 = 1 => Z = -3x_1 + x_2 - 4$

Putting values in constraints:

54) $\begin{cases} -x_1 + x_2 \leq -2 \\ x_1 - x_2 \leq 2 \end{cases} => x_1 - x_2 = 2$

55) $Z = -3x_1 + x_2 - 4 = -3(x_2 + 2) + x_2 - 4 = -2x_2 - 10$

$=> \begin{cases} x_2 = 0 \\ x_1 = 2 \\ Z = -10 \end{cases}$

Putting values in objective function:

56) $x_4 = 0 => Z = -3x_1 + x_2 - 7$
(56)

Putting values in constraints:

57) $\begin{cases} -x_1 + x_2 \leq -1 \\ x_1 - x_2 \leq 2 \end{cases} => x_1 - x_2 \geq 1$

58) $x_1 - x_2 = 1 => x_1 = x_2 + 1 => Z = -3(x_2 + 1) + x_2 - 7 => Z = -2x_2 - 10$

Therefore:

59) $x_2 = 0 => x_1 = 1 \ , \ Z = -10$

60) $x_1 - x_2 = 2 => x_1 = x_2 + 2 => Z = -3(x_2 + 2) +$
$x_2 - 7 \ Z => -2x_2 - 13$

61) $x_2 = 0 => x_1 = 2 \ , \ Z = -13$

Therefore, optimal result is attained from (55), (59):

62) $x_1 = 2 \ , \ x_2 = 0 \ , \ x_3 = 1 \ , \ x_4 = 1 \ , \ Z = -10$

63) $x_1 = 1 \ , \ x_2 = 0 \ , \ x_3 = 1 \ , \ x_4 = 0 \ , \ Z = -10$

End of example B.

C. *Example for "Case B(1)" :*

$$MIN \ Z = 10x_1 + 14x_2 + 21x_3$$
s.t:

64) $\begin{cases} -8x_1 - 11x_2 - 9x_3 \leq -12 \\ -2x_1 - 2x_2 - 7x_3 \leq -14 \\ -9x_1 - 6x_2 - 3x_3 \leq -10 \end{cases}$
$(x_1, x_2, x_3) \in W \ ; \ W=\{0,1,2,\ldots\}$

Resolve:

65) $\begin{cases} 10x_1 + 14x_2 + 21x_3 \leq Z \\ -8x_1 - 11x_2 - 9x_3 \leq -12 \\ -2x_1 - 2x_2 - 7x_3 \leq -14 \\ -9x_1 - 6x_2 - 3x_3 \leq -10 \end{cases}$

Omitting x_1 in apparatus (65), then having:

66) $\begin{cases} x_2 + 39x_3 \leq 4Z - 60 \\ 4x_2 - 14x_3 \leq Z - 70 \\ 66x_2 + 159x_3 \leq 9Z - 100 \end{cases}$

Omitting x_3 in apparatus (66), then having:

67) $\begin{cases} 170x_2 \leq -3570 + 95Z \\ 1560x_2 \leq -12530 + 285Z \end{cases}$
$=> \begin{cases} 102x_2 + 2142 \leq 57Z \\ 312x_2 + 2506 \leq 57Z \end{cases}$

68) $x_2 = 0 => Z \geq 44 =>$
$\begin{cases} 10x_1 + 21x_3 \geq 44 \\ 8x_1 + 9x_3 \geq 12 \\ 2x_1 + 7x_3 \geq 14 \\ 9x_1 + 3x_3 \geq 10 \end{cases}$

69) $x_3 = 0 => x_1 \geq 7 =>$
$x_1 = 7 \ , \ Z = 70$

70) $x_3 = 1 => x_1 \geq 4 =>$
$x_1 = 4 \ , \ Z = 61$

71) $x_3 = 2 => x_1 \geq 1 =>$
$x_1 = 1 \ , \ Z = 52$

Continuing current operation leads to Increasing " Z " , so not to raising " x_3 ".

72) $x_2 = 1 => Z \geq 50 =>$
$\begin{cases} 10x_1 + 21x_3 \geq 36 \\ 8x_1 + 9x_3 \geq 1 \\ 2x_1 + 7x_3 \geq 12 \\ 9x_1 + 3x_3 \geq 4 \end{cases}$

73) For satisfying (72), x_1 and x_3 must be like $\begin{cases} x_1 \geq 0 \\ x_3 \geq 2 \end{cases}$ or like $\begin{cases} x_1 \geq 3 \\ x_3 \geq 1 \end{cases}$, therefore, having Z=56 using 2 first constraints and having Z=65 using 2 second constraints, which both of them are not optimum.

Not to continuing these steps because it leads to raising value of Z over 52.

74) $x_1 = 2 => \quad Z \geq 55$

Hence:

$x_1 = 1 \ , \ x_2 = 0 \ , x_3 = 2 \ , x_4 = 0 \ , \ Z = 52$

End of example C.

D. Example for "Case B(2)"

$MIN\ Z = 2x_1 - 3x_2 - 4x_3$

S.t:

75) $\begin{cases} -x_1 + x_2 + 3x_3 \ \leq 8 \\ 3x_1 + 2x_2 - x_3 \ \leq 10 \end{cases}$

$(x_1, x_2, x_3) \in W$; $W=\{0,1,2,\ldots\}$

Resolve:

76) $\begin{cases} 2x_1 - 3x_2 - 4x_3 \leq Z \\ -x_1 + x_2 + 3x_3 \ \leq 8 \\ 3x_1 + 2x_2 - x_3 \quad \leq 10 \end{cases}$

Omitting x_2 in apparatus (76) , then having:

77) $\begin{cases} -x_1 + 5x_3 \ \leq 24 + Z \\ 13x_1 - 11x_3 \ \leq 30 + 2Z \end{cases}$

Omitting x_1 in apparatus (77), then having:

78) $\begin{cases} 5Z \geq 18x_3 - 114 \\ x_3 = 0 => Z \geq -22 => Z = -22 + K \ , \ K = 0,1,2, \ldots \end{cases}$

79) $\left.\begin{cases} 2x_1 - 3x_2 = -22 + K \\ -x_1 + x_2 \leq 8 \\ 3x_1 + 2x_2 \ \leq 10 \end{cases}\right\} =>$

$\begin{cases} 3x_2 = 2x_1 + 22 - K \\ \\ x_2 \leq 5 => \ 3x_2 \leq 15 \end{cases}$

$=> 2x_1 + 22 - K \leq 15 => K \geq 2x_1 + 7$

By a brief research, finding out optimal result:

$x_1 = 0 \ , \ x_2 = 5 \ , x_3 = 1 \ , \ Z = -19$

End of example D.

IV. Conclusion

The effectiveness of the implied method is that with the aid of simple software, we can make an improvement in the process of solving the pure integer programming problems.

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