Maximum Norm Analysis of a Nonmatching Grids Method for Nonlinear Elliptic Boundary Value Problem $-\Delta u = f(u)$

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Abstract—We provide a maximum norm analysis of a finite element Schwarz alternating method for a nonlinear elliptic boundary value problem of the form $-\Delta u = f(u)$, on two overlapping sub domains with non matching grids. We consider a domain which is the union of two overlapping sub domains where each sub domain has its own independently generated grid. The two meshes being mutually independent on the overlap region, a triangle belonging to one triangulation does not necessarily belong to the other one. Under a Lipschitz assumption on the nonlinearity, we establish, on each sub domain, an optimal L^∞ error estimate between the discrete Schwarz sequence and the exact solution of the boundary value problem.

Keywords—Error estimates, Finite elements, Nonlinear PDEs, Schwarz method.

I. INTRODUCTION

THE Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomains. Extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems can be found in [8], [9], [10] and [11] and the references therein. In this paper, we are interested in the error analysis in the maximum norm for a class of nonlinear elliptic boundary value problems in the context of overlapping nonmatching grids: we consider a domain which is the union of two overlapping subdomains where each subdomain has its own triangulation. Quite a few works on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems are known in the literature cf., e.g., [1], [2] , [4], [6] and [12]. To prove the main result of this paper, we proceed as in [2] and [6]. More precisely, we develop an approach which combines a geometrical convergence result due to [8] and a lemma which consists of estimating the error in the maximum norm between the continuous and discrete Schwarz iterates. The optimal convergence order is then derived making use of standard finite element L^∞-error estimate for linear elliptic equations. In the present paper, the proof of this lemma stands on a Lipschitz continuous dependency with respect to both the boundary condition and the source term for linear elliptic equations see [6]. Now, we give an outline of the paper. In

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Section 2 we state a continuous alternating Schwarz sequences and define their respective finite element counterparts in the context of nonmatching overlapping grids. Section 3 is devoted to the L^∞-error analysis of the method.

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II. PRELIMINARIES

We begin by laying down some definitions and classical results related to linear elliptic equations.

A. Linear Elliptic Equations

Let Ω be a bounded polyhedral domain of $\Re^2$ or $\Re^3$ with sufficiently smooth boundary $\partial\Omega$. We consider the bilinear form

$$a(u,v) = \int_\Omega (\nabla u.\nabla v)dx \qquad (1)$$

the linear form

$$(f,v) = \int_\Omega f(x).v(x)dx, \qquad (2)$$

the right hand side f, a regular function and the space

$$V^{(g)} = \{v \in H^1(\Omega) \text{ such that } v = g \text{ on } \partial\Omega\}, \qquad (3)$$

where g is a regular function defined on $\partial\Omega$. We consider the linear elliptic equation: Find $\zeta \in V^{(g)}$ such that:

$$a(\zeta,v) + c(\zeta,v) = (f,v), \forall v \in V^{(g)}, \qquad (4)$$

where c $\in \Re$ $c > 0$, such that

$$c \geq \beta > 0 \qquad (5)$$

Let V_h be the space of finite elements consisting of continuous piecewise linear functions v vanishing on $\partial\Omega$ and s, $s = 1, 2, ..., m(h)$ be the basis function of V_h. The discrete counterpart of (4) consists of finding $\zeta_h \in V_h^{(g)}$ such that

$$a(\zeta_h,v) + c(\zeta_h,v) = (f,v), \forall v \in V_h^{(g)} \qquad (6)$$

where

$$V_h^{(g)} = \{v \in V_h : v = \pi_h(g) \text{ on } \partial\Omega\} \qquad (7)$$

and π_h is an interpolation operator on $\partial\Omega$.

Theorem 2.1 (cf.[13]) Under suitable regularity of the solution of problem (4), there exists a constant C independent of h such that

$$\|\zeta - \zeta_h\|_{L^\infty(\Omega)} \le Ch^2|logh|. \tag{8}$$

Lemma 2.1 (cf.[11]) Let $w \in H_1(\Omega) \cap \mathcal{C}(\Omega)$ satisfy $a(w, \Phi) + c(w, \Phi) \ge 0$ for all non-negative $\Phi \in H_0^1(\Omega)$ and $w \ge 0$ on $\partial\Omega$. Then $w \ge 0$ on $\overline{\Omega}$.

The proposition below establishes a Lipschitz continuous dependency of the solution with respect to the data. Let $(f;g)$; $(\widetilde{f}, \widetilde{g})$ be a pair of data, and $\zeta = \sigma(f, g)$; $\widetilde{\zeta} = \sigma(\widetilde{f}, \widetilde{g})$ the corresponding solutions to (4).

Proposition 2.1 (cf.[6]) Under conditions of the preceding lemma, we have:

$$\|\zeta - \widetilde{\zeta}\|_{L^\infty(\Omega)} \le max\{(\frac{1}{\beta})\|f - \widetilde{f}\|_{L^\infty(\Omega)}; \|g - \widetilde{g}\|_{L^\infty(\partial\Omega)}\}. \tag{9}$$

Remark 2.1 Lemma 2.1 stays true in the discrete case. Indeed, assume that the discrete maximum principle (d.m.p) hold [5], [7] that is the matrix resulting from the finite element discretization is an M-Matrix. Then we have:

Lemma 2.2 Let $w \in V_h$ satisfy $a(w, s) + c(w, s) \ge 0$ for all $s = 1, 2, ..., m(h)$ and $w \ge 0$ on $\partial\Omega$. Then $w \ge 0$ on $\overline{\Omega}$.

Let (f, g); $(\widetilde{f}, \widetilde{g})$ be a pair of data, and $\zeta_h = \sigma_h(f, g)$; $\widetilde{\zeta}_h = \sigma_h(\widetilde{f}, \widetilde{g})$ the corresponding solutions to (6);

Proposition 2.2 Let the d.m.p hold. Then, under conditions of lemma 2.2, we have

$$\|\zeta_h - \widetilde{\zeta}_h\|_{L^\infty(\Omega)} \le max\{(\frac{1}{\beta})\|f - \widetilde{f}\|_{L^\infty(\Omega)}; \|g - \widetilde{g}\|_{L^\infty(\partial\Omega)}\}. \tag{10}$$

III. SCHWARZ ALTERNATING METHODS FOR NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

Consider the nonlinear elliptic boundary value problem

$$-\Delta u = f(u) \text{ in } \Omega, \tag{11}$$
$$u = 0 \text{ on } \partial\Omega \tag{12}$$

where $f(.)$ is a nondecreasing nonlinearity. Thanks to [3], problem (11) has a unique solution. Let us also assume that $f(.)$ is a Lipschitz continuous on $\Re$; that is

$$|f(x) - f(y)| \le k|x - y|, \forall x, y \in \Re \tag{13}$$

The problem (11) is equivalent to

$$-\Delta u + cu = F(u) \text{ in } \Omega \tag{14}$$
$$u = 0 \text{ on } \partial\Omega \tag{15}$$

where the functional F is defined by: $F(u) = f(u) + cu$ and its Lipschitz constant is $K = k + c$. Evidently we have

$$\beta \le c < K. \tag{16}$$

The related weak form of (14) is defined by

$$a(u, v) + c(u, v) = (F, v), \forall v \in H_0^1(\Omega), \tag{17}$$

Remark 2.2 The main idea of this work consists to applied the multiplicative Schwarz algorithm to the problem (14) equivalent to (11). We decompose Ω into two overlapping

smooth subdomains Ω_1 and Ω_2 such that: $\Omega = \Omega_1 \cup \Omega_2$. We denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$, We assume that the intersection of Γ_i and Γ_j, $i \ne j$ is empty. Let

$$V_i^{(w_j)} = \{v \in H^1(\Omega_i) \text{ such that } v = w_j \text{ on } \Gamma_i\} \tag{18}$$

We associate with problem (17) the following system: Find $(u_1, u_2) \in V_1^{(u_2)} \times V_2^{(u_1)}$ solution of

$$a_1(u_1, v) + c(u_1, v) = (F(u_1), v), \forall v \in V_1^{(u_2)}, \tag{19}$$

$$a_2(u_2, v) + c(u_2, v) = (F(u_2), v), \forall v \in V_2^{(u_1)}, \tag{20}$$

where

$$a_i(u, v) = \int_{\Omega_i} (\nabla u . \nabla v) dx; \tag{21}$$

$$u_i = u/\Omega_i, i = 1, 2. \tag{22}$$

A. The Continuous Schwarz Sequences

Let u^0 be an initialization in $C^0(\Omega)$ (i.e. continuous functions vanishing on $\partial\Omega$) such that

$$a(u^0, v) + c(u^0, v) = (F, v), \forall v \in H_0^1(\Omega). \tag{23}$$

$u_2^0 = u^0/\Omega_2$, we respectively define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in V^{(u_2^n)}$ solves $n \ge 0$

$$a_1(u_1^{n+1}, v) + c(u_1^{n+1}, v) = (F(u_1^{n+1}), v) \forall v \in V_1^{(u_2^n)}; \tag{24}$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in V^{(u_1^{n+1})}$ solves $n \ge 0$

$$a_2(u_2^{n+1}, v) + c(u_2^{n+1}, v) = (F(u_2^{n+1}), v) \forall v \in V_2^{(u_1^{n+1})}; \tag{25}$$

The following is a geometrical convergence of the Schwarz sequences.

Theorem 2.2 (cf. [8] pp. 51-63) The sequences (u_1^{n+1}); (u_2^{n+1}) ; $n \ge 0$ produced by the Schwarz alternating method converge geometrically to the solution (u_1, u_2) of the system (19) (20). More precisely, there exist two constants $k_1, k_2 \in]0; 1[$ which depend on (Ω_1, Γ_2) and (Ω_2, Γ_1) respectively, such that for all $n \ge 0$;

$$\|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} \le k_1^n k_2^n \|u^0 - u\|_{(\Gamma_1)} \tag{26}$$

$$\|u_2 - u_2^{n+1}\|_{L^\infty(\Omega_2)} \le k_1^{n+1} k_2^n \|u^0 - u\|_{(\Gamma_2)}. \tag{27}$$

B. The Discretization

For $i = 1, 2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i; h_i, being the mesh size. The two meshes being mutually independent on $\Omega_1 \cap \Omega_2$, a triangle belonging to one triangulation does not necessarily belong to the other. We consider the following discrete spaces:

$$V_{h_i} = \{v \in \mathcal{C}(\overline{\Omega}_i) \cap H_0^1(\Omega_i) \text{ such that } v/K \in P_1, \forall K \in \tau^{h_i}\} \tag{28}$$

and for every $w \in C(\overline{\Gamma}_i)$, we set:

$$V_{h_i}^{(w)} = \{v \in V_{h_i} : v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega; v = \pi_{h_i}(w) \text{ on } \Gamma_i\} \tag{29}$$

where π_{h_i} denote the interpolation operator on Γ_i.

1) The Discrete Maximum Principle: ([5], [7]) We assume that the respective matrices resulting from the discretiztions of problems (24) and (25) are M-matrices. Note that, as the two meshes h_1 and h_2 are independent over the overlapping subdomains, it is impossible to formulate a global approximate problem which would be the direct discrete counterpart of problem (14).

C. The Discrete Schwarz Sequences

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (24) and (25). Indeed, let u_h^0 be the discrete analog of u^0, defined in (23), let $u_{h_1}^{n+1} \in V_{h_1}^{(u_{h_2}^n)}$ such that

$$a_1(u_{h_1}^{n+1}, v) + c(u_{h_1}^{n+1}, v) = (F(u_{h_1}^{n+1}), v) : \forall v \in V_{h_1}^{(u_{h_2}^n)}; n \geq 0 \tag{30}$$

and $u_{h_2}^{n+1} \in V_{h_2}^{(u_{h_1}^{n+1})}$ such that

$$a_2(u_{h_2}^{n+1}, v) + c(u_{h_2}^{n+1}, v) = (F(u_{h_2}^{n+1}), v) : \forall v \in V_{h_2}^{(u_{h_1}^{n+1})}; n \geq 0 \tag{31}$$

IV. L^∞- ERROR ANALYSIS

This section is devoted to the proof of the main result of the present paper. To that end we begin by introducing two discrete auxiliary sequences and prove a fundamental lemma.

A. Two Auxiliary Schwarz Sequences

For $w_{h_2}^0 = u_{h_2}^0$, we define the sequences $(w_{h_1}^{n+1})$ such that $u_{h_1}^{n+1} \in V_{h_1}^{(u_2^n)}$ solves

$$a_1(w_{h_1}^{n+1}, v) + c(w_{h_1}^{n+1}, v) = (F(u_1^{n+1}), v) : \forall v \in V_{h_1}^{(u_2^n)} \tag{32}$$

and $(w_{h_2}^{n+1})$ such that $w_{h_2}^{n+1} \in V_{h_2}^{(u_1^{n+1})}$ solves

$$a_1(w_{h_2}^{n+1}, v) + c(w_{h_2}^{n+1}, v) = (F(u_2^{n+1}), v) : \forall v \in V_{h_2}^{(u_1^{n+1})} \tag{33}$$

It is then clear that $w_{h_1}^{n+1}$ and $w_{h_2}^{n+1}$ are the finite element approximation of u_1^{n+1} and u_2^{n+1} defined in (24) , (25), respectively. Then, as $F(.)$ is continuous, $\|F(u_i^n)\|_\infty \leq C$ (C independent of n) and, therefore, making use of standard maximum norm estimates for linear elliptic problems, we have

$$\|u_i^n - w_{h_i}^n\|_{L^\infty(\Omega_i)} \leq C h_i^2 |logh_i|. \tag{34}$$

Notation 3.1 From now on, we shall adopt the following notations:

$$\|.\|_1 = \|.\|_{L^\infty(\Omega_1)}, \|.\|_2 = \|.\|_{L^\infty(\Omega_1)} \tag{35}$$

B. The Main Results

The following lemma will play a key role in proving the main result of this paper.

Lemma 3.1 Let u_i^{n+1} the continuous Schwarz sequence and $u_{h_i}^{n+1}$ its discrete counterpart then we have

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \leq \sum_{i=1}^{n+1} \|u_1^i - w_{h_1}^i\|_1 + \sum_{i=0}^n \|u_2^i - w_{h_2}^i\|_2 \tag{36}$$

$$\|u_2^{n+1} - u_{h_2}^{n+1}\|_2 \leq \sum_{i=0}^{n+1} \|u_2^i - w_{h_2}^i\|_2 + \sum_{i=1}^{n+1} \|u_1^i - w_{h_1}^i\|_1 \tag{37}$$

Proof Let us now prove (36) and (37) simultaneously by induction. Indeed for n = 1, using the Proposition 2.2, we have in domain 1

$$\begin{aligned}\|u_1^1 - u_{h_1}^1\|_1 &\leq \|u_1^1 - w_{h_1}^1\|_1 + \|w_1^1 - u_{h_1}^1\|_1 \\ &\leq \|u_1^1 - w_{h_1}^1\|_1 \\ &+ max\{(\frac{1}{\beta})\|F(u_1^1) - F(u_{h_1}^1)\|_1; |u_2^0 - u_{h_2}^0|_2\}\end{aligned}$$

then

$$\begin{aligned}\|u_1^1 - u_{h_1}^1\|_1 &\leq \|u_1^1 - w_{h_1}^1\|_1 \\ &+ max\{(\frac{1}{\beta})\|F(u_1^1) - F(u_{h_1}^1)\|_1; \|u_2^0 - u_{h_2}^0\|_2\}\end{aligned}$$

thus

$$\begin{aligned}\|u_1^1 - u_{h_1}^1\|_1 &\leq \|u_1^1 - w_{h_1}^1\|_1 \\ &+ max\{\rho\|u_1^1 - u_{h_1}^1\|_1; \|u_2^0 - u_{h_2}^0\|_2\}\end{aligned}$$

where

$$\rho = \frac{K}{\beta} > 1 \tag{38}$$

We then have to distinguish between two cases:

$$(1) : max\{\rho\|u_1^1 - u_{h_1}^1\|_1; \|u_2^0 - u_{h_2}^0\|_2\} = \rho\|u_1^1 - u_{h_1}^1\|_1 \tag{39}$$

or

$$(2) : max\{\rho\|u_1^1 - u_{h_1}^1\|_1; \|u_2^0 - u_{h_2}^0\|_2\} = \|u_2^0 - u_{h_2}^0\|_2 \tag{40}$$

Case 1 implies

$$\|u_1^1 - u_{h_1}^1\|_1 \leq \|u_1^1 - w_{h_1}^1\|_1 + \rho\|u_1^1 - u_{h_1}^1\|_1 \tag{41}$$

$$\|u_2^0 - u_{h_2}^0\|_2 \leq \rho\|u_1^1 - u_{h_1}^1\|_1 \tag{42}$$

By adding $\|u_1^1 - w_{h_1}^1\|_1$ in (42) we get

$$\|u_2^0 - u_{h_2}^0\|_2 + \|u_1^1 - w_{h_1}^1\|_1 \leq \|u_1^1 - w_{h_1}^1\|_1 + \rho\|u_1^1 - u_{h_1}^1\|_1 \tag{43}$$

So $\|u_1^1 - w_{h_1}^1\|_1 + \rho\|u_1^1 - u_{h_1}^1\|_1$ is bounded below by $\|u_1^1 - u_{h_1}^1\|_1$ and $\|u_2^0 - u_{h_2}^0\|_2 + \|u_1^1 - w_{h_1}^1\|_1$. This implies that

$$(\mathbf{a}) : \|u_1^1 - u_{h_1}^1\|_1 \leq \|u_2^0 - u_{h_2}^0\|_2 + \|u_1^1 - w_{h_1}^1\|_1 \tag{44}$$

or

$$(\mathbf{b}) : \|u_2^0 - u_{h_2}^0\|_2 + \|u_1^1 - w_{h_1}^1\|_1 \leq \|u_1^1 - u_{h_1}^1\|_1 \tag{45}$$

Which implies

$$\begin{aligned}(\mathbf{a}) : \|u_1^1 - u_{h_1}^1\|_1 &\leq \|u_2^0 - u_{h_2}^0\|_2 + \|u_1^1 - w_{h_1}^1\|_1 \\ &\leq \|u_1^1 - w_{h_1}^1\|_1 + \rho\|u_1^1 - u_{h_1}^1\|_1\end{aligned}$$

or

$$\begin{aligned}(\mathbf{b}) : \|u_2^0 - u_{h_2}^0\|_2 + \|u_1^1 - w_{h_1}^1\|_1 &\leq \|u_1^1 - u_{h_1}^1\|_1 \\ &\leq \|u_1^1 - w_{h_1}^1\|_1 + \rho\|u_1^1 - u_{h_1}^1\|_1\end{aligned}$$

Hence (a) and (b) are true because they both coincide with (41). So, there is either a contradiction and thus case 1 is impossible or case 1 is possible only if

$$\|u_1^1 - u_{h_1}^1\|_1 = \|u_2^0 - u_{h_2}^0\|_2 + \|u_1^1 - w_{h_1}^1\|_1 \tag{46}$$

Case 2 implies

$$\|u_1^1 - u_{h_1}^1\|_1 \leq \|u_1^1 - w_{h_1}^1\|_1 + \|u_2^0 - u_{h_2}^0\|_2 \quad (47)$$

$$\rho\|u_1^1 - u_{h_1}^1\|_1 \leq \|u_2^0 - u_{h_2}^0\|_2 \quad (48)$$

That is

$$\begin{aligned}\|u_1^1 - u_{h_1}^1\|_1 &\leq (\frac{1}{\rho})\|u_2^0 - u_{h_2}^0\|_2 < \|u_2^0 - u_{h_2}^0\|_2\\ &\leq \|u_1^1 - w_{h_1}^1\|_1 + \|u_2^0 - u_{h_2}^0\|_2\end{aligned}$$

Which coincides with (47. So in both cases 1 and 2 we can write

$$\|u_1^1 - u_{h_1}^1\|_1 \leq \|u_1^1 - w_{h_1}^1\|_1 + \|u_2^0 - u_{h_2}^0\|_2 \quad (49)$$

which is equivalent to

$$\|u_1^1 - u_{h_1}^1\|_1 \leq \sum_{i=1}^{1} \|u_1^i - w_{h_1}^i\|_1 + \sum_{i=0}^{0} \|u_2^i - u_{h_2}^i\|_2 \quad (50)$$

Similarly, we obtain in domain 2

$$\begin{aligned}\|u_2^1 - u_{h_2}^1\|_2 &\leq \|u_2^1 - w_{h_2}^1\|_2\\ &+ \ max\{\rho\|u_2^1 - u_{h_2}^1\|_2; \|u_1^1 - u_{h_1}^1\|_1\}\end{aligned}$$

We then have to distinguish between two cases:

$$(1): max\{\rho\|u_2^1 - u_{h_2}^1\|_2; \|u_1^1 - u_{h_1}^1\|_1\} = \rho\|u_2^1 - u_{h_2}^1\|_2 \quad (51)$$

or

$$(2): max\{\rho\|u_2^1 - u_{h_2}^1\|_2; \|u_1^1 - u_{h_1}^1\|_1\} = \|u_1^1 - u_{h_1}^1\|_1 \quad (52)$$

Case 1 implies

$$\|u_2^1 - u_{h_2}^1\|_2 \leq \|u_2^1 - w_{h_2}^1\|_2 + \rho\|u_2^1 - u_{h_2}^1\|_2 \quad (53)$$

$$\|u_1^1 - u_{h_1}^1\|_1 \leq \rho\|u_2^1 - u_{h_2}^1\|_2 \quad (54)$$

By adding $\|u_2^1 - w_{h_2}^1\|_2$ in (54) we get

$$\|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1 \leq \|u_2^1 - w_{h_2}^1\|_2 + \rho\|u_2^1 - u_{h_2}^1\|_2 \quad (55)$$

Hence $\|u_2^1 - w_{h_2}^1\|_2 + \rho\|u_2^1 - u_{h_2}^1\|_2$ is bounded below by both $\|u_2^1 - u_{h_2}^1\|_2$ and $\|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1$ then

$$(\mathbf{a}): \|u_2^1 - u_{h_2}^1\|_2 \leq \|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1 \quad (56)$$

or

$$(\mathbf{b}): \|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1 \leq \|u_2^1 - u_{h_2}^1\|_2 \quad (57)$$

which implies

$$(\mathbf{a}): \begin{aligned}\|u_2^1 - u_{h_2}^1\|_2 &\leq \|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1\\ &\leq \|u_2^1 - w_{h_2}^1\|_2 + \rho\|u_2^1 - u_{h_2}^1\|_2\end{aligned}$$

or

$$(\mathbf{b}): \begin{aligned}\|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1 &\leq \|u_2^1 - u_{h_2}^1\|_2\\ &\leq \|u_2^1 - w_{h_2}^1\|_2 + \rho\|u_2^1 - u_{h_2}^1\|_2\end{aligned}$$

Hence, (a) and (b) are true because they both coincide with (53). So, there is either a contradiction and thus case 1 is impossible or case 1 is possible only if

$$\|u_2^1 - u_{h_2}^1\|_2 = \|u_2^1 - w_{h_2}^1\|_2 + \rho\|u_2^1 - u_{h_2}^1\|_2 \quad (58)$$

Case 2 implies

$$\begin{aligned}\|u_2^1 - u_{h_2}^1\|_2 &\leq \|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1\\ \rho\|u_2^1 - u_{h_2}^1\|_2 &\leq \|u_1^1 - u_{h_1}^1\|_1\end{aligned}$$

So

$$\|u_2^1 - u_{h_2}^1\|_2 \leq \|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1 \quad (59)$$

and

$$\begin{aligned}\|u_2^1 - u_{h_2}^1\|_2 &\leq (\frac{1}{\rho})\|u_1^1 - u_{h_1}^1\|_1 < \|u_1^1 - u_{h_1}^1\|_1\\ &\leq \|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1\end{aligned}$$

which coincides with (59). Then both cases 1 and 2 imply

$$\|u_2^1 - u_{h_2}^1\|_2 \leq \|u_2^1 - w_{h_2}^1\|_2 + \|u_1^1 - u_{h_1}^1\|_1 \quad (60)$$

Using (50) we get

$$\begin{aligned}\|u_2^1 - u_{h_2}^1\|_2 &\leq \|u_2^1 - w_{h_2}^1\|_2\\ &+ (\sum_{i=1}^{1} \|u_1^i - w_{h_1}^i\|_1 + \sum_{i=0}^{0} \|u_2^i - u_{h_2}^i\|_2)\end{aligned}$$

or equivalently

$$\|u_2^1 - u_{h_2}^1\|_2 \leq (\sum_{i=0}^{1} \|u_2^i - u_{h_2}^i\|_2 + \sum_{i=1}^{1} \|u_1^i - w_{h_1}^i\|_1) \quad (61)$$

Now, let us assume that (62) and (63) are trues

$$\|u_1^n - u_{h_1}^n\|_1 \leq \sum_{i=1}^{n} \|u_1^i - w_{h_1}^i\|_1 + \sum_{i=0}^{n-1} \|u_2^i - u_{h_2}^i\|_2 \quad (62)$$

$$\|u_2^n - u_{h_2}^n\|_2 \leq \sum_{i=0}^{n} \|u_2^i - u_{h_2}^i\|_2 + \sum_{i=1}^{n} \|u_1^i - w_{h_1}^i\|_1 \quad (63)$$

and prove that

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \leq \sum_{i=1}^{n+1} \|u_1^i - w_{h_1}^i\|_1 + \sum_{i=0}^{n} \|u_2^i - u_{h_2}^i\|_2 \quad (64)$$

$$\|u_2^{n+1} - u_{h_2}^{n+1}\|_2 \leq \sum_{i=0}^{n+1} \|u_2^i - u_{h_2}^i\|_2 + \sum_{i=1}^{n+1} \|u_1^i - w_{h_1}^i\|_1 \quad (65)$$

Indeed, we have in domain 1

$$\begin{aligned}\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 &\leq \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 + \|w_1^{n+1} - u_{h_1}^{n+1}\|_1\\ &\leq \|u_1^{n+1} - w_{h_1}^{n+1}\|_1\\ &+ \ max\{(\frac{1}{\beta})\|F(u_1^{n+1}) - F(u_{h_1}^{n+1})\|_1; |u_2^n - u_{h_2}^n|_2\}\end{aligned}$$

then

$$\begin{aligned}\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 &\leq \|u_1^{n+1} - w_{h_1}^{n+1}\|_1\\ &+ \ max\{(\frac{1}{\beta})\|F(u_1^{n+1}) - F(u_{h_1}^{n+1})\|_1; \|u_2^n - u_{h_2}^n\|_2\}\end{aligned}$$

thus

$$\begin{aligned}\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 &\leq \|u_1^{n+1} - w_{h_1}^{n+1}\|_1\\ &+ \ max\{\rho\|u_1^{n+1} - u_{h_1}^{n+1}\|_1; \|u_2^n - u_{h_2}^n\|_2\}\end{aligned}$$

We then have to distinguish between two cases:

$$(1): max\{\rho\|u_1^{n+1}-u_{h_1}^{n+1}\|_1; \|u_2^n-u_{h_2}^n\|_2\} = \rho\|u_1^{n+1}-u_{h_1}^{n+1}\|_1 \quad (66)$$

or

$$(2): max\{\rho\|u_1^{n+1} - u_{h_1}^{n+1}\|_1; \|u_2^n - u_{h_2}^n\|_2\} = \|u_2^n - u_{h_2}^n\|_2 \quad (67)$$

Case 1 implies

$$\|u_1^{n+1}-u_{h_1}^{n+1}\|_1 \le \|u_1^{n+1}-w_{h_1}^{n+1}\|_1+\rho\|u_1^{n+1}-u_{h_1}^{n+1}\|_1 \quad (68)$$

$$\|u_2^n - u_{h_2}^n\|_2 \le \rho\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \quad (69)$$

By adding $\|u_1^{n+1} - w_{h_1}^{n+1}\|_1$ in (73) we get

$$\|u_2^n - u_{h_2}^n\|_2 + \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 \le \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 + \rho\|u_1^{n+1} - u_{h_1}^{n+1}\|_1$$

So $\|u_1^{n+1}-w_{h_1}^{n+1}\|_1+\rho\|u_1^{n+1}-u_{h_1}^{n+1}\|_1$ is bounded below by $\|u_1^{n+1}-u_{h_1}^{n+1}\|_1$ and $\|u_2^n-u_{h_2}^n\|_2+\|u_1^{n+1}-w_{h_1}^{n+1}\|_1$. This implies that

$$(\mathbf{a}): \|u_1^{n+1}-u_{h_1}^{n+1}\|_1 \le \|u_2^n-u_{h_2}^n\|_2+\|u_1^{n+1}-w_{h_1}^{n+1}\|_1 \quad (70)$$

or

$$(\mathbf{b}): \|u_2^n-u_{h_2}^n\|_2+\|u_1^{n+1}-w_{h_1}^{n+1}\|_1 \le \|u_1^{n+1}-u_{h_1}^{n+1}\|_1 \quad (71)$$

Which implies

$$(\mathbf{a}): \|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \le \|u_2^n - u_{h_2}^n\|_2 + \|u_1^{n+1} - w_{h_1}^{n+1}\|_1$$
$$\le \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 + \rho\|u_1^{n+1} - u_{h_1}^{n+1}\|_1$$

or

$$(\mathbf{b}): \|u_2^n - u_{h_2}^n\|_2 + \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 \le \|u_1^{n+1} - u_{h_1}^{n+1}\|_1$$
$$\le \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 + \rho\|u_1^{n+1} - u_{h_1}^{n+1}\|_1$$

Hence (a) and (b) are true because they both coincide with (68). So, there is either a contradiction and thus case 1 is impossible or case 1 is possible only if

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 = \|u_2^n - u_{h_2}^n\|_2 + \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 \quad (72)$$

Case 2 implies

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \le \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 + \|u_2^n - u_{h_2}^n\|_2 \quad (73)$$

$$\rho\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \le \|u_2^n - u_{h_2}^n\|_2 \quad (74)$$

That is

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \le (\frac{1}{\rho})\|u_2^n - u_{h_2}^n\|_2 < \|u_2^n - u_{h_2}^n\|_2$$
$$\le \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 + \|u_2^n - u_{h_2}^n\|_2$$

Which coincides with (73). So in both cases 1 and 2 we can write

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \le \|u_1^{n+1} - w_{h_1}^{n+1}\|_1 + \|u_2^n - u_{h_2}^n\|_2 \quad (75)$$

using (63) we get

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \le \|u_1^{n+1} - w_{h_1}^{n+1}\|_1$$
$$+ (\sum_{i=0}^n \|u_2^i - u_{h_2}^i\|_2 + \sum_{i=1}^n \|u_1^i - w_{h_1}^i\|_1)$$

Hence

$$\|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \le \sum_{i=1}^{n+1} \|u_1^i - w_{h_1}^i\|_1 + \sum_{i=0}^n \|u_2^i - u_{h_2}^i\|_2 \quad (76)$$

Which is the desired result (64). Estimate in domain 2 (65) can be proved similarly using estimate (64).

Theorem 3.1 Let $h = max\{h_1, h_2\}$. Then, for n large enough, there exists a constant C independent of both h and n such that

$$\|u_i - u_{h_i}^{n+1}\|_i \le Ch^2|logh|^2; i = 1, 2. \quad (77)$$

Proof Let us give the proof for $i = 1$. The one for $i = 2$ is similar and so will be omitted. Indeed, Let $\kappa = max\{k_1, k_2\}$, hen making use of Theorem 2.2, Lemma 3.1 and (34) respectively, we get

$$\|u_1 - u_{h_1}^{n+1}\|_1 \le \|u_1 - u_1^{n+1}\|_1 + \|u_1^{n+1} - u_{h_1}^{n+1}\|_1 \quad (78)$$

So

$$\|u_1 - u_{h_1}^{n+1}\|_1 \le \kappa^2\|u_1 - u_1^0\|_1$$
$$+ (\sum_{i=1}^{n+1} \|u_1^i - w_{h_1}^i\|_1 + \sum_{i=0}^n \|u_2^i - w_{h_2}^i\|_2)$$

Thus

$$\|u_1 - u_{h_1}^{n+1}\|_1 \le \kappa^2\|u_1 - u_1^0\|_1 + 2(n+1)Ch^2|logh| \quad (79)$$

So, for n large enough, we have

$$\kappa^{2n} = h^2 \quad (80)$$

then

$$n = \frac{|logh|}{|log\kappa|} = \overline{C}|logh| \quad (81)$$

and thus

$$\|u_1 - u_{h_1}^{n+1}\|_1 \le \kappa^2\|u_1 - u_1^0\|_1 + 2(\overline{C}|logh| + 1)Ch^2|logh| \quad (82)$$

So

$$\|u_1 - u_{h_1}^{n+1}\|_1 \le Ch^2|logh|^2 \quad (83)$$

which is the desired result.

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