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Maximizer of the Posterior Marginal Estimate of Phase Unwrapping based on Statistical Mechanics of the Q-Ising Model

Yohei Saika, Tatsuya Uezu

***Abstract*—**We constructed a method of phase unwrapping for a typical wave-front by utilizing the maximizer of the posterior marginal (MPM) estimate corresponding to equilibrium statistical mechanics of the three-state Ising model on a square lattice on the basis of an analogy between statistical mechanics and Bayesian inference. We investigated the static properties of an MPM estimate from a phase diagram using Monte Carlo simulation for a typical wave-front with synthetic aperture radar (SAR) interferometry. The simulations clarified that the surface-consistency conditions were useful for extending the phase where the MPM estimate was successful in phase unwrapping with a high degree of accuracy and that introducing prior information into the MPM estimate also made it possible to extend the phase under the constraint of the surface-consistency conditions with a high degree of accuracy. We also found that the MPM estimate could be used to reconstruct the original wave-fronts more smoothly, if we appropriately tuned hyper-parameters corresponding to temperature to utilize fluctuations around the MAP solution. Also, from the viewpoint of statistical mechanics of the Q-Ising model, we found that the MPM estimate was regarded as a method for searching the ground state by utilizing thermal fluctuations under the constraint of the surface-consistency condition.

***Keywords*—**Bayesian inference, maximizer of the posterior marginal estimate, phase unwrapping, Monte Carlo simulation, statistical mechanics

I. Introduction

WAVE-FRONTS often carry information through noisy channels. Numerous researchers [1]–[6] have studied methodologies of utilizing information on wave-fronts both from theoretical and practical viewpoints. They have also proposed various techniques of reconstructing original wave-fronts using a set of principal values for phase differences observed with optical instruments by using interferometers for this purpose. This problem is called phase unwrapping and various techniques have been proposed to solve this problem, such as the methods of least squares estimation [1], [2], the MAP estimation using the conjugate gradient method [7], the simulated annealing [8],[9] and the method of maximum entropy[9]. Even now, in order to improve the performance of phase unwrapping, various approaches have been tried, such as the edgelist phase unwrapping algorithm [10]. Although many researchers have tried various approaches based on Bayesian inference to phase unwrapping, there have been few systematic approaches to clarifying both static and dynamic properties of the methods for phase unwrapping.

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On the other hand, theoretical physicists have investigated information science [11]-[19] in recent years, such as image restoration and error-correcting codes on the basis of an analogy between statistical mechanics and Bayesian inference using the maximizer of the posterior marginal (MPM) estimates. Researchers have then applied statistical mechanics to problems in information technology, such as information communication and quantum computation. The field of statistical mechanics of information has been developed as an established research field. One of the present authors have utilized statistical mechanics to various problems, such as image restoration [14], inverse halftoning [15], [16], [17] and noise reduction of JPEG-compressed image [18]. Recent years, Saika and Nishimori [19] have investigated phase retrieval based on the analogy between the statistical mechanics of spin glasses and Bayesian inference via MPM estimate.

We formulated the problem of phase unwrapping in SAR interferometry from the viewpoint of statistical mechanics of information in this study. We constructed a method of phase unwrapping based on the statistical mechanics of a three-state Ising model on a square lattice. We used the model of the true prior by suppressing the number of non-zero states of the three-state Ising spin at each sampling point. We then used the likelihood composed of two terms. The first was the term that enhanced the smooth structures of wave-fronts. The second was the surface-consistency conditions of the wave-fronts. We next investigated both static and dynamic properties by using Monte Carlo simulations for a typical wave-front in synthetic aperture radar (SAR) interferometry from the viewpoint of statistical mechanics. Although we used a single artificial model for performance estimation, it was expected that the wave-front has the general property of wave-fronts in the SAR interferometry. We first examined the static properties of an MPM estimate for phase unwrapping based on a phase diagram that represented the region where the MPM estimate succeeded in unwrapping the phase in hyper-parameter space. The term "PU phase" is used in this manuscript to represent the region where the MPM estimate succeeds in phase unwrapping with a high degree of accuracy. Based on the characteristics of the PU phase for typical wave-fronts in SAR interferometry, we found that the surface-consistency conditions for each plaquette effectively extended the range of the PU phase along the T_m axis, if we set to $0.6<h$, and moreover that the surface-consistency conditions did not efficiently extend the range of the PU phase along the T_m axis, if we set to $0.6<h$. We here noted that T_m is a hyper-parameter represents absolute temperature and h is a parameter which adjusts prior information.

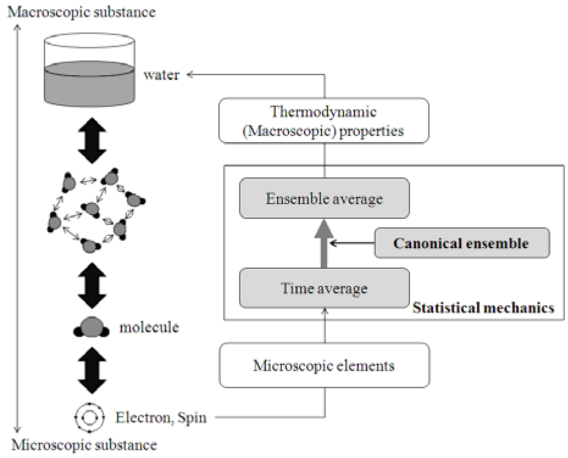

Fig. 1 Statistical mechanics

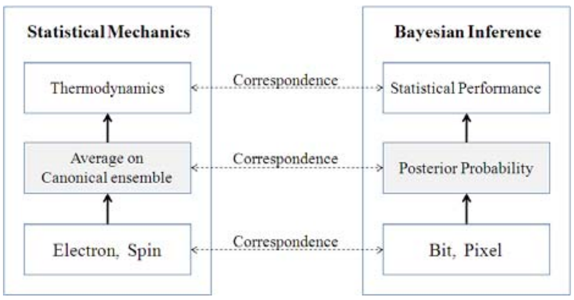

Fig. 2 Statistical mechanics and Bayesian inference

We next examined the efficiency of prior information in the MPM estimate for a phase unwrapping based on the phase diagram in hyper-parameter space. Using Monte Carlo simulations, we found that surface-consistency conditions effectively worked to extend the range of the PU phase along the T_m axis for $0.6<h$, and that the surface-consistency conditions did not efficiently extend the range of the PU phase along the T_m axis for $0<h<0.6$. These results suggested the properties of prior information in that the MPM estimate works robustly for phase unwrapping by introducing prior information on the original wave-fronts under the surface-consistency conditions for each plaquette. We also examined the dynamic properties of the MPM estimate based on the time evolution of performance measures using mean square error utilizing a Monte Carlo simulation for the artificial wave-front which approximated typical wave-fronts in SAR interferometry. We found that the phase was smoothly unwrapped by appropriately utilizing fluctuations around the MAP solution. These results suggested evidence that the probabilistic information processing due to the MPM estimate became a useful tool to achieve smooth phase unwrapping. Also, we discussed static and dynamic properties of the MPM estimate on the basis of statistical mechanics of information. From the viewpoint of statistical mechanics, it was clarified that the MPM estimate is regarded as a method for searching the ground state of the energy function composed of the three-state Q-Ising model by making use of thermal fluctuations. Also, we found that the MPM estimate is effective for phase unwrapping under the constraint of surface-consistency condition which enhances an energy gap above the ground state. Further, we found that the MPM estimate carries out phase unwrapping more smoothly than the MAP estimation via simulated annealing for the typical wave-front in the SAR interferometry.

The content of this paper is organized as follows. First, we outline the framework of statistical mechanics. Then, we first describe a general formulation of the problem of phase unwrapping on the basis of Bayesian inference using the MPM estimate. We next investigate both the static and dynamic properties of the present method using a Monte Carlo simulation for typical wave-fronts in SAR interferometry. The conclusion is devoted to a summary and discussion.

II. Statistical Mechanics

As shown in Fig. 1, a principal goal of statistical mechanics is to clarify thermodynamics of many-body systems starting with interactions between microscopic elements. In general prescription of statistical mechanics, thermal average of macroscopic physical quantity can be estimated as ensemble average over all possible states via a probability distribution:

$$\Pr(\{S\}) = \frac{1}{Z}\exp\left[-\beta H(\{S\})\right] \tag{1}$$

for a given Hamiltonian $H(\{S\})$. In this equation, we use a set of the Ising spin states $\{S\}$ as a set of typical microscopic elements. We take the unit of temperature such that Boltzmann's constant k_B is unity. As a result, $\beta=1/T$. Normalization factor Z is called the "partition function":

$$Z = \sum_{\{S_1\}}\sum_{\{S_2\}}\cdots\sum_{\{S_N\}}\exp\left[-\beta H(\{S\})\right] \tag{2}$$

The probability distribution in eq. (1) is termed the "Boltzmann factor". Using the Gibbs-Boltzmann distribution, we can estimate the thermal average of macroscopic quantity $A(\{S\})$ as

$$\langle A\rangle = \frac{1}{Z}\sum_{\{S_1\}}\sum_{\{S_2\}}\cdots\sum_{\{S_N\}}\exp\left[-\beta H(\{S\})\right]A(\{S\}). \tag{3}$$

Though it is difficult to calculate this macroscopic quantity directly, however it can be estimated by using the approximation theory, such as the mean-field theory and the Bethe approximation. As a recent development of statistical mechanics, researchers have clarified that statistical mechan ics serves a framework and various techniques for probabilistic information processing based on the analogy (Fig. 2) between statistical mechanics and Bayesian inference using the MPM estimate.

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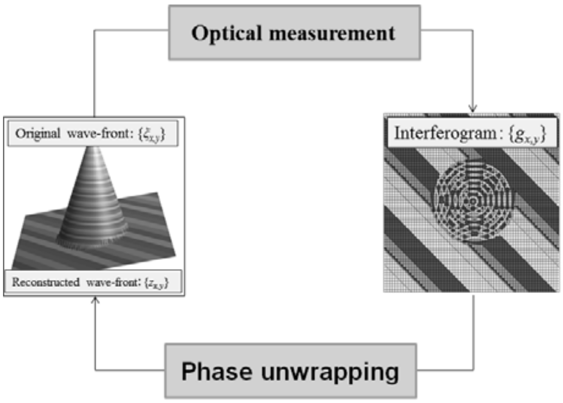


Fig. 3 General formulation for phase unwrapping

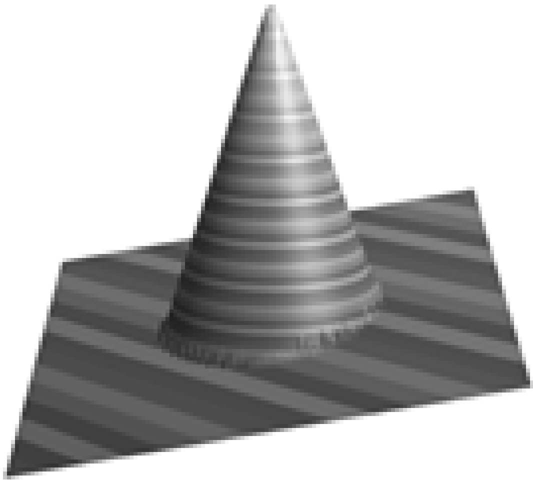


Fig. 4 Typical wave-front in SAR interferometry

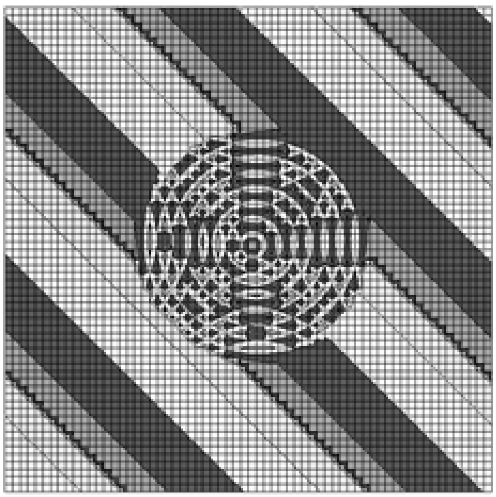


Fig. 5 Interferogram of original wave-front in Fig. 4

III. General Formulation

As shown in Fig. 3, we have presented a general formulation of the problem of phase unwrapping on the basis of Bayesian inference using the MPM estimate in this section.

Let us first consider an original wave-front $\{\xi_{x,y}\}$ ($0<\xi_{x,y}<\infty$, x, y=1,...,L) in this study. If we treat a realistic case, we can use the typical or realistic wave-front in Fig. 4. If we investigate the statistical performance of the present method, on the other hand, we need to consider a set of wave-fronts $\{\xi_{x,y}\}$ that are generated by the true prior expressed as $\Pr(\{\xi_{x,y}\})$. Then, as seen in Fig. 5, we observe the interferogram:

$$\eta_{x,y} = \mathrm{mod}(\xi_{x,y} + \pi, 2\pi) - \pi \,. \tag{4}$$

We then obtain two sets of principal values for phase differences:

$$\tau^x_{x,y} = \mathrm{mod}(\eta_{x+1,y} - \eta_{x,y} + \pi, 2\pi) - \pi + \sigma^x z(x,y) \text{ and} \tag{5}$$

$$\tau^y_{x,y} = \mathrm{mod}(\eta_{x,y+1} - \eta_{x,y} + \pi, 2\pi) - \pi + \sigma^x z(x,y) \tag{6}$$

from the interferogram. These phase differences are, as shown in Fig. 6, corrupted by noise. We especially assumed Gaussian noise $z(x, y)$ as $N(0, 1)$. As shown in Fig. 7(a), the Nyquist sampling theorem holds as

$$|\xi_{x+1,y} - \xi_{x,y}| < \pi, \tag{7}$$

if aliasing occurs at all sampling points. However, the Nyquist sampling theorem in Fig. 7(b) does not hold as

$$|\xi_{x+1,y} - \xi_{x,y}| > \pi, \tag{8}$$

if aliasing occurs at some sampling points. In this case, discontinuity 2π appears at some sampling points in the pattern of the principal values for the phase differences shown in Fig. 6. We next unwrap the phase by utilizing the set of principal values for the phase differences, $\{\tau^x_{i,j}\}$ and $\{\tau^y_{i,j}\}$, on the basis of Bayesian inference using the MPM estimate corresponding to the statistical mechanics of the three-state Q-Ising models $\{n^x_{i,j}\}$ ($n^x_{i,j}$=-1, 0, +1, i=1,…,L-1, j=1,…,L) and $\{n^y_{i,j}\}$ ($n^y_{i,j}$=-1, 0, +1, i=1,…,L, j=1,…,L-1) on the square lattice. We reconstruct the original wave-front with this method to maximize the marginal posterior probability as

$$\hat{n}^x_{x,y} = \arg\max_{n^x_{x,y}} \sum_{\{n^x\}\neq n^x_{x,y}} \sum_{\{n^y\}} \Pr(\{n^x\},\{n^y\} \mid \{\tau^x\},\{\tau^y\}) \tag{9}$$

and

$$\hat{n}^y_{x,y} = \arg\max_{n^y_{x,y}} \sum_{\{n^y\}\neq n^y_{x,y}} \sum_{\{n^x\}} \Pr(\{n^x\},\{n^y\} \mid \{\tau^x\},\{\tau^y\}), \tag{10}$$

where the posterior probability is estimated based on the Bayes formula using the likelihood and the model of the true prior as

$$\Pr\left(\{n^x\},\{n^y\} \mid \{\tau^x\},\{\tau^y\}\right) \propto \frac{\Pr\left(\{n^x\},\{n^y\}\right)\Pr\left(\{\tau^x\},\{\tau^y\} \mid \{n^x\},\{n^y\}\right)}{\sum_{\{\tau^x\}}\sum_{\{\tau^y\}} \Pr\left(\{n^x\},\{n^y\}\right)\Pr\left(\{\tau^x\},\{\tau^y\} \mid \{n^x\},\{n^y\}\right)} \tag{11}$$

We assumed the model of the true prior in this study by suppressing the number of non-zero states of the three-state Ising spins, $\{n^x_{i,j}\}$ and $\{n^y_{i,j}\}$, as

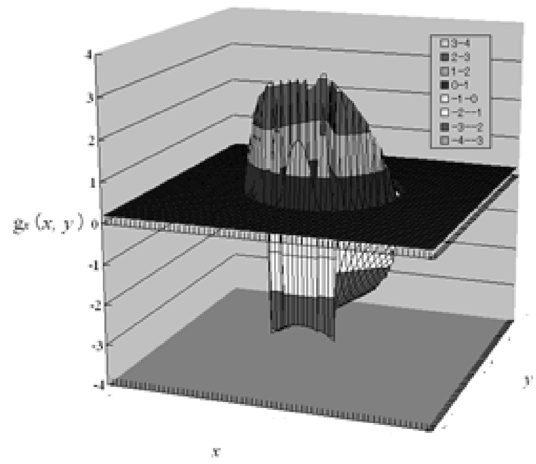
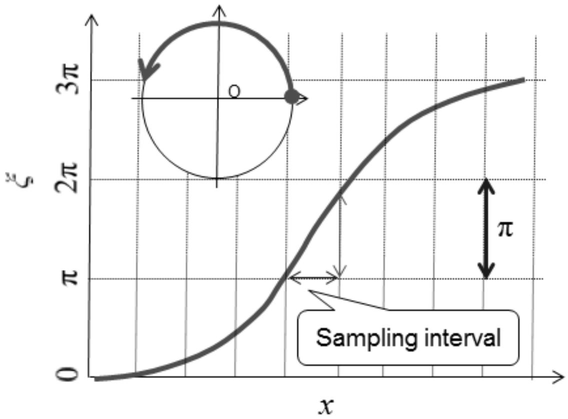
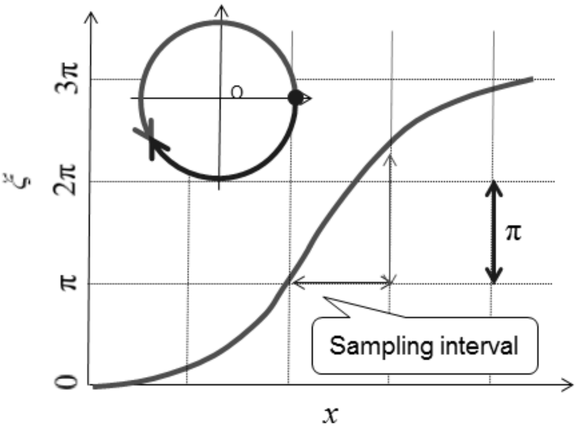


Fig. 6 Set of principal values for phase differences observed from interferogram in Fig. 5



(a)



(b)

Fig. 7 Nyquist sampling theorem. (a) Case in which aliasing does not occur at all sampling points and (b) case in which aliasing occurs at some sampling points

$$\Pr(\{n^x\},\{n^y\}) \propto \exp\left[-\frac{h}{T_m}\sum_{(x,y)}\left\{|n^x_{x,y}|+|n^y_{x,y}|\right\}\right]. \quad (12)$$

We then assumed the likelihood that would enhance the smooth structures seen in natural wave-fronts by suppressing the differences in wave-fronts as

$$\Pr\left(\{\tau^x\},\{\tau^y\}\,|\,\{n^x\},\{n^y\}\right)\propto$$
$$\exp\left[-\frac{1}{T_m}\left\{\frac{J}{(2\pi)^2}H_{\text{int}}(\{n^v{}_{i,j}\})+\frac{\Gamma}{(2\pi)^2}H_{\text{sc}}(\{n\})\right\}\right], \quad (13)$$

where

$$H_{\text{int}}\left(\{n^x\},\{n^y\}\right)=\sum_{(i,j)}\left(\tau^x_{i,j}-\tau^x_{i+1,j}+2\pi(n^x_{i,j}-n^x_{i+1,j})\right)^2$$
$$+\sum_{(i,j)}\left(\tau^y_{i,j}-\tau^y_{i,j+1}+2\pi(n^y_{i,j}-n^y_{i,j+1})\right)^2$$
$$+\alpha\sum_{(i,j)}\left(\tau^x_{i,j}-\tau^x_{i,j+1}+2\pi(n^x_{i,j}-n^x_{i,j+1})\right)^2$$
$$+\alpha\sum_{(i,j)}\left(\tau^y_{i,j}-\tau^y_{i+1,j}+2\pi(n^y_{i,j}-n^y_{i+1,j})\right)^2 \quad (14)$$

$$H_{sc}\left(\{n^x\},\{n^y\}\right)=\sum_{(i,j)}\left(\tau^x_{i,j}+\tau^y_{i,j}-\tau^x_{i+1,j}-\tau^y_{i,j}\right.$$
(14)
$$\left.+2\pi(n^x_{i,j}+n^y_{i+1,j}-n^x_{i,j+1}-n^y_{i,j})\right)^2 \quad (15)$$

As shown in Eqs. (13)–(15), the likelihood we use here is composed of two terms. The first is a term that enhances the smooth structures seen in natural wave-fronts. The second represents surface-consistency conditions for each plaquette.

A reconstructed wave-front is constructed in this study by the expectation value for a wave-front slope as

$$\hat{n}^v_{x,y}=\Theta\left(\overline{n}^v_{x,y}\right) \text{ and} \quad (16)$$
$$\overline{n}^v_{x,y}=\sum_{\{n^x\}}\sum_{\{n^y\}}\Pr\left(\{n^x\},\{n^y\}\,|\,\{\tau^x\},\{\tau^y\}\right)\cdot n^v{}_{x,y}. \quad (17)$$

This is where

$$\Theta(x)=\sum_{k=1}^{3}\theta\left(x-k+\frac{1}{2}\right)-\theta\left(x-k-\frac{1}{2}\right). \quad (18)$$

The wave-fronts are then reconstructed by making use of solutions $\{n_x\}$ and $\{n_y\}$ as

$$z_{x,y}=z_{0,0}+\sum_{l=0}^{x-1}\left(\tau_{l,0}+2\pi n_{l,0}\right)+\sum_{m=0}^{y-1}\left(\tau_{x,n}+2\pi n_{x,m}\right), \quad (19)$$

where $z_{0,0}=\xi_{0,0}=0$.

We will discuss the MPM estimate for phase unwrapping using the three-state Q-Ising model on the square lattice in the last part of this section from the viewpoint of statistical mechanics of information. In the language of statistical mechanical informatics, the maximization of marginal posterior

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probability is regarded as constructing an equilibrium state of the Q-Ising models whose Hamiltonian is at finite temperature T_m. Further, the present method is regarded as a technique of phase unwrapping searching the ground state of the Hamiltonian in Eq. (20) by utilizing thermal fluctuations from an initial state set randomly. Consequently, we explain how thermal fluctuations are useful for phase unwrapping due to the MPM estimate for typical wave-fronts in the SAR interferometery in the parts that follow.

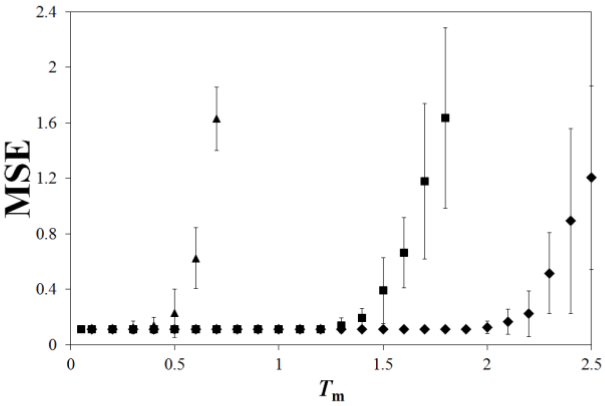

Fig. 8 Mean square error as a function of parameter T_m. First, ▲ denotes the data for J=1, α=0, Γ=0. Then, ■ denotes the data for J=1, α=0, Γ=0.1$(2\pi)^2$. Next, ◆ denotes data J=1, α=0, Γ=0.2 $(2\pi)^2$

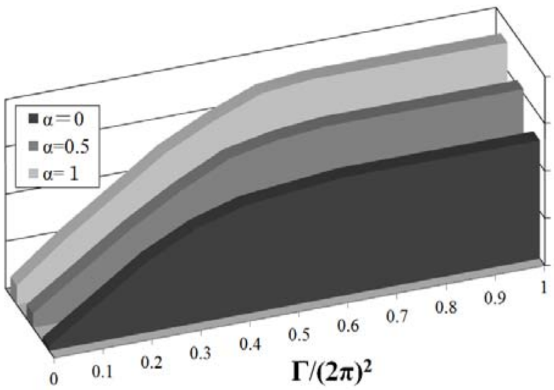

Fig. 9 Phase diagram on T_m-Γ plane of MPM estimate for phase unwrapping for wave-front in Fig. 4 for α=0, 0.5 and 1 at J=1 and h=0

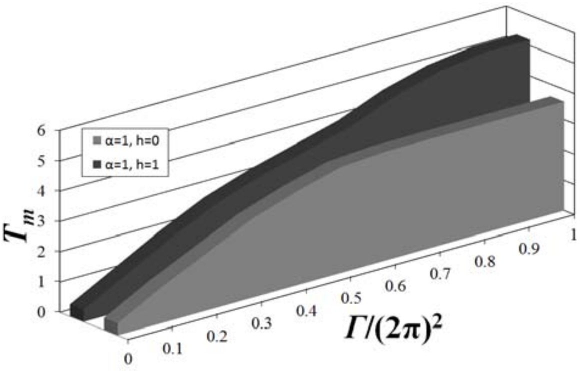

Fig. 10 Phase diagram on T_m-Γ plane of MPM estimate for phase unwrapping for wave-front in Fig. 4 for h=0, 1, at J=1, and α=1

$$H(\{n^x\},\{n^y\}) = h\sum_{(x,y)}(|\,n^x_{x,y}\,|^p + |\,n^y_{x,y}\,|^p) + \frac{J}{(2\pi)^2}H_{\mathrm{int}}\left(\{n^x\},\{n^y\}\right) + \frac{\Gamma}{(2\pi)^2}\cdot H_{sc}\left(\{n^x\},\{n^y\}\right) \quad (20)$$

We numerically evaluate the performance measure based on mean square error to clarify the performance of the present method as

$$\sigma = \frac{1}{L^2}\sum_{x=1}^{L}\sum_{y=1}^{L}\left(z_{x,y} - \xi_{x,y}\right)^2 . \quad (21)$$

Here, $\{z_{x,y}\}$ denotes the wave-front reconstructed with the present method for a realistic image. However, we estimate statistical performance for the set of $\{\xi_{x,y}\}$, which are generated by the true prior expressed as probability distribution $\Pr(\{\xi_{x,y}\})$ as

$$\sigma = \sum_{\{\xi\}}\Pr\left(\{\xi\}\right)\frac{1}{L^2}\sum_{x=1}^{L}\sum_{y=1}^{L}\left(z_{x,y} - \xi_{x,y}\right)^2 . \quad (22)$$

These variables become zero, if all kinds of wave-fronts are completely reconstructed.

IV. Performance

We investigated the performance of the MPM estimate for phase unwrapping using the Monte Carlo simulation for typical wave-fronts in SAR interferometry. We examined both the static and dynamic properties of the MPM estimate using mean square error.

When we estimated the performance of the MPM estimate, we used the original wave-front in Fig. 4. As shown in Fig. 5, we then observed the interferogram $\{\eta_{x,y}\}$ ($|\eta_{x,y}|< \pi$, x, y=1,…,L) using the optical instruments via the interferometer. When we carried out the phase unwrapping, as shown in Fig. 6, we used the principal values of phase differences $\{\tau^x_{x,y}\}$ ($|\tau^x_{x,y}|< \pi$, x=1,…,L-1, y=1,…,L) and $\{\tau^y_{x,y}\}$ ($|\tau^y_{x,y}|< \pi$, x=1,…,L, y=1,…,L-1) both of which were corrupted by additive Gaussian noise with $N(0,\sigma^x=0.02)$ and $N(0,\sigma^y=0.02)$ with each other. We carried out Monte Carlo simulation with 20000 MCS in the procedure of phase unwrapping. We then estimated statistical performance based on the mean square error between the original and reconstructed wave-fronts.

First, we estimated the static properties of the MPM estimate for the wave-front in Fig. 4 based on the phase diagram in hyper-parameter space. We described the phase diagram by evaluating how mean square error depends on parameter Γ for our purposes. We found that MPM estimates were successful in phase unwrapping up to T_m~0.4 (1.2, 2.0) as shown in Fig. 8, if we set $\Gamma/(2\pi)^2$=0 (0.1, 0.2) at J=1, α=0, and h=0. These results indicated that the surface-consistency conditions were useful for extending the range of the PU phase in hyper-parameter space. We described the phase diagram given in Fig. 9 by carrying out the simulations for various values of hyper-parameters, such as T_m, Γ, and α.

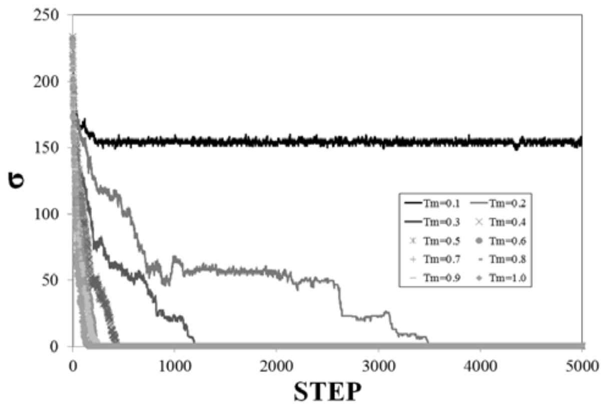

Fig. 11 Time evolution of mean square error obtained by MPM estimate using Monte Carlo simulation for wave-front in Fig. 2

This figure clarifies that the MPM estimate succeeds in phase unwrapping over a wide range under the constraint of surface-consistency condition. We next evaluated how mean square error depends on parameter h to describe the phase diagram in hyper-parameter space to clarify the efficiency of prior information in Eq. (11). As seen in Fig. 10, we clarified that prior information is useful for extending the PU phase under the constraints of surface-consistency conditions. However, we also found that prior information is not useful for extending the PU phase, if we set $h<0.6$. These results indicated that the present method works effectively even at high temperature up to $T_m \sim 5.2$ under the constraint of the surface-consistency condition. Also, we indicated that the prior information on the original wave-fronts is useful for extending the PU phase under the constraint of the surface-consistency condition.

Next, we examined the dynamic properties of the MPM estimate for phase unwrapping. As shown in Fig. 11, we described the time evolution of mean square error using Monte Carlo simulations for the wave-front in Fig. 4. This figure indicates that the MPM estimate smoothly reconstructs the wave-front with great accuracy in the upper region where the MPM estimate is successful in very accurate phase unwrapping. For instance, we found the result that the MPM estimate accurately reconstructs the original wave-fronts in Fig. 4 by using the Monte Carlo simulation with ~230 Monte Carlo steps (MCS) at $T_m=1$, $J=1$, $\alpha=0$, $h=0$ and $\Gamma=0.2(2\pi)^2$. This suggests that the fluctuations around the MAP solution are useful for achieving smooth phase unwrapping due to the MPM estimate.

V. Statistical Mechanical Performance

On the basis of analogy between statistical mechanics and Bayesian inference via the MPM estimate, we discussed the performance of the present method for phase unwrapping for the artificial wave-front in SAR interferometry. In this field of statistical mechanics, researchers have utilized the Monte Carlo simulations to clarify thermodynamic properties of many-body systems, such as the Ising spin system. So, we here utilized the Monte Carlo simulations which numerically approximated both static and dynamic properties based on various methods, such as the Metropolis and heat bath algorithm.

Especially, we examined both static and dynamic properties of the MPM estimate utilizing Monte Carlo simulation via the Metropolis algorithm. As shown in the previous chapter, it was obviously seen that the present method was a method for searching the ground state of the three-state Q-Ising model utilizing thermal fluctuations, and then, phase unwrapping due to the MPM estimate was regarded as relaxation from an initial state to the ground state. Therefore, we estimated the transition probability from 1st excited state to the ground state in relaxation process by making use of the Monte Carlo simulation based on the Metropolis algorithm. In general, it has been well-known that low-temperature property of many-body systems depends both on the ground state and the energy gap $\Delta\varepsilon = \varepsilon_{1st} - \varepsilon_{g.s.}$ above the ground state. So, we here discussed the relation between transition probability between the ground state and 1st excited state and the energy gap between these two states. In the Monte Carlo simulation based on the Metropolis algorithm to construct thermal equilibrium state of many-body systems, we utilized the transition probability from an initial state $\{z_1\}$ to a final state $\{z_2\}$ as

$$\Pr(\{z_1\} \to \{z_2\}) = \begin{cases} 1 & (\varepsilon_2 < \varepsilon_1) \\ \exp[-(\varepsilon_2 - \varepsilon_1)/T_m] & (\varepsilon_2 < \varepsilon_1) \end{cases}, \quad (23)$$

based on the Metropolis algorithm. Here, ε_1 (ε_2) is the energy of the state $\{z_1\}$ ($\{z_2\}$). Then, based on the Metropolis algorithm in Eq. (23), it was obviously seen that the transition probability at $T=T_m$ from the ground state {g. s.} to the 1st excited state {1st excited state} is expressed as

$$\Pr(\{\text{g.s.}\} \to \{1^{\text{st}} \text{ excited state}\}) \propto \exp\left[-\frac{\Delta\varepsilon}{T_m}\right]. \quad (24)$$

This showed the results that transition probability from the ground state to the 1st excited state almost vanishes under the constraint of surface-consistency condition, if $\Delta\varepsilon \ll T_m$. and therefore that the ground state becomes stable even at finite temperature under this condition.

Next, with the use of the knowledge on statistical mechanics, we here examined the static and dynamic properties of the MPM estimate for phase unwrapping. Then, in view of statistical mechanics, the MPM estimate for the present case was regarded as relaxation process to the ground state of the present system with the Hamiltonian in Eq. (20). By considering the elementary excitation from the ground state of this system, we easily obtained the energy gap above the ground state as

$$\Delta\varepsilon = \varepsilon_{1\text{st}} - \varepsilon_{\text{g.s.}} \sim 2J(1+\alpha) + 2\Gamma + h. \quad (25)$$

From this equation, we derived the result that the system has a large energy gap $\Delta\varepsilon \sim 2(2\pi)^2$ above the ground state under the constraint of the surface-consistency condition, if we set to $\Gamma \sim (2\pi)^2$, $J \sim 1$, $\alpha \sim 1$ and $h \sim 1$. This result clarified that the transition probability at $T_m \ll 2(2\pi)^2$ from the ground state to the 1st excited state almost vanishes as

$$\Pr(\{\text{g.s.}\} \to \{1^{\text{st}} \text{ excited state}\}) \propto \exp\left[-\frac{2(2\pi)^2}{T_m}\right], \quad (26)$$

if we set to $\Gamma \sim (2\pi)^2$ in the relaxation process of the present system via the Metropolis algorithm, and therefore that the system is not excited from the ground state by thermal fluctuations under this condition. Actually, as shown in Fig. 9, it was confirmed that the MPM estimate is effective up to T_m=2.6 (3.1, 3.6) under the constraint of surface-consistency condition, $0.6(2\pi)^2<\Gamma$. Next, we examined the efficiency of the prior information for the MPM estimate for phase unwrapping. As seen from Eq. (25), we easily saw that the energy gap above the ground state was further enhanced by introducing the prior information into the model system, and then that the ground state is further stabilized. Actually, as shown in Fig. 10, it was confirmed that that the MPM estimate succeeds in extending the PU phase up to T_m=5.1 by introducing prior information into the three-state Q-Ising model in Eq. (20).

Next, from the statistical mechanical point of view, we considered the dynamical property of the MPM estimate via the Monte Carlo simulation based on the Metropolis algorithm. Because the transition probability between two states generally became larger with the increase in the absolute temperature T_m in the PU phase, it was therefore expected that the equilibrium state was smoothly carried out at high temperature T_m in the PU phase. Based on this knowledge, it was obvious that the convergence to the ground state became smoother at higher temperature in the PU phase. Actually, as shown in Fig. 11, it was confirmed that the MPM estimate smoothly reconstructs the wave-front with great accuracy in the upper region where the MPM estimate is successful in very accurate phase unwrapping.

At the end of this chapter, we compared the performance of the MPM estimate with that of the MAP estimation via simulated annealing [8]. In the MAP estimation via simulated annealing, we optimized the cost function expressed as the Hamiltonian in eq. (17) following the annealing schedule:

$$T_m = T_{\text{initial}} - (T_{\text{initial}} - T_{\text{final}})\frac{n}{\text{STEP}}. \quad (27)$$

Here, $T_{\text{initial}}/T_{\text{final}}$ is initial/final temperature set for simulated annealing. Then, STEP is the number of the Monte Carlo steps (MCS) and n is an integer from 0 to STEP. Using the Monte Carlo simulation for the artificial wave-fronts in Fig. 3, we found that the MAP estimation via the simulated annealing is successful in phase unwrapping with the same accuracy as the MPM estimate, and that the MAP estimation via the simulated annealing succeeds in phase unwrapping for the typical wave-front with ~500 MCS, if we set to J=1, α=1, h=1, T_{initial} =8.0, T_{final}=1.0 and STEP=1000. Although the simulations have been carried out under restricted conditions, we found that it is very difficult to detect the optimal condition to realize smooth phase unwrapping due to complexity of the setting of the annealing schedule. On the other hand, as shown in above, we found that it is not so difficult to find the optimal condition to realize smooth phase unwrapping. These facts suggested that the MPM estimate is more useful than the MAP estimation via simulated annealing for phase unwrapping.

VI. Summary and Discussion

We constructed a method of phase unwrapping via a three-state Q-Ising model arranged on a square lattice on the basis of an analogy between statistical mechanics and Bayesian inference via a MPM estimate.

We then investigated both the static and dynamic properties of the MPM estimate by making use of Monte Carlo simulations for the artificial wave-front which was typical in the SAR interferometry to clarify the performance of the present method for phase unwrapping. First, we examined the static properties of the MPM estimate based on a phase diagram of the MPM estimate in hyper-parameter space, such as the T_m-Γ plane. The phase diagram clarified that the MPM estimate reconstructed the wave-front under the constraints of surface-consistency conditions and that prior information was useful for expanding regions where the MPM estimate effectively worked under the constraint of surface-consistency condition. We then found that the MPM estimate smoothly reconstructed original wave-fronts, which are typical in the SAR interferometry. Next, in view of statistical mechanics, we investigated static and dynamic properties of the MPM estimate for phase unwrapping. We found in view of statistical mechanics that the MPM estimate is regarded as the method for searching the ground state of the three-state Q-Ising model via thermal fluctuations around the MAP solution, and then that phase unwrapping by means of the MPM estimate corresponds to relaxation process of the three-state Q-Ising model. Especially, we found that it is important to enhance energy gap above the ground state to extend the PU phase by introducing the constraint of the surface-consistency condition and that the PU phase is further extended by introducing the model prior under the constraint of the surface-consistency condition. Then, we found that the MPM estimate succeeds in phase unwrapping smoothly, so that the energy gap is set larger than the thermal fluctuations around the MAP estimation. We found that the MPM estimate is useful for phase unwrapping under the constraint of the surface- consistency condition.

It is important to clarify the performance of the present method for realistic cases, such as InSAR images, to solve future problems.

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