

Linear Maps That Preserve Left Spectrum of Diagonal Quaternionic Matrices

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Abstract—In this paper, we discuss some properties of left spectrum and give the representation of linear preserver map the left spectrum of diagonal quaternionic matrices.

Keywords—Quaternionic matrix, left spectrum, linear preserver map.

I. INTRODUCTION

LINEAR preserver problems is an active research area in matrix and operator theory. These problems involve certain linear maps on spaces of matrices or operators. Concretely, if V is a space of matrices over the field F , the problem for given a (scalar-valued, vector-valued, or set-valued) function F on V , studying the linear preservers of F , i.e., those linear maps on V satisfying

$$F(\varphi(A)) = F(A) \text{ for all } A \in V$$

is said to be the linear preserver problem.

Over a hundred years ago, Frobenius studied the following problem. Let $M_{(m,n)}(F)$ be the set of $m \times n$ matrices over a field F and $M_n(F) = M_{(n,n)}(F)$. If φ is a linear map on $M_n(C)$ satisfying

$$\det(\varphi(A)) = \det(A) \quad (1)$$

for all $A \in M_n(C)$, what can be said of φ ? Frobenius [2] showed that every linear map satisfying (1) must be of the form

$$\varphi(A) = MAN \text{ or } \varphi(A) = MA^t N \quad (2)$$

for some $M, N \in M_n(C)$ with $\det(MN) = 1$.

The linear operator φ satisfying (1) is called a linear preserver of the determinant function or simply a determinant preserver. In the last few decades, thousands of papers were published on linear preservers. Many interesting results were obtained. See [3] for a detailed discussion.

Quaternions are very useful. For example, in mathematical physics, quantum mechanics based on quaternionic analysis are mainstream physics. In engineering, quaternions usually used

in control systems and computer science, play a role in computer graphics. Quaternion formalism is also used in studies of molecular symmetry.

Quaternion as an algebraic system is a noncommutative division algebra, linear preserver problem on matrices with quaternion entries is fragmentary. Due to the noncommutativity of quaternions, the eigenvalues of a quaternionic matrix are two types of eigenvalues, which are left eigenvalues of quaternionic matrices and right eigenvalues of quaternionic matrices. Right eigenvalues are well studied in literature [1], [5]. On the contrary, left eigenvalues are less known.

Zhang [8] commented that left eigenvalues is not easy to handle, and few results have been obtained. Wood [7] used a topological method to confirm that the left eigenvalue always exists and demonstrated that the left eigenvalues of a 2×2 matrix can be found by solving a quaternionic quadratic equation. So [6] considered the existence of a left eigenvalue of a 3×3 quaternionic matrix. Recently, Huang and So gave a characterization of 2×2 matrices having an infinite number of left eigenvalues in [4]. It is still an open problem whether the approach of Algebra Fundamental Theorem works for general $n \times n$ matrices for $n \geq 4$.

Note that the left spectrum is not similarly invariant, thus it is an interesting question for discussing the representation of the linear preserver map of the left spectrum. In this paper, we discuss some properties of the left spectrum and give the representation of linear preserver the left spectrum of certain quaternionic matrices.

II. PRELIMINARIES

Let R and C be the set of real numbers and complex numbers, respectively. Q is the set of quaternions of the form $a_1 + a_2i + a_3j + a_4k$, where $a_i \in R$ and

$$i^2 = j^2 = k^2 = ijk = -1.$$

Let $M_n(Q)$ be the set of all $n \times n$ matrices over Q , $DM_n(Q)$ be the set of all diagonal matrices in $M_n(Q)$, and E denotes $n \times n$ identity matrix.

Definition 1. Given $A \in M_n(Q)$, $\lambda \in Q$ is called a left eigenvalue of A if

$$Ax = \lambda x$$

for some nonzero $x \in \mathcal{Q}^n$. The set of distinct left eigenvalues is called the left spectrum of A , denoted by $\sigma_l(A)$.

Definition 2. The map $\Phi : M_n(\mathcal{Q}) \rightarrow M_n(\mathcal{Q})$ is called quaternion linear map if Φ satisfies:

$$\begin{aligned} \Phi(A+B) &= \Phi(A) + \Phi(B) \\ \Phi(Aq) &= \Phi(A)q \end{aligned}$$

for all $A, B \in M_n(\mathcal{Q})$ and $q \in \mathcal{Q}$.

For convenience, we list a result in [4] as our Lemma 1.

Lemma 1 ([4]). Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(a) If $bc = 0$, then $\sigma_l(A) = \{a, d\}$;

(b) If $bc \neq 0$, then

$$\sigma_l(A) = \{a + b\lambda, \lambda^2 + b^{-1}(a-d)\lambda - b^{-1}c = 0\}.$$

Lemma 2. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $\sigma_l(A) = \{0, 1\}$, then A has forms in the following.

- i. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$;
- ii. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$;
- iii. $A = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$, $b \neq 0$;
- iv. $A = \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix}$, $b \neq 0$;
- v. $A = \begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}$, $c \neq 0$;
- vi. $A = \begin{bmatrix} 1 & 0 \\ c & 0 \end{bmatrix}$, $c \neq 0$;
- vii. $A = \begin{bmatrix} bt & b \\ dt & d \end{bmatrix}$, $t = b^{-1}a$, $bdt \neq 0$, $bt + bdb^{-1} = 1$.

Proof. If $bc = 0$, by Lemma 1(a), it is easy to imply that (i), (ii), (iii), (iv), (v) and (vi) are valid.

If $bc \neq 0$, by Lemma 1 (b), then there exist $\lambda_1, \lambda_2 \in \mathcal{Q}$ such that

$$\lambda_1^2 + b^{-1}(a-d)\lambda_1 - b^{-1}c = 0 \tag{3}$$

$$\lambda_2^2 + b^{-1}(a-d)\lambda_2 - b^{-1}c = 0 \tag{4}$$

where $a + b\lambda_1 = 0$ and $a + b\lambda_2 = 1$.

Since $a + b\lambda_1 = 0$, $bc \neq 0$, one has $\lambda_1 = -b^{-1}a$, by (3) and $\lambda_1 = -b^{-1}a$, then

$$b^{-1}ab^{-1}a - b^{-1}ab^{-1}a + b^{-1}db^{-1}a - b^{-1}c = 0,$$

Thus $db^{-1}a = c$.

Let $t = b^{-1}a$, then

$$A = \begin{bmatrix} bt & b \\ dt & d \end{bmatrix}.$$

By (4), then

$$\lambda_2^2 + b^{-1}(a-1)\lambda_2 + b^{-1}(1-d)\lambda_2 - b^{-1}c = 0.$$

Since $a + b\lambda_2 = 1$, that is $\lambda_2 = -b^{-1}(a-1)$, we can imply that

$$\begin{aligned} (1-d)\lambda_2 &= c; \\ \lambda_2 - d\lambda_2 &= c; \\ \lambda_2 + db^{-1}(a-1) &= c; \\ \lambda_2 + db^{-1}a - db^{-1} &= c; \\ \lambda_2 + c - db^{-1} &= c. \end{aligned}$$

Hence

$$\lambda_2 = db^{-1}.$$

Note that $a + bdb^{-1} = 1$, then $bt + bdb^{-1} = 1$. The proof is completed.

Lemma 3. Let

$$\begin{aligned} A &= \Phi \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \\ B &= \Phi \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \end{aligned}$$

and

$$\sigma_l(A) = \sigma_l(B) = \{0, 1\}.$$

If $\Phi(E) = E$, then

- (i) $\Phi \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\Phi \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$;
- (ii) $\Phi \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Phi \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$;

$$(iii) \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b_1 \\ 0 & 1 \end{pmatrix}, \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_2 \\ 0 & 0 \end{pmatrix};$$

and $b_1 \neq 0, b_2 \neq 0, b_1 + b_2 = 0$.

$$(iv) \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 0 \end{pmatrix}, \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b_2 \\ 0 & 1 \end{pmatrix};$$

and $b_1 \neq 0, b_2 \neq 0, b_1 + b_2 = 0$.

$$(v) \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c_1 & 1 \end{pmatrix}, \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_2 & 0 \end{pmatrix};$$

and $c_1 \neq 0, c_2 \neq 0, c_1 + c_2 = 0$.

$$(vi) \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c_1 & 0 \end{pmatrix}, \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c_2 & 1 \end{pmatrix};$$

and $c_1 \neq 0, c_2 \neq 0, c_1 + c_2 = 0$.

$$(vii) \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_1 t_1 & b_1 \\ d_1 t_1 & d_1 \end{pmatrix}, \Phi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_2 t_2 & b_2 \\ d_2 t_2 & d_2 \end{pmatrix};$$

where $b_1, t_1, d_1; b_2, t_2, d_2$ satisfy the conclusion of the Lemma 2 and

$$b_1 t_1 + b_2 t_2 = 1, b_1 + b_2 = 0, d_1 t_1 + d_2 t_2 = 1, d_1 + d_2 = 0,$$

Proof. By $\Phi(E) = E$ and Lemma 2, by simple computation, the lemma follows. Here, the proof is omitted.

III. REPRESENTATION OF LINEAR PRESERVER MAP OF LEFT SPECTRUM

The following theorem is the main result of this paper.

Theorem. Let quaternion linear map $\Phi: M_2(D) \rightarrow M_2(Q)$ satisfy $\Phi(E) = E$, that $\sigma_l(A) = \sigma_l(\Phi(A))$ for all $A \in DM_n(Q)$, then Φ has one of the following forms.

- (i) $\Phi(D) = D$ for all $D \in DM_n(Q)$;
- (ii) $\Phi(D) = UDU$ for all $D \in DM_n(Q)$;
- (iii) $\Phi(D) = UXPDOPYU + U(Y^T QDQX^T)^T U$;
- (iv) $\Phi(D) = XPDPY + (Y^T QDQX^T)^T$;
- (v) $\Phi(D) = U(XPDPY)^T U + U(Y^T QDQX^T)U$;
- (vi) $\Phi(D) = (XPDPY)^T + (Y^T QDQX^T)$;
- (vii) $\Phi(D) = A(T_1 D U + T_2 U D)U$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X = \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 \\ b_2 & 1 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, T_1 = \begin{pmatrix} t_1 & 0 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 1 \\ t_2 & 0 \end{pmatrix}, A = \begin{pmatrix} b_1 & b_2 \\ d_1 & d_2 \end{pmatrix}$$

$$b_1 t_1 + b_2 t_2 = 1, b_1 + b_2 = 0, d_1 t_1 + d_2 t_2 = 1, d_1 + d_2 = 0,$$

Proof.

(1) By Lemma 3 (i), then $\Phi(D) = D$ for all $D \in DM_n(Q)$;

(2) By Lemma 3 (ii), we get

$$\begin{aligned} \Phi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} &= \Phi \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} b \right) = \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} a + \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} b \\ &= \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Let $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then

$$\Phi(D) = UDU$$

for all $D \in DM_n(Q)$.

(3) Since $\Phi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} a + \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} b$

by Lemma 3 (iii), we know

$$\Phi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} b & b_1 a + b_2 b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b_1 a + b_2 b & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that

$$\begin{aligned} \begin{pmatrix} a & 0 \\ b_1 a & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_2 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ b_2 b & b \end{pmatrix} &= \left(\begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \right)^T \end{aligned}$$

Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 \\ b_2 & 1 \end{pmatrix},$$

then

$$\Phi(D) = UXPDOPYU + U(Y^T QDQX^T)^T U;$$

For these cases (iv) (v) (vi) in Lemma 3, by the analogous proof of (iii), we get the conclusions (iv)(v)(vi) in Theorem is valid, here, the proof is omitted.

(4) By Lemma 3 (vii), note that

$$\begin{aligned} \Phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} b_1 t_1 & b_1 \\ d_1 t_1 & d_1 \end{pmatrix}, & b_1 t_1 + b_1 d_1 b_1^{-1} &= 1. \\ \Phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} b_2 t_2 & b_2 \\ d_2 t_2 & d_2 \end{pmatrix}, & b_2 t_2 + b_2 d_2 b_2^{-1} &= 1. \end{aligned}$$

Therefore

$$\begin{aligned} \Phi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) &= \Phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)a + \Phi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)b \\ &= \begin{bmatrix} b_1 t_1 a + b_2 t_2 b & b_1 a + b_2 b \\ d_1 t_1 a + d_2 t_2 b & d_1 a + d_2 b \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix} \begin{bmatrix} t_1 a & a \\ t_2 b & b \end{bmatrix} \\ &= \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix} \left(\begin{bmatrix} t_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) + \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ t_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix} \begin{bmatrix} t_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ t_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

Let

$$T_1 = \begin{bmatrix} t_1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 \\ t_2 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix},$$

then T_1 and T_2 are invertible.

If there exists a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that

$$\begin{cases} b_1 x_1 + b_2 x_2 = 0 \\ d_1 x_1 + d_2 x_2 = 0 \end{cases}$$

Since $b_1 = -b_2$, we have $x_1 - x_2 = 0$, by

$$d_1 + d_2 = 1,$$

then $x_1 = x_2 = 0$. Hence A is invertible.

By above arguments, we have

$$\begin{aligned} \Phi(D) &= AT_1 D + AT_2 U D U \\ &= (AT_1 D U + AT_2 U D) U \\ &= A(T_1 D U + T_2 U D) U. \end{aligned}$$

The proof is completed.

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